Secularity in $u(\theta)$ and Bertrand's Thm

(↳ closed orbits ONLY for Kepler or HO; else secular)

$$\frac{d^2u}{d\theta^2} + u = -r^2 f \quad \text{(Normalized)}$$

$$u = 1/r \qquad f \to \frac{1}{r^\alpha} \Rightarrow \boxed{X \equiv \alpha - 2}$$

$$\text{let } r^2 f \to -\frac{1}{r^X} \to -u^X$$

$$\Rightarrow \boxed{u'' + u = u^X}$$

$X = 0 \Rightarrow 1/r^2$ Kepler

$X = -3 \Rightarrow r$ H.O.

Circular orbit $\Rightarrow u = 1$

let $u = 1 + \tilde{u}$, $|\tilde{u}| \ll 1$

$$\Rightarrow \tilde{u}'' + \tilde{u} = (1+\tilde{u})^X - 1$$

$$= X\tilde{u} + X(X-1)\frac{\tilde{u}^2}{2} + X(X-1)(X-2)\frac{\tilde{u}^3}{6} + \cdots$$

let $\beta^2 = 1 - X$, β = rational

$X = 0 \Rightarrow$ No h.o.

$X = 2 \Rightarrow \alpha = 4$ unstable

$$\Rightarrow \boxed{\frac{\tilde{u}''}{\beta^2} + \cancel{\beta^2}\tilde{u} = -X\frac{\tilde{u}^2}{2} - X(X-2)\frac{\tilde{u}^3}{6} + \cdots}$$

let $\tilde{u} = u_1 + \underbrace{u_2}_{\epsilon} + \underbrace{u_3}_{\epsilon^2} + \cdots$, $|u_{n+1}| \ll |u_n|$

$$\Rightarrow \tilde{u}^2 \to (u_1 + u_2)^2 \to u_1^2 + 2u_2u_1 + O(\epsilon^4)$$

$$\tilde{u}^3 \to u_1^3 + O(\epsilon^4)$$

So $u_1'' + \beta^2 u_1 = 0$

$$\Rightarrow \boxed{u_1 = a_1 \cos(\beta\theta)}$$

$$\frac{u_2''}{\beta^2} + u_2 = -\frac{x a_1^2}{4}\left(1 + \cos(2\beta\theta)\right)$$

$$\Rightarrow \boxed{u_2 = -\frac{x a_1^2}{4} + \frac{x a_1^2}{12}\cos(2\beta\theta)}$$

$$\boxed{\frac{u_3''}{\beta^2} + u_3 = -x\, u_2 u_1 - x(x-2)\frac{u_1^3}{6}}$$

$u_2 u_1$ will have terms like $\cos(\beta\theta)$

u_1^3 " " " " "

$\Rightarrow$ secular behavior, $u_3 \sim t\sin(\beta\theta)$

$\to \infty$ as $t \to \infty$; perturbation fails

unless coefficient of $\cos(\beta\theta)$ term vanishes.

$\boxed{\text{• Clearly, } x = 0 \text{ is one possibility}}$

Also, Consider $u_2 u_1 + \frac{(x-2)}{6} u_1^3 \to \frac{x}{12}\cos(\beta\theta)\left[\cos(2\beta\theta) - 3\right]$

$$\frac{(x-2)}{6}\cos^3(\beta\theta) \to \frac{x}{12}\left[\frac{1}{2}\cos(\beta\theta) - 3\cos(\beta\theta)\right]$$

$$+\frac{(x-2)}{6}\frac{3}{4}\cos(\beta\theta) \to \frac{x}{12}\left(\frac{-5}{2}\right) + \frac{(x-2)}{8} \to -5x + 3(x-2)$$

$= -2x - 6 = -2(x+3)$. $\boxed{\text{This is zero for } x = -3}$

$\boxed{\therefore \text{ Closed orbits for } x = 0 \text{ or } x = -3 \Rightarrow f = \frac{-1}{r^2} \text{ or } f = -r}$

Secular behavior (example)

Suppose $u_3'' + \beta^2 u_3 = A\cos(\beta\phi)$

Try $u_3 = C\phi \sin(\beta\phi)$

$u_3'' = -C\phi\beta^2 \sin(\beta\phi) + 2\beta C\cos(\beta\phi)$

$$\Leftrightarrow \underbrace{-C\phi\beta^2 \cancel{\sin(\beta\phi)} + 2C\beta\cos(\beta\phi) + \beta^2 C\phi \cancel{\sin(\beta\phi)}}_{\text{cancel}} = A\cos(\beta\phi)$$

$\Rightarrow 2C\beta = A \quad \Rightarrow \boxed{C = \frac{A}{2\beta}}$

$\Rightarrow u_3 = \frac{A}{2\beta}\phi\sin(\beta\phi)$

$u_3 > u_2$ as $\phi \to \infty \Rightarrow$ failure of expansion.