EM forces and Lagrange Eqns

① $\vec{E}$ and $\vec{B}$ can be written as $\{\vec{A}, \varphi\}$

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$

let $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ ⓐ $\Rightarrow \vec{\nabla}\times\vec{E} = -\partial_t \vec{\nabla}\times\vec{A}$

$$= -\vec{\nabla}\times\partial_t \vec{A}$$

$$\Rightarrow \vec{\nabla}\times\left[\vec{E} + \partial_t\vec{A}\right] = 0 \Rightarrow \boxed{\vec{E} = -\vec{\nabla}\varphi - \partial_t\vec{A}}$$ ⓑ

② Rewrite EM force in terms of $\{\vec{A}, \varphi\}$

$$\vec{F}_{EM}/q = \vec{E} + \vec{v}\times\vec{B} = -\vec{\nabla}\varphi - \partial_t\vec{A} + \vec{v}\times(\vec{\nabla}\times\vec{A})$$

$$= -\vec{\nabla}\varphi - \partial_t\vec{A} - \vec{v}\cdot\vec{\nabla}\vec{A} + (\vec{\nabla}\vec{A})\cdot\vec{v}$$

$$= -\vec{\nabla}\varphi - d\vec{A}/dt + \vec{\nabla}(\vec{A}\cdot\vec{v}), \text{ using } (\vec{\nabla}\vec{v}) = 0,$$

and $\frac{d}{dt}\vec{A}[\vec{r}(t), t] = \vec{v}\cdot\frac{\partial\vec{A}}{\partial\vec{r}} + \frac{\partial\vec{A}}{\partial t}$

$$\boxed{\vec{F}_{EM} = -q\left[\frac{d\vec{A}}{dt} + \vec{\nabla}(\varphi - \vec{v}\cdot\vec{A})\right]}$$ ⓒ

③ Show using $\vec{F}_{EM}$ in © that Lagrange Eqns can be redefined to reduce to "standard" form

We have $$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) - \left(\frac{\partial L}{\partial \vec{r}}\right) = \vec{F}_{EM}$$

where $L = T - V(\vec{r})$

using © $\Rightarrow$ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) - \left(\frac{\partial L}{\partial \vec{r}}\right) = -q\frac{d\vec{A}}{dt} - q\vec{\nabla}(\varphi - \vec{v}\cdot\vec{A})$$

Collect $\Rightarrow$ $$\frac{d}{dt}\left[\frac{\partial L}{\partial \vec{v}} + q\vec{A}\right] - \frac{\partial}{\partial \vec{r}}\left[L - q(\varphi - \vec{v}\cdot\vec{A})\right]$$

But $q\vec{A} \equiv \frac{\partial}{\partial \vec{v}}(q\vec{v}\cdot\vec{A})$, since $\vec{A} = \vec{A}(\vec{r}, t)$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \vec{v}}\left(L + q\vec{v}\cdot\vec{A}\right)\right] - \frac{\partial}{\partial \vec{r}}\left[L + q\vec{v}\cdot\vec{A} - q\varphi\right] = 0$$

By inspection, using $\partial\varphi/\partial\vec{v} = 0$, we get

$$\boxed{\frac{d}{dt}\left(\frac{\partial L'}{\partial \vec{v}}\right) - \left(\frac{\partial L'}{\partial \vec{r}}\right) = 0}$$

where $\boxed{L' = T - V - q(\varphi - \vec{v}\cdot\vec{A})}$