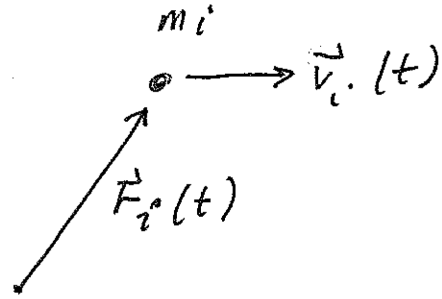
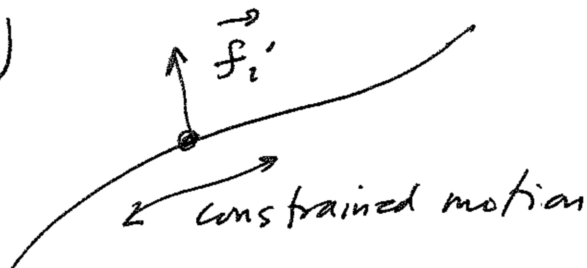
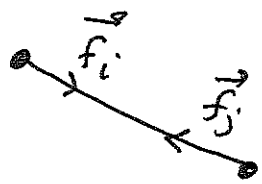
Derivation of Lagrange's Eqns from Newton's Eqns plus constraints

(some summations are suppressed, Einstein convention)



- N particles, $i = 1, \ldots, N$
- $3N$ degrees of freedom, max
- n degrees of freedom left after constraints, $n \leq 3N$ ($3N - n$ constraints)
- ~~define~~ "$\delta \vec{r}_i$" are virtual displacements consistent with constraints





- $\vec{f}_i$ are forces of constraint;

  assume either $\vec{f}_i \cdot \delta\vec{r}_i = 0$ (constraints do no work)

  OR $\sum\limits_{\substack{\text{some} \\ \text{connected masses}}} \vec{f}_i \cdot \delta \vec{r}_i = 0$, (rigid constraints between masses equal & opposite)

- Assume holonomic constraints, ie,

$$f(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t) = 0,$$

ie, coordinates are related. Thus, if one coordinate changes, another one will be forced to change

- n degrees of freedom left after constraints
$\Rightarrow$ introduce n generalized coordinates $q_j(t)$, $j = 1, \ldots, n$. The $3N$ $\vec{r}_i$ can now be expresed in terms of the n $\vec{q}_j$'s (and assume vice versa)

$$\Rightarrow \text{let } \boxed{\vec{r}_i = \vec{r}_i\,(q_j, t)} \quad (0) \qquad \begin{array}{l} i = 1, \ldots, N \\ j = 1, \ldots, n \\ n \leq 3N \end{array}$$

- The t dependence means we allow this condition to change explicitly with time

Derivation

- $$m_i \dot{\vec{V}}_i = \vec{F}_i$$

let $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$ constraints

use $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0 \Rightarrow$

$$\boxed{\sum_i m_i \dot{\vec{V}}_i \cdot \delta \vec{r}_i = \sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i} \quad (1)$$

- Starting point. Motivation: $\vec{f}_i$ have been eliminated. 2nd Motivation: even though this is only <u>one</u> equation (we need n eqns!), we could "vary" each $\delta \vec{r}_i$ "independently" and set each $(\delta \vec{r})_k$ coordinate eqn separately to zero.

– However, not so fast, since $\delta \vec{r}_i$ are not all independent. $\therefore$ re express $\delta \vec{r}_i$ in terms of δq_j, $j = 1, \cdots, n$, and then do independent virtual displacements $\Rightarrow$ n equations.

- Use $\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$. ✓ summation

  Note virtual displacement $\Rightarrow$ "δt" $= 0$

Use this in ①.

$$\Rightarrow \text{RHS} = \vec{F}_i \cdot \delta \vec{r}_i = \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

let $\boxed{Q_j \equiv \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}}$, "generalized force"

$$\Rightarrow \text{RHS} = Q_j \, \delta q_j$$

In the case of conservative forces, Q_j simplifies;

Suppose $\vec{F}_i = -\frac{\partial V}{\partial \vec{r}_i}$. Then,

$$\vec{F}_i \cdot \delta \vec{r}_i = -\frac{\partial V}{\partial \vec{r}_i} \cdot \delta \vec{r}_i = -\delta V(\vec{r}_i), \text{ chain rule}$$

and $\delta V(q_j) = \frac{\partial V}{\partial q_j} \delta q_j$

$$\Rightarrow \quad \boxed{Q_j = -\frac{\partial V}{\partial q_j}}$$

for conservative forces $\vec{F}_i = -\frac{\partial V}{\partial \vec{r}_i}$

$$V = \sum_i V_i(q_j)$$

- LHS is messier. First note that

$$\boxed{\vec{V}_i \equiv \dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}} \quad \text{(2)}, \text{ from } \text{(0)}$$

Henceforth,* assume that all partials are with respect to 3 "variables", $\{\dot{q}, q, t\}$, ie, allow $\dot{q}$ as an extra variable. ← Key step for phase space

Thus, from (2), we have

* This is not guaranteed to work if constraints were non-holonomic

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \qquad \text{(2a)}$$

and

$$\frac{\partial \vec{v}_i}{\partial q_\ell} = \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_\ell} \dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial q_\ell \partial t} \qquad \text{(2b)}$$

- Now manipulate RHS as follows

$$\dot{\vec{V}}_i \cdot \delta\vec{r}_i = \dot{\vec{V}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j \,;$$

then,
$$\dot{\vec{V}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \underbrace{\frac{d}{dt}\left[\vec{V}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right]}_{A} - \underbrace{\vec{V}_i \cdot \frac{d}{dt}\left[\frac{\partial \vec{r}_i}{\partial q_j}\right]}_{B}$$

Now
$$A = \frac{d}{dt}\left[\vec{V}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right] = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{V_i^2}{2}\right)\right]$$

where we have used 2a ;

and
$$B = \vec{V}_i \cdot \frac{d}{dt}\left[\frac{\partial \vec{r}_i}{\partial q_j}\right] = \vec{V}_i \cdot \left[\frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_l}\dot{q}_l + \frac{\partial^2 \vec{r}_i}{\partial t\, \partial q_j}\right]$$

$$= \vec{V}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\frac{v_i^2}{2}\right), \text{ using } 2b$$

Inserting these into above expression

$$\Rightarrow \text{LHS} = \sum_i m_i \dot{\vec{V}}_i \cdot \delta\vec{r}_i = \sum_i m_i \left[\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{V_i^2}{2}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{V_i^2}{2}\right)\right]\delta q_j$$

$$\therefore \text{LHS} = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right)\right]\delta q_j$$

where $T = \sum_i \frac{1}{2} m_i v_i^2$

- Now, each δq_j is independent $\Rightarrow$ Lagrange's equations are n equations; viz,

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right) = Q_j} \quad j = 1, \ldots, n$$

Constraint forces do not appear

and, if $\vec{F}_i = -\frac{\partial V}{\partial \vec{r}_i}$, then

$$Q_j = -\frac{\partial V}{\partial q_j}, \quad V = V(q_j, t)$$

all PE

$$\Rightarrow \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = 0} \quad j = 1, \ldots, n$$

$$L = T - V$$

$$T = \sum_i \frac{1}{2} m_i v_i^2, \quad V = \underbrace{V(q_j, t)}_{\text{external PE } \forall \text{ masses}}$$