<u>Phys601/F11/Problem Set 10</u> ***Due November 23, 2011***

Subject to upgrade; note new due date

Midterm in class: Nov 16
Midterm take-home: ~Nov 17-20

<u>10.1H</u> Geometric optics of MHD waves*

This problem considers the geometric optics of an MHD wave about a magnetic X-point. "Magnetosonic waves" in a low pressure magneto-plasma satisfy the wave equation

$$\partial_t^{\,2}\psi = V^2\nabla^2\psi .$$

V is the Alfven velocity, here assumed varying slowly in position **r** = (x,y) as **V = V**(**r**), where **V** = **B**/$(\mu_0\rho)^{1/2}$, **B** is the magnetic field, μ_0 is μ_0, and ρ is the mass density of the medium. Assume that ρ is a constant. For this problem, we are considering an ambient magnetic field which is an "X-point". Assume that the magnetic field is derivable from a vector potential A_z = Hxy. For normalization, set $H^2/(\mu_0\rho) = 1$. To be precise, V^2 = **V.V**.

We are interested in ray trajectories about the X-point. *<u>Assume that the motion is 2-D only, ie, dz/dt = 0, k_z = 0. Work in Cartesian coordinates to begin.</u>* Proceed as follows:

1. Make a sketch of the **B** field and calculate B^2. Note the azimuthal symmetry in 2D.

2. Write down the WKB local dispersion relation, using $\vec{\nabla}S$ = **k**, where k^2 = **k.k**, **k** = (k_x, k_y) in 2D Cartesian coordinates, and ω is the fixed driver frequency.

3. What are the equations satisfied by the rays, **r**(t), and the conjugate momenta, **k**(t)? Are there any obvious cyclic Cartesian coordinates? Identify the one obvious constant of the motion.

4. Show by direct differentiation that D = **k.r** is another constant of the motion.

5. By direct time differentiation of the object r^2 = **r.r**, and using the constants of the motion, find an equation of motion satisfied by r(t), ie, only r(t) and its derivatives enter. Solve this equation assuming that r(0) = 1. Show that the solution is of the form exp[αt]. Specify α in terms of the constants of the motion and show that |α| ≤ 1. Describe the radial location of the wavepacket as t increases for the 3 cases α = 0, α > 0, and α < 0.

6. From the remaining equations, find a set of coupled ODE's that couple only the set {x, k_x, r}. By using r(t) from (5), manipulate this set to find a single ODE that governs x(t). Solve for the general solution of x(t) and take the real part. By using the form of r(t) found above, deduce the angular location of the wavepacket in polar coordinates. Now describe the 2D trajectory of the wavepacket as t increases for the 3 cases $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$.

*Geometric optics of MHD waves was considered in general in the paper Eikonal Method in Magnetohydrodynamics, by Steven Weinberg, Phys. Rev. 126, 1899–1909 (1962)

<u>10.2H</u> Hamilton-Jacobi theory of free fall under g

Consider a mass m in free fall in $\mathbf{g} = -g\hat{\mathbf{x}}$. Let the energy = α. Find p(t) and q(t) using Hamilton-Jacobi Theory. For the P coordinate, use $P = \alpha$. Use $Q = \beta$, where β is also a constant of the motion.

1. Solve explicitly for $p(\alpha, \beta, t)$ and $q(\alpha, \beta, t)$. Identify the physical meaning of β.

2. Invert the above and find $\alpha(p, q, t)$ and $\beta(p, q, t)$. Make sketches of the α and β coordinates for $t < 0$, $t=0$, and $t > 0$. Describe the coordinates and the embedded particle motion.