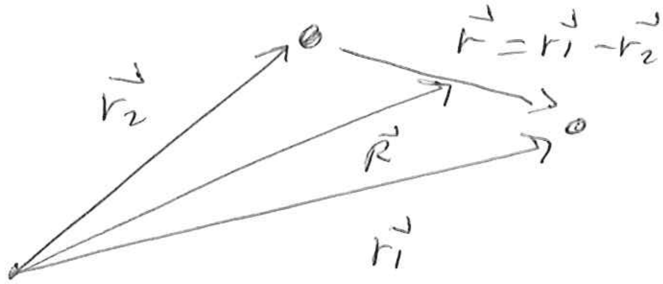
Solns PS8

Phys 410 / F16

8.1



$$\sum m \vec R \equiv \sum (m \vec r)$$

Check $\vec r_1 = \vec R + \frac{m_2}{M} \vec r$?

$$M\vec r_1 = M\vec R + m_2 \vec r$$
$$= m_1 \vec r_1 + \cancel{m_2 \vec r_2} + m_2(\vec r_1 - \cancel{\vec r_2})$$
$$= M\vec r_1 \quad \checkmark$$

$\vec r_2 = \vec R - \frac{m_1}{M} \vec r$?

$$M\vec r_2 = \cancel{m_1 \vec r_1} + m_2 \vec r_2 - m_1(\cancel{\vec r_1} - \vec r_2) \quad \checkmark$$

$$T = \frac{1}{2} m_1 \left(|\dot{\vec r}_1|^2\right) + \frac{1}{2} m_2 \left(|\dot{\vec r}_2|^2\right)$$
$$= \frac{1}{2} \cancel{m_1} \left[m_1 |\dot{\vec R}|^2 + \frac{m_1 m_2^2}{M^2} |\dot{\vec r}|^2 + m_2 |\dot{\vec R}|^2 + \frac{m_2 m_1^2}{M^2} |\dot{\vec r}|^2 \right]$$

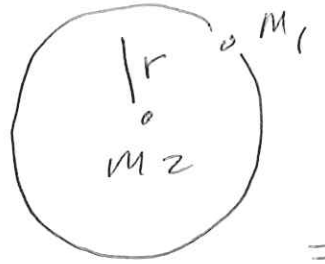
cross terms vanish

$$= \frac{1}{2} M |\dot{\vec R}|^2 + \frac{1}{2} \mu |\dot{\vec r}|^2$$

since $\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2} = \frac{m_1 m_2}{M} \equiv \mu$

8.7

(a)



$$m_1 r \omega^2 = \frac{G m_1 m_2}{r^2}$$

$$\Rightarrow \boxed{\omega^2 = \frac{G m_2}{r^3}, \quad T = \frac{2\pi}{\omega}} \text{ (a)}$$

(b) $$\mu \ddot{\vec{r}} = -\frac{G \overset{m_1 m_2}{\cancel{\mu M}}}{r^3} \vec{r}$$

$\therefore$ mass μ, $|F| = G m_1 m_2 / r^2$

"elementary $\Rightarrow$ Newtonian Mechanics"

$$\underbrace{\mu r \omega^2}_{\text{acceleration}} = \frac{G m_1 m_2}{r^2}$$

$$\Rightarrow \boxed{\omega^2 = \frac{GM}{r^3}} \text{ (b)}$$

as $m_2 \to \infty$, $\underline{M \to m_2}$.

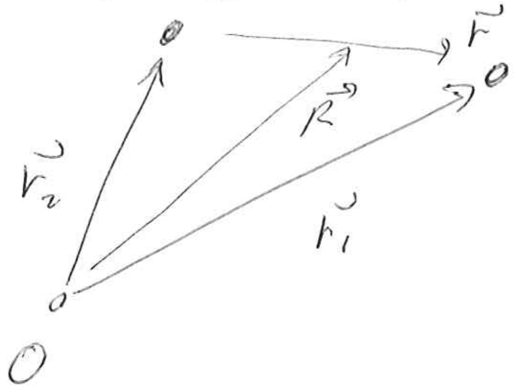
$\therefore$ (b) $\to$ $\boxed{\omega^2 \simeq \frac{G m_2}{r^3}}$

This is the same as (a), i.e., orbital period is like m_2 fixed; m_1 drops out in this limit.

8·10 $\quad U_1 = \frac{1}{2} k r_1^2, \quad U_2 = \frac{1}{2} k r_2^2$

about some fixed force center

Place origin @ this fixed center, O



Also,

$$U_{12} = \frac{1}{2} k \alpha r^2$$

Thus, each mass has 2 forces on it.

As before, $\boxed{T = \frac{1}{2} M |\dot{\vec{R}}|^2 + \frac{1}{2} \mu |\dot{\vec{r}}|^2 \\ U = \frac{1}{2} k r_1^2 + \frac{1}{2} k r_2^2 + \frac{1}{2} k \alpha r^2}$

But keep in mind $\quad \vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r}$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r}$$

$$\Rightarrow \boxed{r_1^2 = R^2 + \left(\frac{m_2}{M}\right)^2 r^2 \\ r_2^2 = R^2 + \left(\frac{m_1}{M}\right)^2 r^2} \boxed{+ \text{cross term } (2\vec{R} \cdot \vec{r}\, m_2/M) \\ + \text{cross term } (-2\vec{R} \cdot \vec{r}\, m_1/M)}$$

* Since $m_1 = m_2$, cross terms vanish

also, $\mu = \frac{m_1 m_2}{m_1 + m_2} \rightarrow \frac{m^2}{2m} = \frac{m}{2}$

So, $\cancel{T = \frac{1}{2} M |\dot{\vec{R}}|^2 + \frac{m}{4} |\dot{\vec{r}}|^2}$

$$\Rightarrow \boxed{T = \underbrace{\tfrac{1}{2}M}_{m} |\dot{\vec{R}}|^2 + \frac{m}{4}|\dot{\vec{r}}|^2}$$

$$\frac{U}{(1/2k)} = 2R^2 + \underbrace{\left(\frac{m_1^2 + m_2^2}{M^2}\right)}_{\frac{2m^2}{(2m)^2} = \frac{1}{2}} r^2 + \alpha r^2$$

$\mathcal{L}(\dot{\vec{R}}, \dot{\vec{r}}, R, r)$

$$\boxed{\frac{U}{1/2k} = 2R^2 + \frac{1}{2}r^2 + \alpha r^2} = 2R^2 + \left(\frac{1}{2}+\alpha\right) r^2$$

Note $\vec{R}$ not ignorable here

$\vec{R}$:

$$\frac{\partial \mathcal{L}}{\partial \vec{R}} = \vec{D}_{\vec{R}}\mathcal{L} = -\hat{R}\frac{\partial}{\partial R}(kR^2) = -2k\vec{R}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}} = 2m\dot{\vec{R}}$$

$$\Rightarrow 2m\ddot{\vec{R}} = -2k\vec{R}$$

$$\Rightarrow \boxed{\ddot{\vec{R}} = -\frac{k}{m}\vec{R}}$$

CM oscillates @ $\omega^2 = k/m$

$\vec{r}$:

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} = \vec{D}_{\vec{r}}\mathcal{L} = -k\left(\frac{1}{2}+\alpha\right)\vec{r}$$

$$\partial \mathcal{L}/\partial \dot{\vec{r}} = \frac{1}{2}m\dot{\vec{r}} \Rightarrow \boxed{\frac{1}{2}m\ddot{\vec{r}} = -k\left(\frac{1}{2}+\alpha\right)\vec{r}}$$

$\vec{r}$ oscillates @ $\boxed{\omega^2 = \frac{k}{m}(1+2\alpha)}$

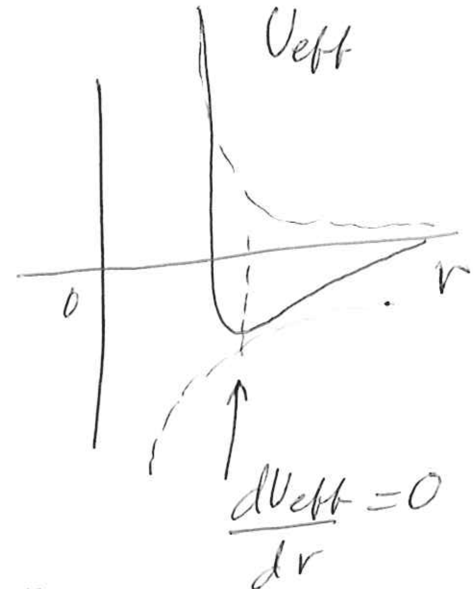
$\alpha = 0 \Rightarrow$ same frequency, similar to 8.8

$\alpha \to \infty \Rightarrow \omega^2 \to 2\alpha k/m$

8.12

$$U_{eff} = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

$$\mu \ddot{r} = -\frac{dU_{eff}}{dr}$$



(a) "Equilibrium" in r

(i.e., $\ddot{r} = 0 \Rightarrow$ circular orbit since $r_{max} = r_{min}$)

$\ddot{r} = 0 \Leftrightarrow dU_{eff}/dr = 0$

$$\frac{dU_{eff}}{dr} = \frac{G m_1 m_2}{r^2} - \frac{\ell^2}{\cancel{2}\mu r^3}$$

(Note that dU_{eff}/dr is just $-F_{eff}(r)$. So we are saying $F_{eff}(r) = 0$)

$$\frac{dU_{eff}}{dr} = 0 \Rightarrow \boxed{r_0 = \frac{\ell^2}{G m_1 m_2 \mu}} \quad (3)$$

(b) Find $\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = \frac{-2G m_1 m_2}{r^3} + \frac{3\ell^2}{\mu r^4}$

$$= -\frac{2G m_1 m_2}{r_0^3} + \frac{3\cancel{\ell^2}}{\mu r_0^3}\frac{G m_1 m_2 \mu}{\cancel{\ell^2}}$$

$$= \frac{G m_1 m_2}{r_0^3} > 0 \Rightarrow \text{stable}$$

Complete story.

Expand about $r = r_0$.

let $r = r_0 + \tilde{r}$

$$\Rightarrow \mu \ddot{\tilde{r}} \simeq -\underbrace{\left(\frac{dV}{dr}\right)_{r_0}}_{=0 \text{ by def}} - \left(\frac{d^2V}{dr^2}\right)_{r_0} \tilde{r}$$

$$\Rightarrow \boxed{\mu \ddot{\tilde{r}} = -\frac{Gm_1m_2}{r_0^3}\tilde{r}}$$

$\therefore$ stable oscillation @ $\boxed{\omega^2 = \frac{Gm_1m_2}{\mu r_0^3}}$ (1)

r_0 defined above

(c) Orbital period of planet:

$$\dot{\phi} = \frac{l}{\mu r_0^2} \Rightarrow \frac{2\pi}{T} = \frac{l}{\mu r_0^2}$$

$$\Rightarrow \boxed{\omega^2 = \frac{l^2}{\mu^2 r_0^4}} \quad (2)$$

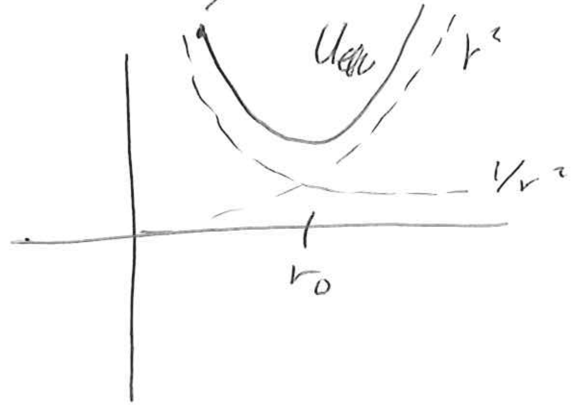
Compare (1) : (2) $\quad$ (2) $\rightarrow \frac{Gm_1m_2}{\mu r_0^3} = \frac{l^2}{\mu^2 r_0^4}$

using (3). They agree.

8.13 This is parallel to 8.12 except $U = \frac{1}{2}kr^2$; opposed to $U \propto -\frac{1}{r}$ as in 8.12

We have $U_{eff} = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}$

& $\mu \ddot{r} = -\frac{dU_{eff}}{dr}$



(a) Circular orbit

$\Rightarrow \frac{dU_{eff}}{dr} = 0$

$\Rightarrow kr - \frac{l^2}{\mu r^3} = 0 \quad \Rightarrow \boxed{r_0^4 = \frac{l^2}{\mu k}}$

(b) small perturbations

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = k + \frac{3l^2}{\mu r_0^4} = k + 3k = 4k > 0$$

$\Rightarrow$ stable

$r = r_0 + \tilde{r}$

$$\Rightarrow \mu \ddot{\tilde{r}} = -\left.\frac{dU_{eff}}{dr}\right|_{r_0} - \left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} \tilde{r}$$

$\Rightarrow \boxed{\mu \ddot{\tilde{r}} = -4k\tilde{r}} \quad \Rightarrow \boxed{\omega^2 = \frac{4k}{\mu}}$ Twice the rate of rotation