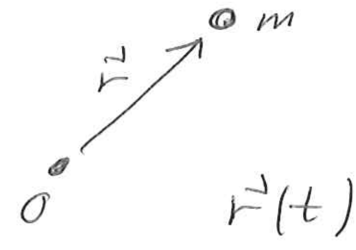
Solns PS 7

Phys 410 / F16

Central force motion in H.O. potential

$$\boxed{m\ddot{\vec r} = -k\vec r}$$



① <u>$\vec F$ is conservative:</u> $\vec r = (x, y, z)$

because $\vec\nabla \times \vec F = 0$

eg $(\vec\nabla \times \vec F)_z = \partial_x F_y - \partial_y F_x$

$= \partial_x(-ky) - \partial_y(-kx) = 0$

② <u>Find U</u>

$$-dU = \vec F \cdot d\vec r = -k(x,y,z)\cdot(dx, dy, dz)$$

$$= -k(x\,dx + y\,dy + z\,dz)$$

$$= -\frac{k}{2}\, d(x^2+y^2+z^2)$$

$$\Rightarrow \boxed{U = \frac{1}{2} k r^2} \qquad r^2 \equiv x^2+y^2+z^2$$

③ <u>From WE Thm</u>

$$\mathcal{E} = \frac{1}{2} m v^2 + \frac{1}{2} k r^2 = \text{const}$$

$$\vec v \equiv \dot{\vec r}, \quad v^2 = |\dot{\vec r}|^2$$

(4) $\vec{F}$ is central $\Rightarrow \vec{L} = \text{const}$

$$\vec{F} = -k\vec{r}$$

$$m\ddot{\vec{r}} = -k\vec{r}$$

Cross w $\vec{r}$ $\Rightarrow$ $m\vec{r}\times\ddot{\vec{r}} = 0$ — (1)

Note $\frac{d}{dt}(\vec{A}\times\vec{B}) = \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}$

$$\Rightarrow \frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = \underbrace{\dot{\vec{r}}\times\dot{\vec{r}}}_{=0} + \vec{r}\times\ddot{\vec{r}}$$

$\therefore$ From (1), $\frac{d}{dt}(m\vec{r}\times\dot{\vec{r}}) = 0$

$$\Rightarrow \boxed{\vec{L} \equiv m\vec{r}\times\vec{v} = \text{const vector}}$$

Note $\vec{r}(t)$ and $\vec{v}(t)$ but

$d\vec{L}/dt = 0$ for all t

i.e., $\vec{L} = \vec{L}(t=0)$

(5) $\vec{L} = \text{const} \Rightarrow \vec{r}(t)$ stays in 2D plane

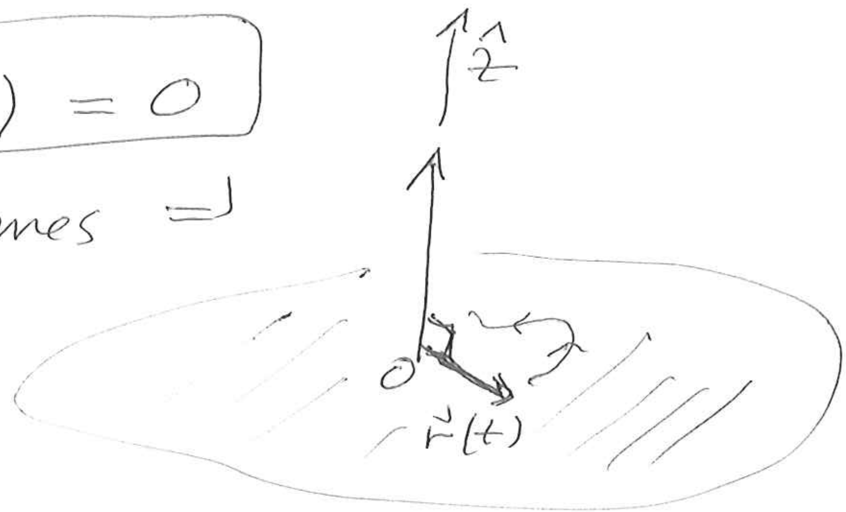
$\vec{L} = m\vec{r}\times\vec{v} = \text{const}$

$\Rightarrow \vec{L}\cdot\vec{r} = m\vec{r}\times\vec{v}\cdot\vec{r} = 0$

Since $(\vec{A}\times\vec{B})\cdot\vec{B} = 0$

$$\Rightarrow \boxed{\vec{L}\cdot\vec{r}(t) = 0}$$

true for all times $\Rightarrow$



$\Rightarrow \vec{r}(t)$ stays in (x, y) plane if $\vec{L} \parallel \hat{z}$.

$$\Rightarrow \boxed{\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y}}$$

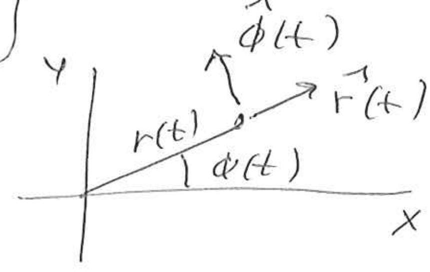
Problem reduced to 2-D

⑥ <u>Rewrite $\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}$ in polar coords</u>

(polar coords are more natural)

(refer to Taylor Sec 1.7)

$$\Rightarrow \boxed{\vec{r}(t) = r(t)\,\hat{r}(t)}$$



As per Taylor

$$\dot{\vec{r}}(t) = \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$

$$\Rightarrow \boxed{\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}}$$

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}}$$

use $\dot{\hat{r}} = \dot{\phi}\hat{\phi}$, $\dot{\hat{\phi}} = -\dot{\phi}\hat{r}$

$$\Rightarrow \boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + \frac{(r^2\dot{\phi})^{\cdot}}{r}\hat{\phi}}$$

where $\dfrac{(r^2\dot{\phi})^{\cdot}}{r} = 2\dot{r}\dot{\phi} + r\ddot{\phi}$

(7) <u>Write $\vec{F} = m\vec{a}$ in polar</u>

$$\Rightarrow \boxed{\begin{aligned} m\ddot{r} &= mr\dot{\phi}^2 - kr \quad (2) \\ \frac{(mr^2\dot{\phi})^{\cdot}}{r} &= 0 \quad (3) \end{aligned}}$$

From (3), $\boxed{mr^2\dot{\phi} = l = \text{const}}$ (4)

$\Rightarrow$ magnitude of $\vec{L}$, $|\vec{L}|$, is also a constant.

(8) Eliminate $\dot{\phi}$ to get an ODE for r

$$\Rightarrow \boxed{m\ddot{r} = \frac{\ell^2}{mr^3} - kr} \quad (5)$$

2nd order ODE for $r(t)$, nonlinear in r

(9) Use $\mathcal{E}$ = constant

$$\mathcal{E} = \frac{1}{2} m|\dot{\vec{r}}|^2 + \frac{1}{2} kr^2$$

Use $\dot{\vec{r}}$ from polar

$$\Rightarrow \mathcal{E} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2} kr^2$$

Substitute for $\dot{\phi}$

$$\boxed{\mathcal{E} = \frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + \frac{1}{2} kr^2} \quad (6)$$

$\Rightarrow$ $\mathcal{E}$ involves $\{\dot{r}, r\}$ only

$$U_{eff} \equiv \frac{1}{2} kr^2 + \frac{\ell^2}{2mr^2}$$

(10) Check that $\dot{\mathcal{E}} = 0$ recovers $\ddot{r}$ eqn

$$0 = \dot{\mathcal{E}} = m\dot{r}\ddot{r} - \frac{\ell^2}{mr^3}\dot{r} + kr\dot{r}$$

$\dot{r}$ common

$$\Rightarrow m\ddot{r} = \frac{\ell^2}{mr^3} - kr \quad \checkmark$$

(11) Rewrite $\mathcal{E}$ = constant as $\dot{r}$ eqn

From (6)

$$\boxed{\dot{r} = \pm\sqrt{\frac{2\mathcal{E}}{m} - \frac{\ell^2}{m^2r^2} - \frac{kr}{m}}} \quad (7)$$

1st order ODE for $r(t)$, separable

$$\Rightarrow \frac{dr}{\sqrt{}} = \pm\, dt$$

Can integrate

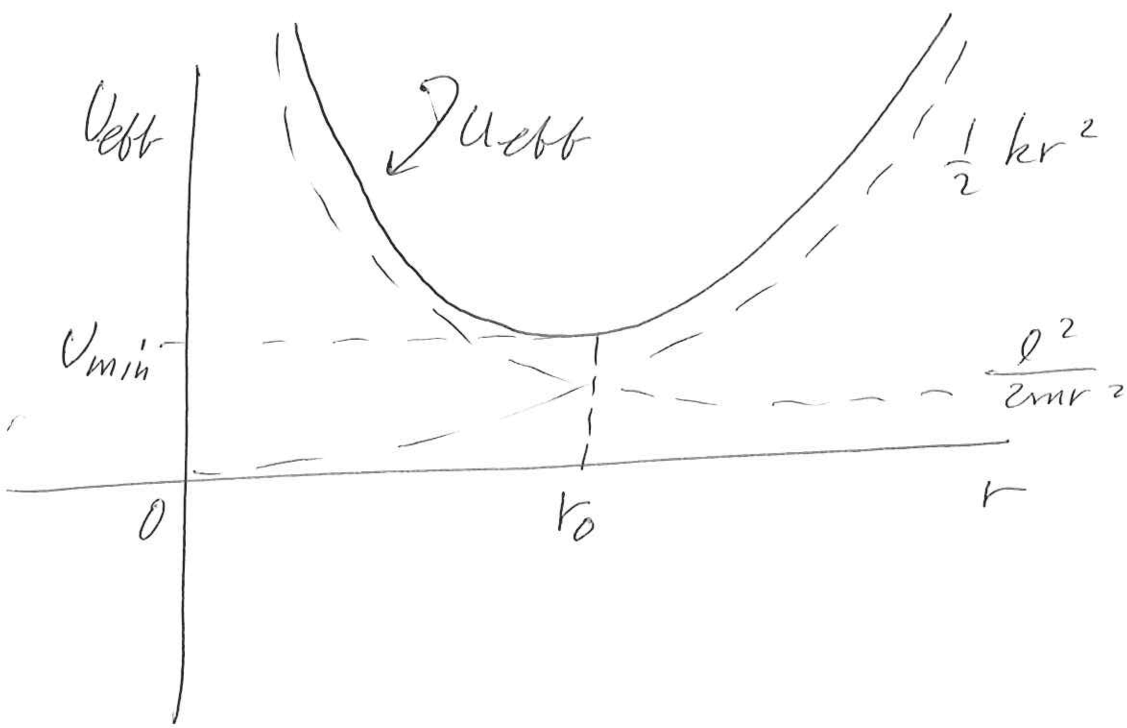
(12) Define $U_{eff}(r)$

$U_{eff} = U(r) + \ell^2$ term

$\Rightarrow \boxed{U_{eff}(r) = \frac{1}{2}kr^2 + \frac{1}{2}\frac{\ell^2}{mr^2}}$ (8)

$\frac{1}{2}m\dot{r}^2 \geq 0$

$\Rightarrow \frac{1}{2}m\dot{r}^2 = E - U_{eff}(r) > 0$



(13) $\mathcal{E} = U_{min} \Rightarrow r = r_0 \Rightarrow$ circular orbit

$$\frac{dU_{eff}}{dr} = kr - \frac{l^2}{mr^3} = 0$$

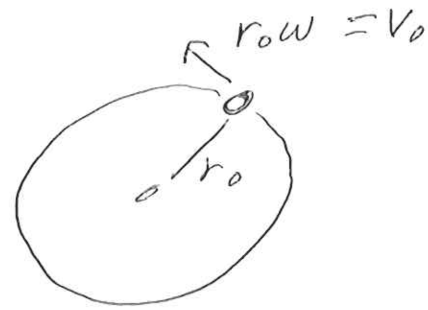
$\Rightarrow U_{min}$

$$\Rightarrow \boxed{r_0^4 = \frac{l^2}{mk}} \quad (9)$$

From (4) $\quad m r_0^2 \dot{\phi} = l \quad \Rightarrow \dot{\phi} = \text{const} = \omega_0$

$$\Rightarrow \boxed{\omega_0 = \frac{l}{m r_0^2}} \quad (10)$$

(14) Circular motion



$F_{in} = m v_0^2 / r_0$

$F_{in} = k r_0$

$\Rightarrow \quad k r_0^2 = m v_0^2 = m r_0^2 \omega_0^2$

$$\Rightarrow \boxed{\omega_0^2 = k/m} \quad (11)$$

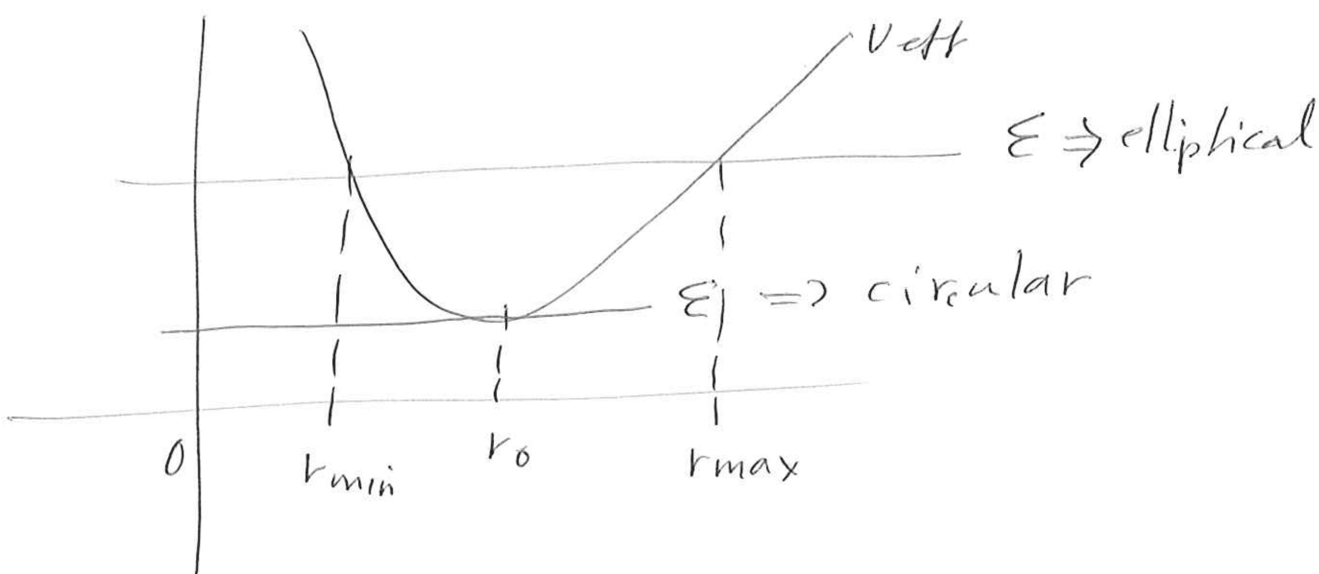
~~Check Plug in (10)~~ [illegible]

From (9), (10), eliminate l

$$\Rightarrow r_0^4 = \frac{\omega_0^2 m^2 r_0^4}{mk}$$

$$\Rightarrow \boxed{\omega_0^2 = k/m}$$ agrees

Circular orbits & elliptical (bound) orbits



(15) Small perturbations about circular orbit

$$m\ddot{r} = \frac{l^2}{mr^3} - kr$$ from (5)

equilibrium for $\ddot{r} = 0$:

$$\frac{\ell^2}{m r_0^3} = k r_0 \Rightarrow r_0^4 = \frac{\ell^2}{mk}$$

as before

Let $r = r_0 + \tilde{r}$, Taylor expand

$$\Rightarrow m\ddot{\tilde{r}} = -\frac{3\ell^2}{m r_0^4}\tilde{r} - k\tilde{r}$$

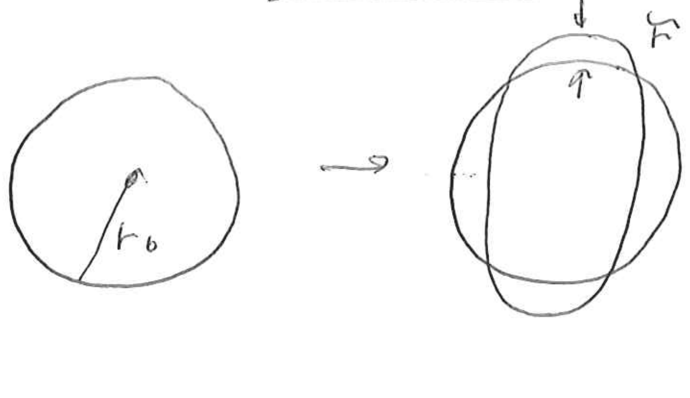
$$\text{RHS} \propto \frac{-3\, m k r_0^4}{m r_0^4} - k = -4k$$

$$\Rightarrow \ddot{\tilde{r}} = -4\frac{k}{m}\tilde{r} \Rightarrow \cos(\omega t)$$

$$\Rightarrow \text{small oscillations @ } \omega = 2(k/m)^{1/2}$$

$$\Rightarrow \boxed{\frac{\omega}{\omega_0} = 2}$$

oscillations @ twice the circular rotation rate



faster oscillation rate by $\times 2$