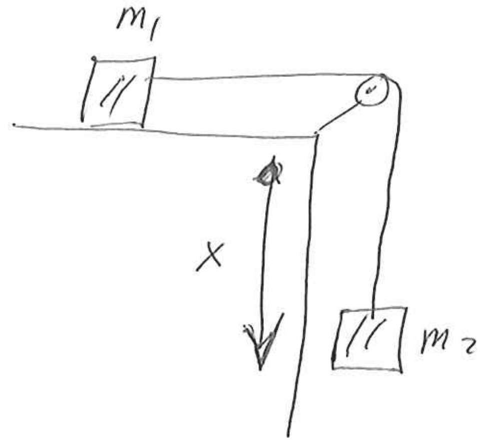
Solns PS 6

Phys410/F16

7.15

$x \geq 0$




$$U = -m_2 g x$$

$$T = \frac{1}{2} m_1 \left|(\ell - x)^{\circ}\right|^2 + \frac{1}{2} m_2 \overset{\circ}{x}^2$$

$$T = \frac{1}{2}(m_1 + m_2)\overset{\circ}{x}^2$$

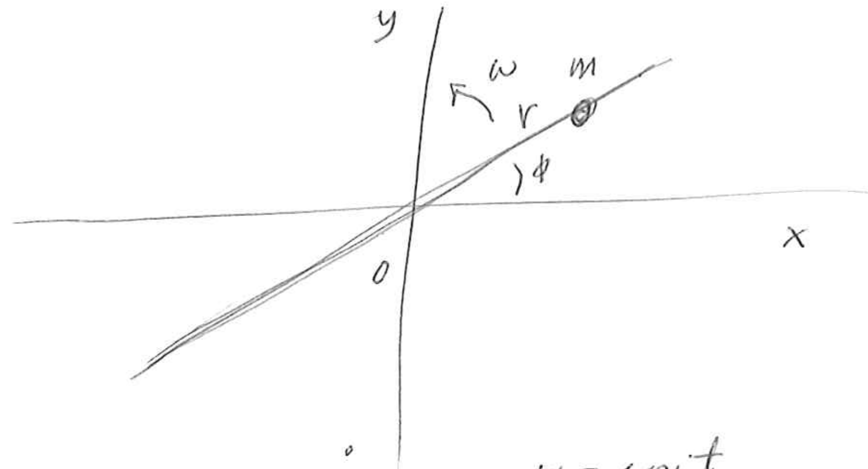
$$\mathcal{L} = T - U$$

$$\frac{\partial \mathcal{L}}{\partial x} = m_2 g \quad , \quad \frac{\partial \mathcal{L}}{\partial \overset{\circ}{x}} = (m_1 + m_2)\overset{\circ}{x}$$

$$\Rightarrow \boxed{(m_1 + m_2)\overset{\circ\circ}{x} = m_2 g}$$

$$\Rightarrow \frac{a}{g} = \frac{m_2}{m_1 + m_2}$$

7.21




$X = r\cos\phi$

$Y = r\sin\phi$

$\phi = \omega t$

$\omega = $ const

$\dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$

$\dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{m}{2}(\dot{r}^2 + r^2\omega^2)$

$U = 0$ (No forces apart from N)

$\frac{\partial \mathcal{L}}{\partial r} = mr\omega^2$ , $\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$

$m\ddot{r} = mr\omega^2$

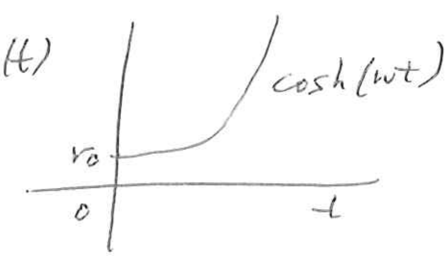
$\Rightarrow \ddot{r} = r\omega^2 \Rightarrow r(t) \sim e^{\pm\omega t}$

$r = Ae^{\omega t} + Be^{-\omega t}$ $\quad \dot{r}(0) = 0$

$\Rightarrow r(t) = r_0\cosh(\omega t)$ , $r(0) = r_0$, $\dot{r}(0) = 0$

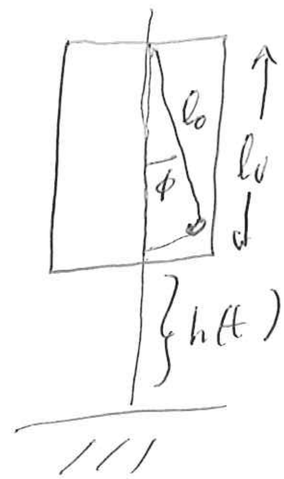
If $r_0 = 0 \Rightarrow r = 0$ → but this is not stable for small perturbations

If $r_0 > 0 \Rightarrow r(t)$



7.22

For pendulum bob,




$$x = l_0 \sin\phi$$
$$y = h(t) + l_0 - l_0\cos\phi$$

$$\dot{x} = l_0 \cos\phi\, \dot{\phi}$$
$$\dot{y} = \dot{h} + l_0 \sin\phi\, \dot{\phi}$$

$$\boxed{\begin{aligned} U &= mgy = mgh + mgl_0(1-\cos\phi) \\ T &= m\frac{1}{2} l_0^2 \dot{\phi}^2 + \frac{m}{2}\dot{h}^2 + m\dot{h} l_0 \sin\phi\, \dot{\phi} \end{aligned}}$$

$$\Rightarrow \mathcal{L} = T - U = \mathcal{L}(\phi, \dot{\phi}, t)$$
$$\ddot{h}(t) = a, \quad \dot{h} = at, \quad h = \frac{1}{2}at^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = m\dot{h} l_0 \cos\phi\, \dot{\phi} - mgl_0 \sin\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml_0^2 \dot{\phi} + m\dot{h} l_0 \sin\phi$$

cancel

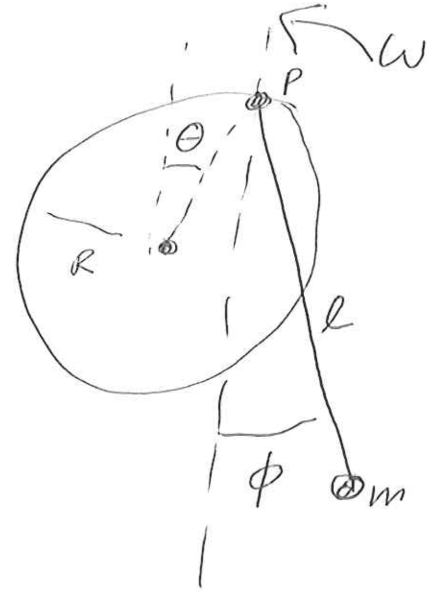
$$\Rightarrow ml_0^2 \ddot{\phi} + m\ddot{h} l_0 \sin\phi + \cancel{m\dot{h} l_0 \cos\phi\, \dot{\phi}} = \cancel{m\dot{h} l_0 \cos\phi\, \dot{\phi}} - mgl_0 \sin\phi$$

$$\Rightarrow ml_0^2 \ddot{\phi} = -ml_0 \sin\phi\, (g + \ddot{h})$$

$$\boxed{\ddot{\phi} = -\sin\phi \frac{(g + \ddot{h})}{l_0}}$$

effective gravity

7.29




$X = R\sin\theta + l\sin\phi$

$$\boxed{Y = l\cos\phi - R\cos\theta}$$

Y increases downward from O

$\dot\theta = \omega = \text{const}$

$\dot X = R\cos\theta\,\omega + l\cos\phi\,\dot\phi$

$\dot Y = -l\sin\phi\,\dot\phi + R\sin\theta\,\omega$

$\dot x^2 + \dot y^2 = (R\omega)^2 + (l\dot\phi)^2 + 2Rl\omega\dot\phi \, *$

$(\cos\theta\cos\phi - \sin\theta\sin\phi) \to \cos(\theta+\phi)$

$$\boxed{T = \frac{1}{2} m(\dot x^2 + \dot y^2)}$$

$\dot x^2 + \dot y^2 = (R\omega)^2 + (l\dot\phi)^2 + 2Rl\omega\dot\phi\cos(\theta+\phi)$

$$\boxed{\theta = \omega t}$$

$$\boxed{U = -mgY}$$

$\mathcal{L} = \mathcal{L}(\phi, \dot\phi, t)$

$$\frac{\partial\mathcal{L}}{\partial\phi} = mRl\omega\dot\phi(-)\sin(\theta+\phi) + mgl(-)\sin\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m l^2 \dot{\phi} + m R l \omega \cos(\theta + \phi)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m l^2 \ddot{\phi} - m R l \omega \sin(\theta + \phi)(\omega + \dot{\phi})$$

$$m l^2 \ddot{\phi} - m R l \omega \sin(\theta + \phi)(\omega + \dot{\phi})$$

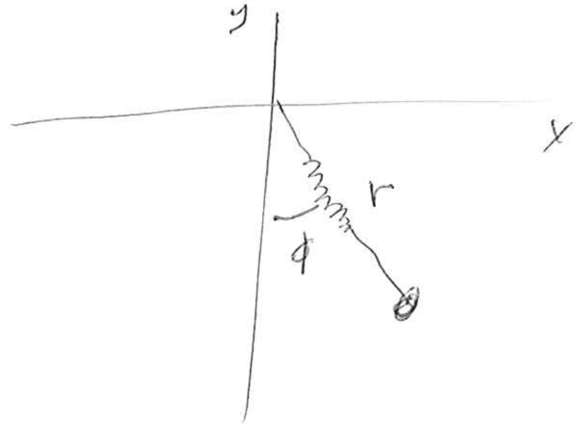
$$= - m R l \omega \dot{\phi} \sin(\theta + \phi) \qquad \text{cancel}$$

$$- m g l \sin\phi$$

$$\boxed{\ddot{\phi} - \frac{R}{l}\omega^2 \sin(\theta + \phi) = -\frac{g}{l}\sin\phi} \quad \boxed{\theta = \omega t}$$

$$\omega = 0 \Rightarrow \ddot{\phi} = -\frac{g}{l}\sin\phi, \text{ as expected}$$

7036




Spr const $= k$

Spr length $= l_0$

$U_g = -mgy = -mgr\cos\phi$

$U_k = \frac{1}{2}k(r-l_0)^2$

$x = r\sin\phi \qquad \dot{x} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$

$y = -r\cos\phi \qquad \dot{y} = -\dot{r}\cos\phi + r\sin\phi\,\dot{\phi}$

$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$

(a) $\boxed{\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + mgr\cos\phi - \frac{1}{2}k(r-l_0)^2}$

$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi})$

(b) $\frac{\partial \mathcal{L}}{\partial r} = r\dot{\phi}^2 + mg\cos\phi - k(r-l_0)$

$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} \quad \Rightarrow \boxed{m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi - k(r-l_0)}$ (1)

cent force, $g\downarrow$, spring

$\frac{\partial \mathcal{L}}{\partial \phi} = -mgr\sin\phi\,, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi}$

$\Rightarrow \boxed{m(r^2\dot{\phi})^{\cdot} = -mgr\sin\phi}$ (2)

ang mom, torque

$r = \text{const}$

$\Rightarrow \left(\frac{g}{l_0}\right)$ oscillations

(c) equilibrium $\quad \ddot{r} = 0, \; \ddot{\phi} = \dot{\phi} = 0$

$$\Rightarrow \boxed{mg = k(r_0 - l_0)} \Rightarrow r_0 \quad (\text{for } \phi_0 = 0]$$

$$mgr_0 \sin\phi_0 = 0 \Rightarrow \phi_0 = 0$$

$$\therefore \boxed{r = r_0 \text{ and } \phi = 0}$$

small perturbations $\quad r = r_0 + \tilde{r}$

$$\phi = \tilde{\phi}$$

$$mg\cos\phi - k(r - l_0)$$

$$\cancel{= mg\cos(0) + mg\cos(\tilde{\phi}) - k(r_0 + \tilde{r} - l_0)}$$

$$= mg\cos[0 + \tilde{\phi}] - k(r_0 + \tilde{r} - l_0)$$

$$\simeq mg[\underbrace{\cos(0)}_{\text{cancels}} - \underbrace{\sin(0)}_{\text{zero}}\tilde{\phi} - \cos(0)\underbrace{\frac{\tilde{\phi}^2}{2}}_{\text{small}} \ldots] \underbrace{- k(r_0 - l_0)}_{\text{cancels}} - k\tilde{r}$$

$\rightarrow -k\tilde{r}$ $\quad$ Also $mr\dot{\phi}^2 \simeq mr\dot{\tilde{\phi}}^2 \rightarrow$ small

Also, $mgr\sin\phi \rightarrow mg(r_0 + \tilde{r})[\sin(0) + \cos(0)\tilde{\phi}]$

$$= mg(r_0 + \underbrace{\tilde{r}}_{\text{small}})\tilde{\phi} \simeq mgr_0\tilde{\phi}$$

So, (1) $\Rightarrow m\ddot{\tilde{r}} \simeq -k\tilde{r} \Rightarrow$

& (2) $\Rightarrow mr_0^2\ddot{\tilde{\phi}} \simeq -mgr_0\tilde{\phi} \Rightarrow$

$$\boxed{\begin{aligned} \ddot{\tilde{r}} &= -\frac{k}{m}\tilde{r} \\ \ddot{\tilde{\phi}} &= -\frac{g}{r_0}\tilde{\phi} \end{aligned}}$$

← k oscillns

← g oscillns

uncoupled oscillators