Phys 410 / F16

Solns PS3

4.21 $\vec{F} = -\frac{GMm\,\hat{r}}{r^2}$

let $GMm = 1$ $\quad \vec{F} = -\frac{\vec{r}}{r^3}$

$$(\vec{\nabla}\times\vec{F})_z = -\partial_x\left(\frac{y}{r^3}\right) + \partial_y\left(\frac{x}{r^3}\right)$$

$$= +y\frac{3}{r^4}\partial_x r - x\frac{3}{r^4}\partial_y r$$

$$\partial_x r^2 = \partial_x(x^2+y^2+z^2) = 2x$$

$$2r\partial_x r = 2x \qquad r\partial_x r = x$$

$$\Rightarrow (\vec{\nabla}\times\vec{F})_z = \frac{3y}{r^4}\frac{x}{r} - \frac{3x}{r^4}\frac{y}{r} = 0$$

Likewise, y + x components.

- Try $U = \frac{-1}{r}$

$$-\vec{\nabla}U = -\frac{1}{r^2}\vec{\nabla}r = -\frac{\hat{r}}{r^2}$$ agrees.

$$\Rightarrow \boxed{-\vec{\nabla}U = \frac{-\vec{r}}{r^3} = \vec{F}}$$

4.23 (a) $\vec{F} = (x, 2y, 3z)$ $k=1$

$(\vec{\nabla} \times \vec{F})_x = \partial_y 3z - \partial_z 2y = 0$

likewise y & z.

$\therefore$ conservative

$\vec{F} = -\vec{\nabla} U$

$\partial_x U = -x$

$U = -\int_{000}^{x00} dx\, x = -\frac{x^2}{2}$

$\partial_y U = -2y$ $U = -\int_{x00}^{xy0} dy\, 2y = -\frac{2y^2}{2}$

$\partial_z U = -3z$ $U = -\int dz\, 3z = -\frac{3z^2}{2}$

Try $\boxed{U = -\frac{1}{2}(x^2 + 2y^2 + 3z^2)}$ ✓

$-\vec{\nabla} U = (x, 2y, 3z)$

(b) $\vec{F} = k(y, x, 0)$ $(\vec{\nabla} \times \vec{F})_z = \partial_x x - \partial_y y = 0$.

$\Rightarrow$ cons

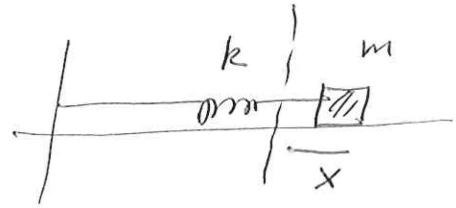
$\partial_x U = -y$ $U \to -xy$

$\partial_y U = -x$, $U \to -xy$

$\boxed{U = -xy}$ $-\vec{\nabla} U = \vec{\nabla}(xy) = y\hat{x} + x\hat{y}$ ✓

(c) $(\vec{\nabla} \times \vec{F})_z = 1 + 1 = 2 \neq 0$

4.28



$$\mathcal{E} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

When $\dot{x} = 0$, $x = A \Rightarrow \mathcal{E} = \frac{1}{2} k A^2$

$$\Rightarrow \boxed{A^2 = x^2 + \dot{x}^2/\omega^2} \qquad \omega^2 \equiv k/m$$

(a) and (b)

$$\frac{\dot{x}}{\omega} = \pm\sqrt{A^2 - x^2}.$$ Initial kick to right $\Rightarrow$ pick + sign

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt.$$ Find position at time t away from origin.

$$\Rightarrow \int_0^x \frac{dx'}{\sqrt{A^2 - x'^2}} = \omega t$$

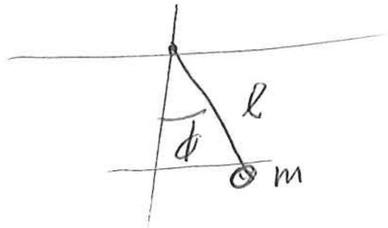
let $s = x'/A \Rightarrow \int_0^{x/A} \frac{ds}{\sqrt{1 - s^2}} = \omega t$

Use integral $\int \frac{ds}{\sqrt{1-s^2}} = \operatorname{arc}(\sin(x))$

$$\Rightarrow \boxed{x(t) = A \sin \omega t} \qquad 0 < t\omega < \pi/2$$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

4.34



$$U = mg(l - l\cos\phi)$$

$$U = mgl(1-\cos\phi)$$

$$U(\phi=0) = 0 \,, \quad U(\phi=\pi/2) = mgl$$

P energy increases as m goes up. ✓

$$v = l\dot{\phi}$$

$$\boxed{\mathcal{E} = \frac{1}{2} m\, l^2 \dot{\phi}^2 + mgl(1-\cos\phi)}$$

$\dot{\mathcal{E}} = 0$ since $\mathcal{E} \neq \mathcal{E}(t)$, its constant

$$\Rightarrow 0 = m l^2 \dot{\phi}\ddot{\phi} + mgl\sin\phi\,\dot{\phi}$$

$$\Rightarrow m l^2 \ddot{\phi} = - mgl\sin\phi$$

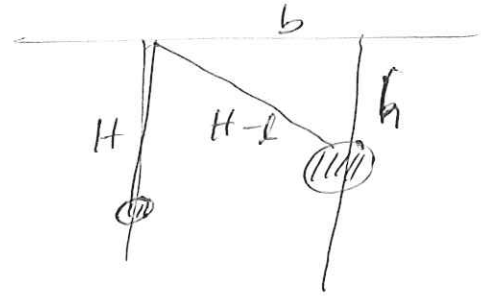
$$\boxed{I = ml^2} \Rightarrow \boxed{I\ddot{\phi} = -mgl\sin\phi}$$

$$\phi \ll \pi/2 \Rightarrow \sin\phi \simeq \phi \qquad I\ddot{\phi} = -mgl\,\phi$$

$$\Rightarrow \cos(\omega t) \qquad \omega^2 = \frac{mgl}{ml^2} = g/l$$

$$\tau_0 = \frac{2\pi}{\omega} = 2\pi\left(l/g\right)^{1/2}$$

4.36



$$U = -MgH - mgh, \qquad U = 0 \text{ at } y = 0$$

$$l - H = \frac{b}{\sin\theta} \qquad \tan\theta = \frac{b}{h}$$

$$\Rightarrow \frac{U}{g} = -M\left(l - \frac{b}{\sin\theta}\right) - \frac{mb}{\tan\theta}$$

$$\frac{U}{Mg} = -l + \frac{b}{\sin\theta} - \frac{m}{M}\frac{b}{\tan\theta}$$

$$\frac{U}{Mgb} = -\frac{l}{b} + \frac{1}{\sin\theta} - \frac{m}{M}\frac{\cos\theta}{\sin\theta}$$

$$\boxed{\frac{U}{Mgb}(\theta) = -\frac{l}{b} + \frac{\left(1 - \frac{m}{M}\cos\theta\right)}{\sin\theta} \qquad \theta \neq 0}$$

Equilibrium? only if $dU/d\theta = 0$

$$\frac{U'}{Mgb} = -\left(1 - \frac{m}{M}\cos\theta\right)\frac{\cos\theta}{\sin^2\theta} + \frac{m}{M}\frac{\sin\theta}{\sin\theta}$$

$$\cancel{0 = -\left(1 - \frac{m}{M}\cos\theta\right)\frac{\cos\theta}{\sin\theta} + \frac{m}{M}\sin\theta}$$

$$\cancel{\left(1 - \frac{m}{M}\cos\theta\right)\cos\theta = \frac{m}{M}\sin^2\theta}$$

$$\frac{U'}{Mgb} = -\frac{\cos\theta}{\sin^2\theta} + \frac{m}{M}\frac{\cos^2\theta}{\sin^2\theta} + \frac{m}{M}$$

$$= -\frac{\cos\theta}{\sin^2\theta} + \frac{m}{M}\frac{1}{\sin^2\theta}$$

$$\boxed{\frac{U'}{Mgb} = \frac{\frac{m}{M} - \cos\theta}{\sin^2\theta}}$$

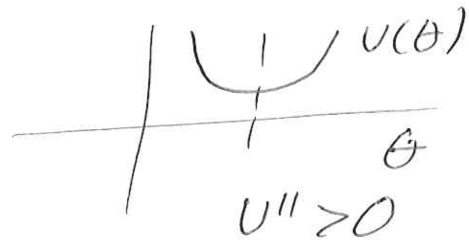
$U' = 0 \iff \cos\theta = m/M$. Only poss for $\boxed{\frac{m}{M} < 1}$

$$\frac{U''}{Mgb} = \underbrace{\left(\frac{m}{M} - \cos\theta\right)}_{=0 \text{ from previous}} \frac{d}{d\theta}\left(\frac{1}{\sin^2\theta}\right) + \frac{\sin\theta}{\sin^2\theta}$$

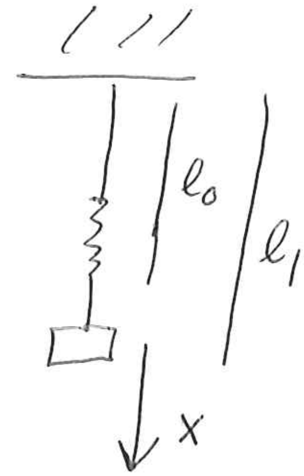
$$\Rightarrow \frac{U''}{Mgb} = \frac{1}{\sin\theta} = \cancel{\frac{\sqrt{M^2-m^2}}{M}} \frac{M}{\sqrt{M^2-m^2}}$$

$$\Rightarrow \frac{U''}{Mgb} = \frac{1}{\sqrt{1-m^2/M^2}} > 0$$

$\Rightarrow$ stable



Ch 5.1



Equilibrium:

$$mg = k(l_1 - l_0)$$

$$\Rightarrow l_1$$

small oscillations: (x>0 down, F down)

$$F = -k(l_1 + x - l_0) + mg$$

$$= -kx - k(l_1 - l_0) + mg$$

$$\Rightarrow \boxed{F = -kx}$$

$$-\frac{dU}{dx} = F(x) = -kx$$

$$\Rightarrow U = \frac{1}{2}kx^2$$

5.2 $U(r) = A\left[(f-1)^2 - 1\right]$

$f(r) = e^{\frac{R-r}{s}}$ $\qquad s \ll R$

$f'(r) = -\frac{1}{s} f$

$r \to +\infty \Rightarrow f \to 0$, fast

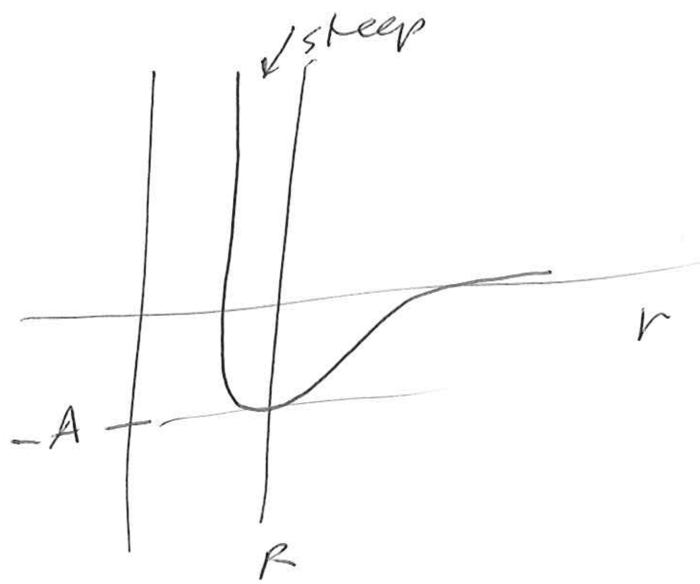
$\Rightarrow U \to A[1-1] \to 0$

$r \to 0 \Rightarrow f \to e^{R/s} \gg 1$

$\Rightarrow U(r) \to A e^{R/s} \gg 1$

$f(r=R) = 1 \Rightarrow U(r=R) = -A$

Sketch



Equilibrium

$$\frac{dV}{dr} = 0 \Rightarrow \frac{1}{A}\frac{dU}{dr} = 2(f-1)\left(-\frac{f}{s}\right)$$

$$\Rightarrow f = 1 \Rightarrow \boxed{r = R}$$

Let $r = r_0 + x = R + x$

$$\Rightarrow \quad R - r = -x$$

$$\frac{U(x)}{A} = \left(e^{-\frac{x}{s}} - 1\right)^2 - 1$$

$$\simeq \left(1 - \frac{x}{s} - 1\right)^2 - 1$$

$$= -1 + \left(\frac{x}{s}\right)^2$$

$$U(x) \simeq -A + \frac{A}{s^2} x^2$$ Hooke's Law

$$\Rightarrow \frac{1}{2} k = A/s^2$$

<u>5.9</u> let $x(t) = A\sin(\omega t)$

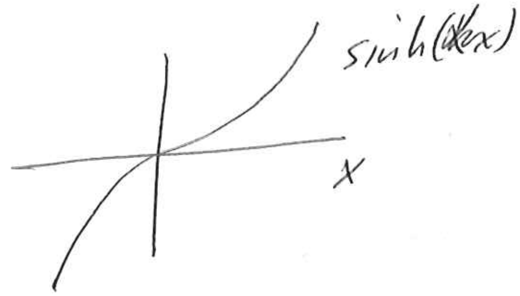
$x_{max} = A$ when $\omega t = \pi/2$

$\dot{x} = A\omega \cos(\omega t)$

$\dot{x}_{max}(\omega t = \pi/2) = A\omega$

$A = x_{max}$
$\omega A = \dot{x}_{max}$
$\Rightarrow \frac{1}{\omega} = \frac{0.2}{1.2} s = \frac{1}{6} s$

$\Rightarrow T = \frac{2\pi}{6} \Rightarrow \boxed{T = \frac{\pi}{3} s}$



<u>5.10</u> $F = -F_0 \sinh(\alpha x)$

$-\frac{dU}{dx} = -F_0 \sinh(\alpha x)$

$U = \frac{F_0}{\alpha} \cosh(\alpha x)$

$\omega^2 = \frac{F_0 \alpha}{m}$

$x \to 0 \quad U(x) \simeq \frac{F_0}{\alpha}\left(1 + \frac{\alpha^2 x^2}{2}\right)$

$U(x) \simeq \frac{F_0}{\alpha} + \frac{F_0}{2}\alpha x^2$

5.23

$$m\ddot{x} = -b\dot{x} - kx \quad \text{— ①}$$

let $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \neq \text{const}$

$$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x}$$

$$= \dot{x}(m\ddot{x} + kx)$$

$$= -b\dot{x}^2 \text{, from ①} \Rightarrow \dot{E} < 0$$

E lost

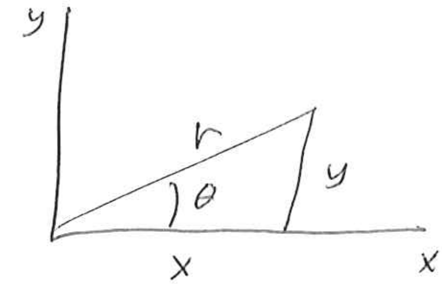
W done by friction $= -b\dot{x}\,dx$

$\frac{dW}{dt} = -b\dot{x}^2$ = energy lost to thermal energy

$$\Rightarrow \boxed{\dot{E} = \frac{dW}{dt}}$$

5.35

(a)



Coordinate transformation

$$x = r\cos\theta \,, \quad y = r\sin\theta$$

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

Euler $e^{i\theta} = \cos\theta + i\sin\theta$

$$\Rightarrow \quad z = re^{i\theta}$$

(b) $|z|^2 \equiv r^2 \,, \quad z^* \equiv x - iy \,.$

Then $zz^* = (x+iy)(x-iy) = x^2+y^2 = r^2$

(c) $z^* = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

(d) $(zw)^* = |z|\,|w|\,(e^{i\theta}e^{i\alpha})^* = |z|\,|w|(e^{i(\theta+\alpha)})^*$

$$= |z||w|e^{-i(\theta+\alpha)} = z^* w^*$$

$$\left(\frac{1}{z}\right)^* = \left(\frac{e^{-i\theta}}{|z|}\right)^* = \frac{e^{i\theta}}{|z|} = \frac{1}{z^*}$$

(e)

$$z = \frac{a}{b+ic} \,, \quad |z|^2 = \frac{a}{b+ic}\,\frac{a}{b-ic} = \frac{a^2}{b^2+c^2}$$

5.1H

$$x(0) = 0$$
$$\dot{x}(0) = -1$$
$$\ddot{x}(0) = -2$$

$$\dddot{x} - \ddot{x} + \dot{x} - x = 0$$

$$x = e^{\alpha t}$$

$$\alpha^3 - \alpha^2 + \alpha - 1 = 0$$
$$\alpha^2(\alpha - 1) + (\alpha - 1) = 0$$
$$(\alpha^2 + 1)(\alpha - 1) = 0$$

$$\Rightarrow \{e^{\pm it}, e^t\} \rightarrow \left\{ \begin{matrix} \cos t \\ \sin t \end{matrix} \right\}, e^t$$

let $x(t) = A\cos t + B\sin t + Ce^t$

Initial

$$0 = A + C$$
$$\Rightarrow x = A\cos t + B\sin t - Ae^t$$
$$\dot{x} = -A\sin t + B\cos t - Ae^t$$
$$\ddot{x} = -A\cos t - B\sin t - Ae^t$$

$$\Rightarrow -1 = B - A \Rightarrow B = 0$$
$$-2 = -A - A \Rightarrow A = 1$$

$$\Rightarrow \boxed{x(t) = \cos t - e^t}$$