Solns PS 13

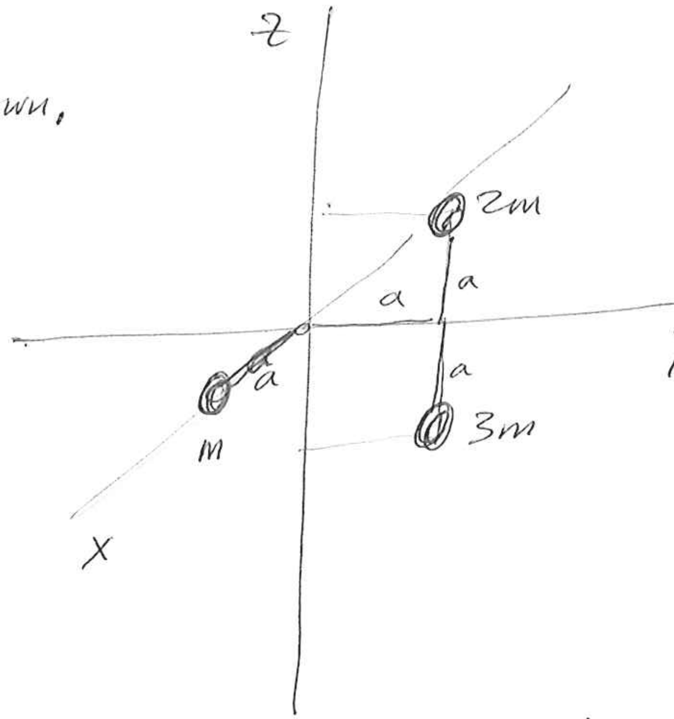
Phys 410/F16

10.35

Find $\underset{\sim}{I}$ wrt O

and (x, y, z) as shown.

let $m = 1$, $a = 1$.



$\sum m(xY)$

$\rightarrow 2(0 \times 1) + 3(0 \times +1)$

$= 0$

$\sum m(xz) = 1(1 \times 0) + 2(0 \times 1) + 3(0 \times -1) = 0$

$\sum m(yz) = 1(0 \times 0) + 2(1 \times 1) + 3(1 \times -1) = -1$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ & \cdot & +1 \\ & & \cdot \end{pmatrix}$$

$\sum m(y^2 + z^2) = 1(0+0) + 2(1^2 + 1^2) + 3(1^2 + (-1)^2)$

$= 4 + 6 = 10$

$\sum m(x^2 + z^2) = 1(1^2 + 0^2) + 2(0 + 1^2) + 3(0 + 1^2)$

$= 6$

$\sum m(x^2 + y^2) = 1(1^2 + 0^2) + 2(0^2 + 1^2) + 3(0^2 + 1^2)$

$= 6$

(a) $I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & +1 \\ 0 & +1 & 6 \end{pmatrix}$

(b) $I\omega = \lambda\omega$

$\Leftrightarrow |I - \lambda| = 0$

$(10-\lambda)\left[(6-\lambda)(6-\lambda) - 1\right] = 0$

$\Rightarrow \boxed{\lambda = 10}, \quad (6-\lambda)^2 = 1, \quad 6 - \lambda = \pm 1$

$\Rightarrow \boxed{\begin{array}{l} \lambda = 5 \\ \lambda = 7 \end{array}}$

$\underline{\lambda = 10}$ $\quad \Rightarrow \quad \begin{array}{l} 4\omega_2 = \omega_3 \\ 4\omega_3 = \omega_2 \end{array} \quad \Rightarrow \quad \omega_3 = \omega_2 = 0 \quad \Rightarrow (1, 0, 0)$

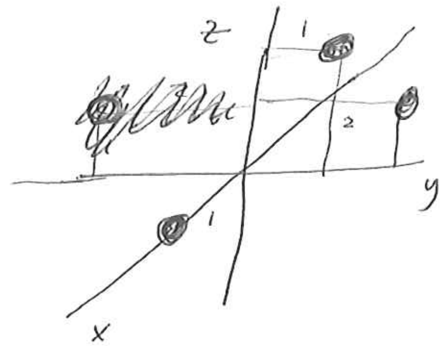
$\underline{\lambda = 7}$ $\quad \begin{array}{l} 3\omega_1 = 0 \\ \omega_2 = \omega_3 \end{array} \quad \Rightarrow \quad (0, 1, 1)$

$\underline{\lambda = 5}$ $\quad \begin{array}{l} 5\omega_1 = 0 \\ \omega_2 = -\omega_3 \end{array} \quad \Rightarrow \quad (0, 1, -1)$

Unnormalized Principal axes.

orthogonal, as can be checked.

10.36



(a) Let $m = 1$
$a = 1$

$(1, 0, 0)$
$(0, 1, 2)$
$(0, 2, 1)$

$$\sum xy = 1\cdot 0 + 0\cdot 1 + 0\cdot 2 = 0$$

$$\sum xz = 1\cdot 0 + 0\cdot 2 + 0\cdot 1 = 0$$

$$\sum yz = 0\cdot 0 + 1\cdot 2 + 2\cdot 1 = 4$$

$$\sum y^2+z^2 = 0^2+0^2 + 1^2+2^2 + 2^2+1^2 = 10$$

$$\sum x^2+z^2 = 1^2+0^2 + 0^2+2^2 + 0^2+1^2 = 6$$

$$\sum x^2+y^2 = 1^2+0^2 + 0^2+1^2 + 0^2+2^2 = 6$$

$$\Rightarrow I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$$

(b) $I\omega = \lambda\omega \Rightarrow \overline{10}\left[5\cdot\overline{6} - 4^2\right] = 0$

$\Rightarrow \overline{10} = 0$ or $\left[5\cdot\overline{6} - 4^2\right] = 0$

$\Rightarrow \boxed{\lambda = 10}$ or $(5-\lambda)(6-\lambda) - 16 = 0$

$30 - 11\lambda + \lambda^2 - 16 = 0$

$\lambda^2 - 11\lambda + 14 = 0$

(b). $det = 0 \Rightarrow \bar{10}[\bar{6}^2 - 4^2] = 0$

$\bar{10} = 0 \Rightarrow \boxed{\lambda = 10}$

$\bar{6}^2 - 4^2 = (\bar{6}-4)(\bar{6}+4) = 0$

$\Rightarrow (2-\lambda)(10-\lambda) = 0$

$\Rightarrow \boxed{\lambda = 2} \quad \boxed{\lambda = 10}$

$\underline{\lambda = 2} \Rightarrow \quad 8w_1 + 0w_2 + 0w_3 = 0 \Rightarrow w_1 = 0$

$0w_1 + 4w_2 - 4w_3 = 0 \Rightarrow w_2 = w_3 \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = 10}$

$-4w_2 - 4w_3 = 0 \quad w_2 = -w_3$

$-4w_2 - 4w_3 = 0 \Rightarrow \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$

a is arbitrary

$\Rightarrow$ choose $(a, 1, -1)$, $(b, 1, -1)$ $\Rightarrow$ orthogonal

$\Rightarrow ab + 1 + 1 = 0 \Rightarrow ab = -2$

$\Rightarrow$ choose $(1, 1, -1)$, $(-2, 1, -1)$ and $(0, 1, 1)$

10 · 44

let $\omega_{30} = 0$

$\omega_{10} \neq 0$

$$\lambda_1 \dot{\omega}_1 = \omega_2 \omega_3 (\lambda_2 - \lambda_3)$$

$$\lambda_2 \dot{\omega}_2 = \omega_3 \omega_1 (\lambda_3 - \lambda_1)$$

$$\lambda_3 \dot{\omega}_3 - \omega_1 \omega_2 \underbrace{(\lambda_1 - \lambda_2)}_{=0} = \Gamma_3 = \text{const}$$

$\lambda_1 = \lambda_2$

$$\Rightarrow \boxed{\omega_3 = \frac{\Gamma_3}{\lambda_3} t}$$

from rest

$\omega_3 = \alpha t$, $\alpha \equiv \Gamma_3 / \lambda_3$

$$\Rightarrow \lambda_1 \dot{\omega}_1 = \omega_2 (\lambda_1 - \lambda_3) \omega_3(t)$$

$$\lambda_1 \dot{\omega}_2 = \omega_1 (\lambda_3 - \lambda_1) \omega_3(t)$$

$$\boxed{\begin{aligned} \dot{\omega}_1 &= \omega_2 K \omega_3 \\ \dot{\omega}_2 &= -\omega_1 K \omega_3 \end{aligned}}$$

$K \equiv \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right)$

$$\frac{1}{\omega_3} \frac{d\omega_1}{dt} = \omega_2 K$$

Let $\int_0^t \omega_3(t)\, dt = \int_0^\tau d\tau$

$$\Rightarrow \tau = \alpha t^2/2$$

$$d\tau = \omega_3\, dt$$

$$\Rightarrow \frac{d\omega_1}{d\tau} = K\omega_2$$

$$\frac{d\omega_2}{d\tau} = -K\omega_1$$

$$\Rightarrow \frac{d^2\omega_1}{d\tau^2} = -K^2 \omega_1 \Rightarrow \boxed{\omega_1 = \omega_{10} \cos(K\alpha t^2/2)}$$

B.1 H

The mathematical point here is if one solves $I\vec{\omega} = \lambda\vec{\omega}$, one is guaranteed to find 3 $\vec{\omega}^{(n)}$'s. Note that for these $\vec{\omega}^{(n)}$'s, $\vec{L}$ is $\parallel$ to $\vec{\omega}^{(n)}$.

1. Thus, our definition of a P-axis is consistent with the above theorem.

2. To find P-axes, solve for $\vec{\omega}^{(n)}$ from $I\,\vec{\omega}^{(n)} = \lambda^{(n)}\vec{\omega}^{(n)}$. Again, self-consistent. The 3 $\vec{\omega}^{(n)}$ are the P-axes.

3. In coordinates chosen w.r.t. P-axes, Let $I = I_{kl}$, in its most general form. Then, for each $\omega^{(n)}$, say $\omega^{(1)} = (1, 0, 0)$, we have $(I_{xx}, I_{yx}, I_{zx}) = \lambda^{(1)}(1, 0, 0)$. Comparing sides, $\Rightarrow I_{xx} = \lambda^{(1)}$, $I_{yx} = 0$, $I_{zx} = 0$. Doing this for all n, we find I = diagonal.

4. Thus, in P-coordinate axes, I is diagonal.

- Now $\omega^{(n)} \parallel$ P-axes. e.g., suppose $\omega^{(1)} = (1, 0, 0)$. Then $I\vec{\omega}^{(1)} = \lambda^{(1)}\vec{\omega}^{(1)}$.
  Thus, $\vec{L} = I\vec{\omega}^{(1)}$ is $\parallel$ to $\vec{\omega}^{(1)}$. Etc.

- For the R to L proof, start with $\vec{L} \parallel \vec{\omega}$, for diagonal I. Thus, start with $I\vec{\omega} = \lambda\vec{\omega} \Rightarrow (\lambda_1\omega_x, \lambda_2\omega_y, \lambda_3\omega_z) = \lambda(\omega_x, \omega_y, \omega_z)$. Equating sides,

we have
$$\lambda_1\omega_x = \lambda\omega_x \qquad (1)$$
$$\lambda_2\omega_y = \lambda\omega_y \qquad (2)$$
$$\lambda_3\omega_z = \lambda\omega_z \qquad (3)$$

Suppose, from (1), $\lambda = \lambda_1$, $\omega_x \neq 0$.
Then, (2) $\Rightarrow \lambda_2\omega_y = \lambda_1\omega_y$. Assuming distinct λ's, $\Rightarrow \omega_y = 0$. Likewise, $\omega_z = 0$.

$\therefore \vec{L} \parallel \vec{\omega} \Rightarrow \vec{\omega} \parallel$ P-axes.

15.79

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad (1)$$

$$\vec{p} = \gamma m \vec{v}$$

$$\gamma^2 = \frac{1}{1 - \vec{v}\cdot\vec{v}/c^2}$$

$$\Rightarrow \gamma^2 - \left(\gamma \frac{\vec{v}}{c}\right)\cdot\left(\gamma \frac{\vec{v}}{c}\right) = 1$$

$$\Rightarrow 2\gamma\dot{\gamma} = 2\left(\gamma \frac{\vec{v}}{c}\right)\cdot\left(\gamma \frac{\vec{v}}{c}\right)^{\circ} \qquad (2)$$

From (1), $\dot{\gamma} m \vec{v} + \gamma m \vec{a} = \vec{F}$

Use (2) $\Rightarrow$ $\frac{\vec{v}}{c}\cdot\left(\gamma\frac{\vec{v}}{c}\right)^{\circ} m \vec{v} + \gamma m \vec{a} = \vec{F}$

Use (1) again $\Rightarrow$ $\cancel{m}\frac{\vec{v}}{c}\frac{\vec{v}}{c}\cdot\vec{F} + \gamma m \vec{a} = \vec{F}$

$$\Rightarrow \boxed{\vec{F} = \gamma m \vec{a} + \vec{v}\,\vec{v}\cdot\vec{F}/c^2}$$

15.80

Here, $\vec{F} = q\vec{v} \times \vec{B}$

$$\Rightarrow \quad \frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B}$$

Use 15.79

$$\Rightarrow \quad \gamma m\vec{a} + \underbrace{\frac{q\vec{v} \times \vec{B} \cdot \vec{v}}{c^2}}_{=0} = q\vec{v} \times \vec{B} \quad \text{— (1)}$$

Now $|\vec{a}| = v^2/r$, centripetal acceleration

assuming circular orbit

$$(1) \Rightarrow \quad \gamma \frac{mv^2}{r} = qvB$$

$$\Rightarrow \quad r = \frac{\gamma m v}{qB} \quad \Rightarrow \quad \boxed{r = \frac{|\vec{p}|}{qB}}$$