Solns PS 12

Phys 410 / F16

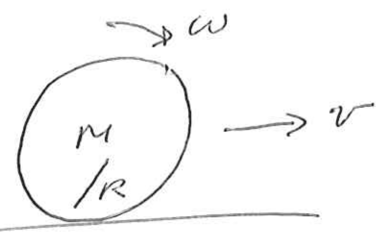
10.2

From Eq 10.18, and prior,

$$T = T(\text{motion of CM}) + T(\text{rot}^n \text{ about CM}) \quad (A)$$

if fixed pt R is at rest,

$$T = \frac{1}{2} \sum m_\alpha |\dot{r}_\alpha|^2 . \quad (B)$$



No slip

$\Rightarrow$ $v = R\omega$

1st, use (A) $\Rightarrow T = \frac{1}{2} M \dot{R}_{cm}^2 + T_{\text{wrt CM}}$

$$= \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} M R^2 \omega^2 + \frac{1}{2}\frac{1}{2} M R^2 \omega^2$$

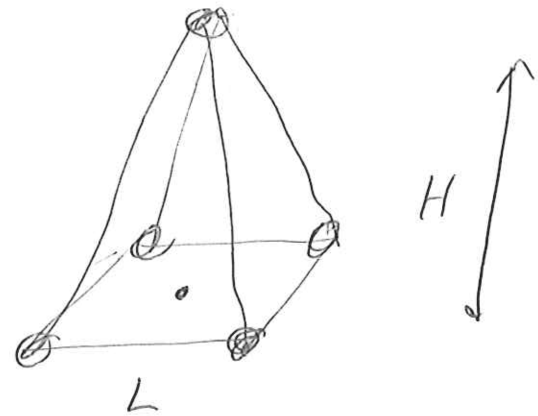
$$T = \frac{3}{4} M R^2 \omega^2 ; \leftarrow$$

2nd, use B, wrt instantaneous fixed point

$$T = \frac{1}{2} I_0' \omega^2 = \frac{1}{2}\frac{3}{2} M R^2 \omega^2$$

$$T = \frac{3}{4} M R^2 \omega^2 , \leftarrow \text{Agree}$$

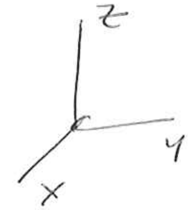
10.3



$\sum M_\alpha \vec{R} \equiv \sum M_\alpha \vec{r_\alpha}$

$\sum M_\alpha = 5 \quad , \; m = 1$

Let $\vec{R} = (X, Y, Z)$, Let $L = 1$



O @ center of square

$\Rightarrow 5X = \sum M_\alpha x_\alpha = \sum x_\alpha$

But $\sum x_\alpha = 0$ from symm

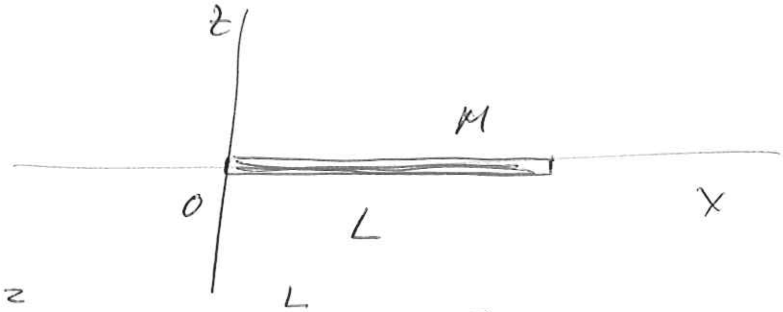
Also, $5Y = \sum y_\alpha = 0$ " "

$5Z = \sum z_\alpha = H$

$\Rightarrow \vec{R} = (0, 0, H/5)$

10.10

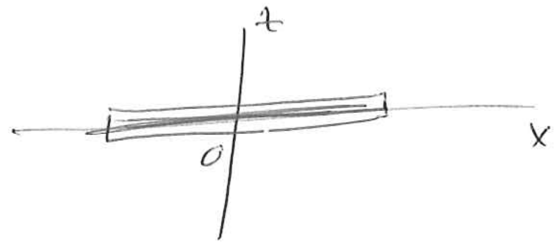
(a)



$$\frac{I}{M} = \frac{\sum m x^2}{\sum m} \rightarrow \frac{\int_0^L dx\,\mu\, x^2}{\int_0^L dx\,\mu} = \frac{1}{3} L^2$$

$$\boxed{I = \frac{1}{3} M L^2}$$

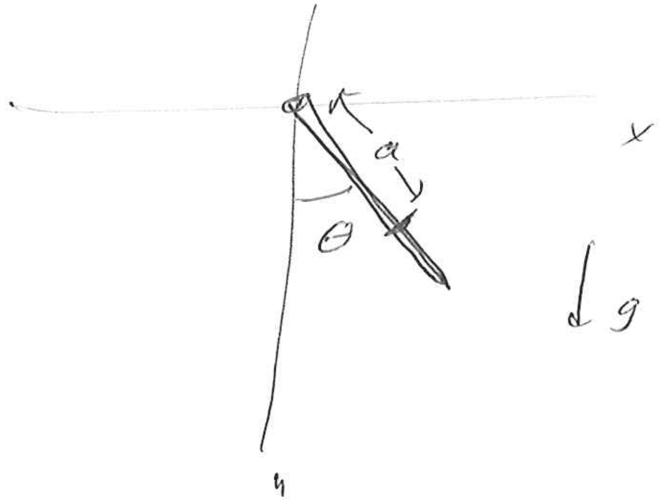
(b) $$\frac{I}{M} \rightarrow \frac{\int_{-L/2}^{L/2} dx\,\mu x^2}{\int_{-L/2}^{L/2} dx\,\mu}$$



$$= L^2 \left(\frac{x^3}{3}\right)_{-1/2}^{1/2} = L^2 \frac{2}{3}\frac{1}{8} = L^2 \frac{1}{12}$$

$$\boxed{I = \frac{1}{12} M L^2}$$

10.13



mass = m

MI about z = I

$L = I\omega$

$\dot{L} = \Gamma$

$\Gamma = Mga\sin\theta$, opposite sign to L

$\Rightarrow \boxed{I\dot{\omega} = -Mga\sin\theta} \qquad \omega = \dot{\theta}$

$\theta \to 0 \Rightarrow \boxed{I\ddot{\theta} \simeq -Mga\theta}$

$\therefore$ $\boxed{\text{angular frequency of osc} = \sqrt{\frac{Mga}{I}}}$

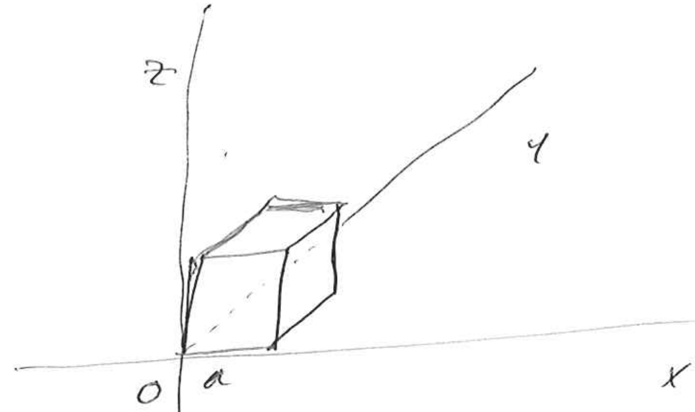
(a)

For pendulum, ang freq $= \sqrt{\frac{g}{l}}$

(b)

$\Rightarrow \frac{g}{l_{equiv}} = \frac{Mga}{I} \Rightarrow \boxed{\left(\frac{l}{a}\right)_{equiv} = \frac{I}{Ma^2}}$

10.15



rotate about z

$\vec{\omega} = \omega \hat{z}$

MI about this axis, I_{zz}

$$\frac{I_{zz}}{M} = \frac{\sum m (x^2+y^2)}{\sum m}$$

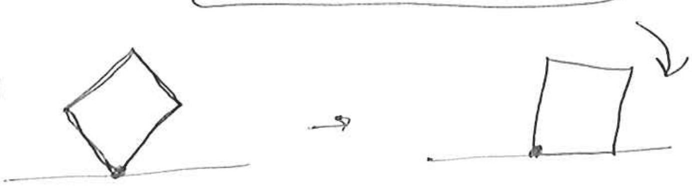
$$\frac{I_{zz}}{M} = \frac{\int dV (x^2+y^2) \rho}{\int dV \rho} = \frac{\int_0^1 dz \int_0^1 dx \int_0^1 dy (x^2+y^2)}{\int dz \int dx \int dy}$$

$$= \frac{\int dx \left[x^2 y + y^3/3 \right]_0^1}{\int dx} \Rightarrow \int dx (x^2 + 1/3)$$

$$= \left[\frac{x^3}{3} + \frac{1}{3} x \right] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

(a) $\Rightarrow \boxed{I_{zz} = \frac{2}{3} M a^2}$

(b)



PE(initial) $= Mg \frac{\sqrt{2} a}{2}$

PE(final) $= Mg \frac{1}{2} a$

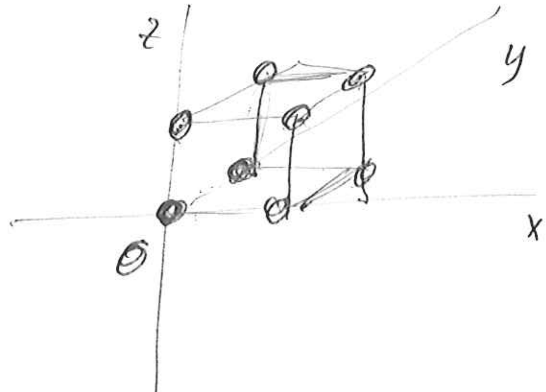
$KE = \frac{1}{2} I \omega^2$

$\Rightarrow \boxed{\omega^2_{final} = \frac{Mga}{\cancel{2}\, I} (\sqrt{2}-1)}$

OR $\boxed{\omega^2_{final} = (\sqrt{2}-1) \frac{3}{2} \frac{g}{a}}$

10.22

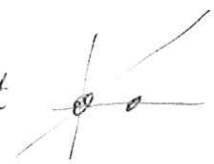
(a) about corner



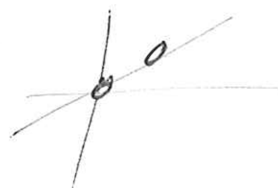
$$\sum mxy \to \sum xy = (11) + (11) = 2$$

$$\sum mxz \to \sum xz = (11) + (11) = 2$$

$$\sum myz \to \sum yz = (11) + (11) = 2$$

$\sum m(y^2+z^2) =$ all but 

$$= (1^2) + (1^2) + (1^2) + (1^2) + (1^2+1^2) + (1^2+1^2) = 8$$

$\sum m(x^2+z^2) =$ all but 

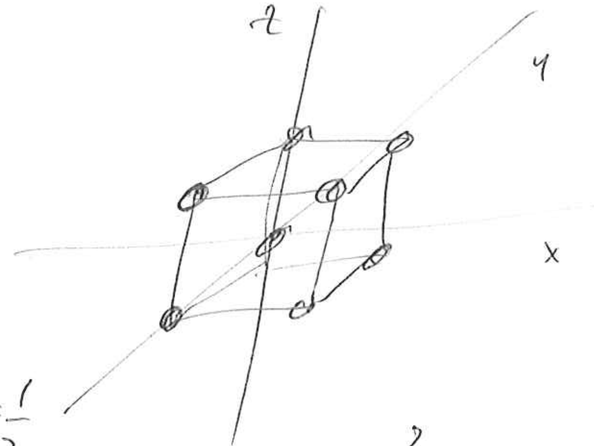
$$= (1^2) + (1^2) + (1^2) + (1^2) + (1^2+1^2) + (1^2+1^2) = 8$$

likewise $\sum m(x^2+y^2) = 8$

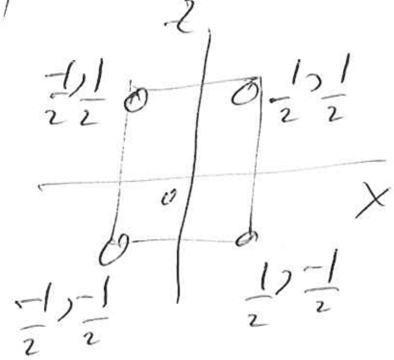
$$\Rightarrow \quad I = ma^2 \begin{pmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

(b) about center

High symmetry



$\sum xz$, consider @ $y = \frac{1}{2}$



Fore every $x_+ z_+$

there is and $x_- z_+$

and viceversa

$\Rightarrow$ No off diagonal

likewise others $\Rightarrow$ Diagonal matrix

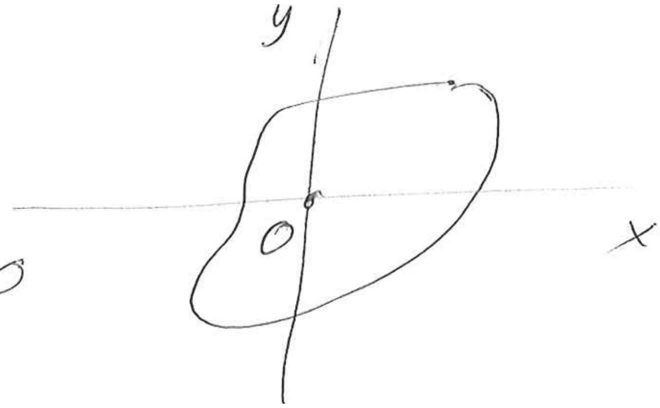
$\sum (x^2 + z^2)$ all indep of sign

$$\Rightarrow \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] \times 8 = \frac{2}{4} \cdot 8 = 4$$

$$\Rightarrow I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} 4ma^2$$

10.23

choose x, y axes as shown, about O ($z = 0$ plane)



$$\Rightarrow \sum xz = 0 \qquad \Rightarrow \begin{pmatrix} 0 & -xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sum yz = 0$$

$$I_{zz} = \sum m(x^2+y^2)$$

$$I_{xx} + I_{yy} = \sum m(y^2+z^2) + \sum m(x^2+z^2)$$

$$= \sum m(y^2) + \sum m(x^2)$$

$$= \sum m(x^2+y^2)$$

$$\Rightarrow \boxed{I_{xx} + I_{yy} = I_{zz}}$$