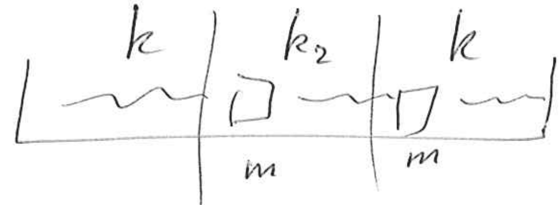
Phys 410/F16

Solns PS #0

[Note: solutions sometimes cast in normalized units, as discussed in class and Taylor. You do not have to know this — but it's useful and good to learn.]

11.4



- $m\ddot{x}_1 = -kx_1 + k_2(x_2 - x_1)$

  $m\ddot{x}_2 = -kx_2 + k_2(x_1 - x_2)$

- let $m=1$, $k=1$ (But cannot set* $k_1 = 1$ since same dimension as k)

$$\Rightarrow \quad \ddot{x}_1 = -x_1 + k_2(x_2 - x_1)$$
$$\ddot{x}_2 = -x_2 + k_2(x_1 - x_2)$$

Note the system depends on a single parameter, k_2, ie, k_2/k.

- Try $e^{i\omega t} \Rightarrow$

$$-\omega^2 x_1 = -x_1 + k_2(x_2 - x_1)$$
$$-\omega^2 x_2 = -x_2 + k_2(x_1 - x_2)$$

* this means t is normalized to $(m/k)^{1/2}$, and k_2 normalized to k.

$$\Rightarrow \boxed{\begin{aligned} (1+k_2-\omega^2)\,x_1 &= k_2\,x_2 \\ (1+k_2-\omega^2)\,x_2 &= k_2\,x_1 \end{aligned}} \qquad (A)$$

- $|\mathrm{Det}| = 0 \Rightarrow$

$$(1+k_2-\omega^2)^2 = k_2^2$$

$$\Rightarrow 1+k_2-\omega^2 = \pm k_2$$

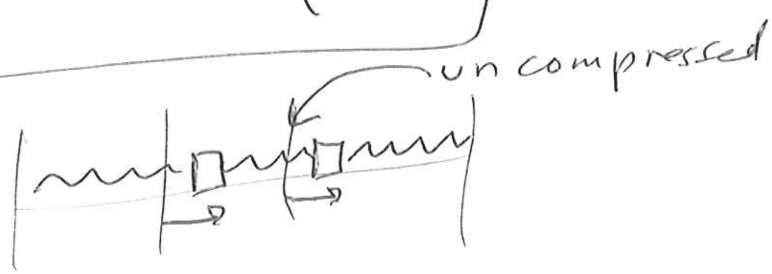
$$\Rightarrow \boxed{\begin{aligned} &1-\omega^2 = 0 \qquad \underline{\#1} \\ \text{OR}\quad &1+2k_2-\omega^2 = 0 \qquad \underline{\#2} \end{aligned}}$$

- $\#1$ $\quad \underline{\omega^2 = 1} \Rightarrow$ (from A)

$$k_2 x_1 = k_2 x_2 \Rightarrow x_1 = x_2$$

$$\Rightarrow \boxed{\omega^2 = 1, \quad \vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\Rightarrow$

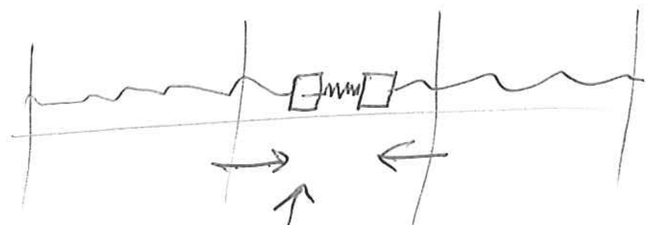


(#2) $\omega^2 = 1 + 2k_2 \Rightarrow$

$$x_1 = -x_2$$

$$\boxed{\omega^2 = 1 + 2k_2, \quad \vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$\Rightarrow$



k_2 compressed

Unormalizing

(#1) $\boxed{\omega^2 = k/m}$

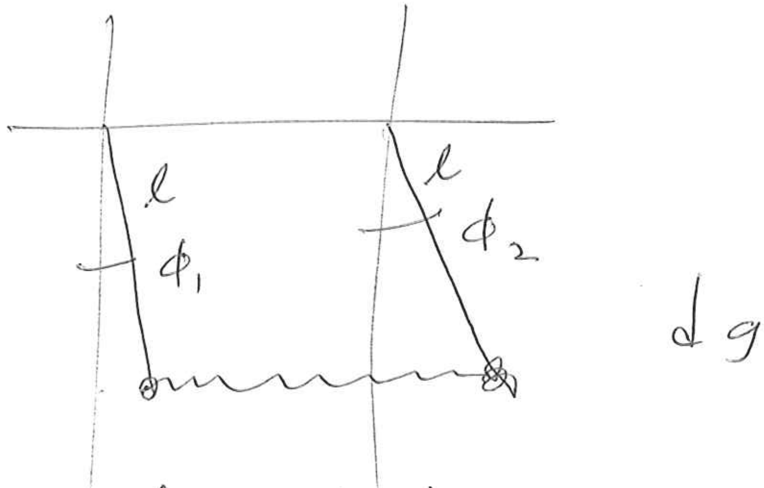
since $k = 1$
$m = 1$
$\Rightarrow k/m = 1$
$\Rightarrow$ Normalization to "k mode".

(#2) $\omega^2 = \frac{k}{m} + \frac{2k_2}{m}$

OR $\boxed{\frac{\omega^2}{k/m} = 1 + 2\left(\frac{k_2}{k}\right)}$

system parameterized by (k_2/k).

11.14



- $m_1 = m_2 = m$

$$T = \frac{1}{2} m \ell^2 \left(\dot\phi_1^2 + \dot\phi_2^2\right)$$

$$U_s = \frac{1}{2} k \ell^2 (\phi_2 - \phi_1)^2 \qquad \text{(A)}$$

$$U_g = \frac{1}{2} m g \ell \left(\phi_1^2 + \phi_2^2\right)$$

- let $m = 1$, $\ell = 1$, $k = 1$
  - $\Rightarrow$ length normalized to ℓ
  - $\Rightarrow$ time normalized to $(m/k)^{1/2}$
  - $\Rightarrow$ Energy normalized to $k\ell^2$

$$\Rightarrow \quad T = \frac{1}{2} \left(\dot\phi_1^2 + \dot\phi_2^2\right)$$

$$U_s = \frac{1}{2} (\phi_2 - \phi_1)^2 \qquad \text{(B)}$$

$$U_g = \frac{1}{2} g \left(\phi_1^2 + \phi_2^2\right)$$

- Note Cannot set $g = 1$.
  - (a) g has same dimensions as $\ell k/m$
  - (b) cannot set more than 3 things to unity.

$\mathcal{L} = T - U_s - U_g$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \dot{\phi}_1 \; ; \; \frac{\partial \mathcal{L}}{\partial \phi_1} = (\phi_2 - \phi_1) - g\phi_1$$

$$\Rightarrow \boxed{\begin{aligned} \ddot{\phi}_1 &= -g\phi_1 + (\phi_2 - \phi_1) \\ \ddot{\phi}_2 &= -g\phi_2 - (\phi_2 - \phi_1) \end{aligned}}$$

System Depends only on one parameter g where

$$g \to \frac{mgl}{kl^2} = \frac{mg}{kl}$$

$$\Rightarrow \boxed{\begin{aligned} (g+1-\omega^2)\phi_1 &= \phi_2 \\ (g+1-\omega^2)\phi_2 &= \phi_1 \end{aligned}}$$

$|Det| = 0 \Rightarrow (g+1-\omega^2)^2 = 1$

$$g+1-\omega^2 = \pm 1$$

#1 $\quad \omega^2 = g \Rightarrow \phi_1 = \phi_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

#2 $\quad \omega^2 = g+2 \Rightarrow \phi_1 = -\phi_2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- More systematic normalization

- Start from (A)

let $\hat{U}_s = \frac{U_s}{kl^2}$, $\hat{U}_g = \frac{U_g}{kl^2}$, $\hat{T} = \frac{T}{kl^2}$

$$\Rightarrow \hat{T} = \frac{1}{2}\frac{m}{k}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right)$$

$$\hat{U}_s = \frac{1}{2}\left(\phi_2 - \phi_1\right)^2$$

$$\hat{U}_g = \frac{1}{2}\frac{mg}{kl}\left(\phi_1^2 + \phi_2^2\right)$$

- let $\hat{t} = t\left(\frac{k}{m}\right)^{1/2}$

$$\Rightarrow \frac{d\phi_1}{dt} = \frac{d\hat{t}}{dt}\frac{d\phi_1}{d\hat{t}} = \left(\frac{k}{m}\right)^{1/2}\frac{d\phi_1}{d\hat{t}}$$

$$\Rightarrow \frac{1}{2}\frac{m}{k}\left(\frac{d\phi_1}{dt}\right)^2 = \frac{1}{2}\left(\frac{d\phi_1}{d\hat{t}}\right)^2$$

$$\Rightarrow \hat{T} = \frac{1}{2}\left(\hat{\dot{\phi}}_1^2 + \hat{\dot{\phi}}_2^2\right)$$

Drop all λ's

$\Rightarrow$

$$T = \frac{1}{2}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right)$$

$$U_s = \frac{1}{2}\left(\phi_2 - \phi_1\right)^2$$

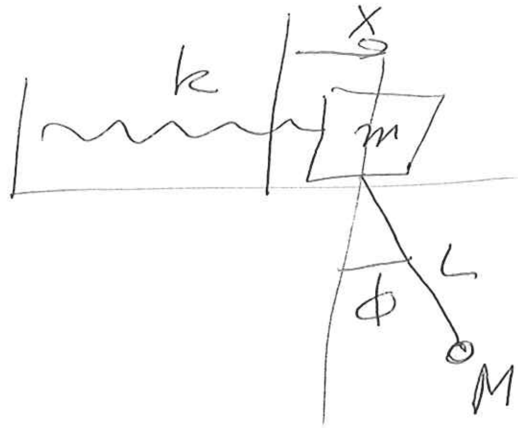
$$U_g = \frac{1}{2}\left(\frac{mg}{kl}\right)\left(\phi_1^2 + \phi_2^2\right)$$

— exactly as our starting point (B)

setting $k=1$, $l=1$, $m=1$,

with single param $g \to \frac{mg}{kl}$

11.19



(a) small ϕ

$$T_m = \frac{1}{2} m \dot{X}^2$$

$$T_M = \frac{1}{2} M \dot{(X + L\phi)}^2$$

$$= \frac{1}{2} M (\dot{X} + L\dot{\phi})^2$$

$$U_m = \frac{1}{2} k X^2$$

$$U_M = \frac{1}{2} M g L \phi^2$$

- let $m = 1$, $k = 1$, $L = 1$

$$\Rightarrow \boxed{\begin{aligned} T &= \frac{1}{2}\dot{X}^2 + \frac{1}{2} M (\dot{X} + \dot{\phi})^2 \\ U &= \frac{1}{2} X^2 + \frac{1}{2} M g \phi^2 \end{aligned}}$$

2 params: $M \to \frac{M}{m}$, and $g \to \frac{mg}{kL}$

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} + M(\dot{x} + \dot{\phi})$$

$$\frac{\partial L}{\partial x} = -x$$

$$\Rightarrow \boxed{\ddot{x} + M(\ddot{x} + \ddot{\phi}) = -x}$$

$$\frac{\partial L}{\partial \dot{\phi}} = M(\dot{x} + \dot{\phi})$$

$$\frac{\partial L}{\partial \phi} = -Mg\phi$$

$$\Rightarrow \boxed{\ddot{x} + \ddot{\phi} = -g\phi}$$

- Try $e^{+i\omega t} \Rightarrow$

$$+\omega^2(1+M)\,x + \omega^2\phi = +x$$

$$+\omega^2 x + \omega^2\phi = +g\phi$$

$$\Rightarrow \boxed{\begin{aligned} &[\omega^2(1+M) - 1]\,x = -\omega^2\phi \\ &(\omega^2 - g)\,\phi = -\omega^2 x \end{aligned}} \quad (A)$$

Determinant = 0

$$\Rightarrow \left[\omega^2(1+M) - 1\right](\omega^2 - g) = \omega^4$$

Lim $M = 1$, $g = 1/2$

$$\Rightarrow (\omega^2 2 - 1)(\omega^2 - 1/2) = \omega^4$$

$$(\omega^2 - 1/2)^2 = \omega^4/2$$

$$\boxed{\omega^2 - 1/2 = \pm\, \omega^2/\sqrt{2}}$$

#1 $\quad \omega^2(1 - 1/\sqrt{2}) = 1/2$

#2 $\quad \omega^2(1 + 1/\sqrt{2}) = 1/2$

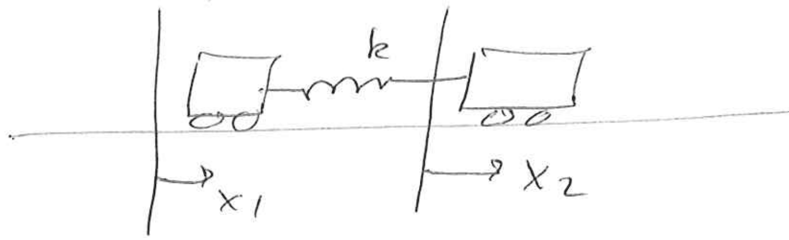
From (A) $\quad \cancel{(2\omega^2 - 1)}$

$$(\omega^2 - 1/2)\,\phi = -\omega^2 X$$

#1 $\Rightarrow \dfrac{\omega^2}{\sqrt{2}}\phi = -\omega^2 X \Rightarrow \phi = -\sqrt{2} X \quad \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} X \\ \phi \end{pmatrix}$

#2 $\Rightarrow -\dfrac{\omega^2}{\sqrt{2}}\phi = -\omega^2 X \Rightarrow \phi = \sqrt{2} X \quad \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

11.27



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} k (x_2 - x_1)^2$$

$k = 1, \quad m = 1$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \dot{x}_1, \quad \frac{\partial \mathcal{L}}{\partial x_1} = (x_2 - x_1)$$

$$\Rightarrow \quad \ddot{x}_1 = (x_2 - x_1)$$

$$\ddot{x}_2 = -(x_2 - x_1)$$

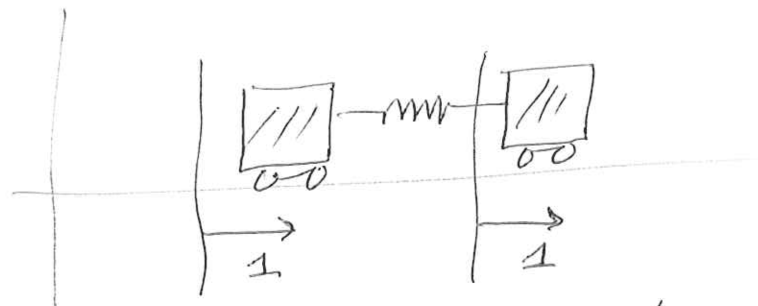
$$\Rightarrow \boxed{\begin{aligned} (1-\omega^2)\, x_1 &= x_2 \\ (1-\omega^2)\, x_2 &= x_1 \end{aligned}} \qquad \text{A}$$

$$(1-\omega^2)^2 = 1 \Rightarrow (1-\omega^2) = \pm 1$$

#1 $\quad 1-\omega^2 = 1 \Rightarrow \omega^2 = 0$

From (A) $\quad x_1 = x_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

This means

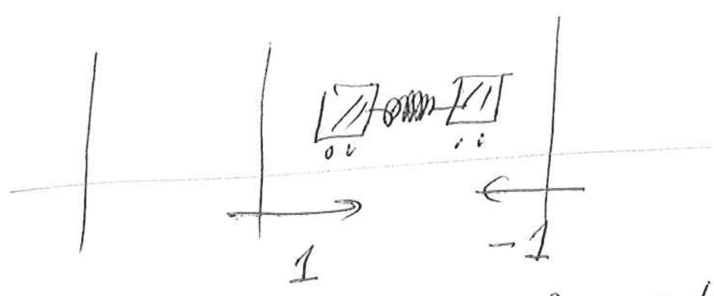


both carts move at constant, equal velocity, so spring uncompressed, so ~~but~~ No oscillation.

eigenvector $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, equal displacement

#2 $\quad 1-\omega^2 = -1 \Rightarrow \omega^2 = 2$

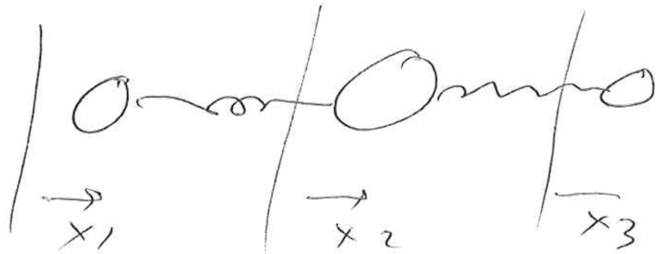
$\Rightarrow$



$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Compression $\Rightarrow \omega^2 = 2k/m$

11.32



let $m = 1,\ k = 1,\ M \neq m = 1.$

$$T = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} M \dot{x}_2^2$$

$$U = \frac{1}{2}(x_2 - x_1)^2 + \frac{1}{2}(x_3 - x_2)^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \dot{x}_1, \quad \frac{\partial \mathcal{L}}{\partial x_1} = (x_2 - x_1)$$

x_3 is symmetric $\Rightarrow$

$$\boxed{\begin{aligned} \ddot{x}_1 &= -(x_1 - x_2) \\ \ddot{x}_3 &= -(x_3 - x_2) \end{aligned}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = M\dot{x}_2, \quad \frac{\partial \mathcal{L}}{\partial x_2} = -(x_2 - x_1) - (x_2 - x_3)$$

$$\boxed{M\ddot{x}_2 = -(x_2 - x_1) - (x_2 - x_3)}$$

$e^{i\omega t} \Rightarrow$

$$(1-\omega^2)X_1 = X_2$$
$$(1-\omega^2)X_3 = X_2$$

$$(2-M\omega^2)X_2 = X_1 + X_3$$

$$\begin{vmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-M\omega^2 & -1 \\ 0 & -1 & 1-\omega^2 \end{vmatrix} = 0 \quad \text{A}$$

$$(1-\omega^2)\left[(2-M\omega^2)(1-\omega^2) - 1\right]$$

$$+(1-\omega^2) = 0$$

$$\Rightarrow \boxed{(1-\omega^2) = 0} \quad \#1$$

OR $(2-M\omega^2)(1-\omega^2) - 1 = -1$

$$(2-M\omega^2)(1-\omega^2) = 2$$

$$2 - \omega^2(1+M) + M\omega^4 = 2$$

$$\Rightarrow \quad M\omega^4 \overset{-}{\cancel{+}} \omega^2(1+M) = 0$$

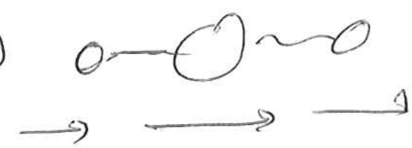
$$\Rightarrow \quad \omega^2\left[M\omega^2 \overset{-}{\cancel{+}} (1+M)\right] = 0$$

$$\Rightarrow \quad \boxed{\omega^2 = 0} \qquad \#2$$

$$\text{or} \quad \boxed{\omega^2 = 1 + \frac{1}{M}} \qquad \#3$$

$$\#2 \quad \omega^2 = 0 \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} = 0$$

$$\Rightarrow \quad x_1 = x_2 = x_3 \quad \Rightarrow$$



translation mode

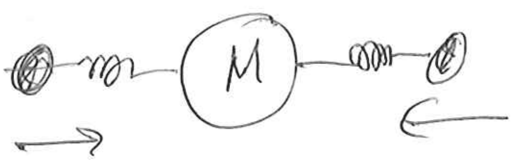
springs uncompressed

#1 $\omega^2 = 1$

$$\Rightarrow \boxed{X_2 = 0}$$

$$\begin{pmatrix} -1 & 2-M & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$\Rightarrow \boxed{X_3 = -X_1} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$



M stays inert

#3 $M\omega^2 = (1+M)$