Solns Midterm 2

Phys 410/F16

Phys410/F16/Hassam/Midterm 2
1 hr 40 min **1 sheet 2-sided** **Box all important results**

Problem 1 (20 points)

Find the equation of the path joining the origin (0,0) to the point (1,1) in the x-y plane that makes the integral $\int dx\,(y'^2 + y^3 y' + y'^2)$ an extremum, where $y'=dy/dx$.

y^2 typo

Problem 2 (20 points)

A new physics equation is proposed to update Newton's Equation. The new equation takes the form

$m\,(d/dt)[x^{\cdot} + \varepsilon \sinh(x^{\cdot})] = F(x)$, where $F(x) = -\,dU/dx$, and $x^{\cdot} = dx/dt$.

If this equation is to be taken seriously, one must be able to recast it into standard Euler-Lagrange form, $(d/dt)(\partial L/\partial x^{\cdot}) = (\partial L/\partial x)$, thus ensuring the existence of a variational principle. Here, $L = L(x,x^{\cdot})$. Lagrange was able to accomplish this for $\varepsilon = 0$ by introducing the idea that x could be treated as an independent variable thus allowing derivatives of the form $(\partial f/\partial x^{\cdot})_x$ to replace any functions of $x^{\cdot}$. Here, in general, $f = f(x,x^{\cdot})$.

$\dot{x}$
typo

Using Lagrange's approach, investigate if you can recast the above equation with constant ε into E-L form. If so, define the new $L(x,x^{\cdot})$. For partial credit, you may present the $\varepsilon = 0$ formulation.

Problem 3 (30 points)

A particle is constrained to move on the surface of a circular cone. The cone axis is the vertical z-axis, the vertex is at the origin (pointing down), and the half angle is α. There is gravity g pointing down. Mass = m.

1. Write down the Lagrangian, L, in terms of spherical polar coordinates (r,ϕ).

2. Find two equations of motion.

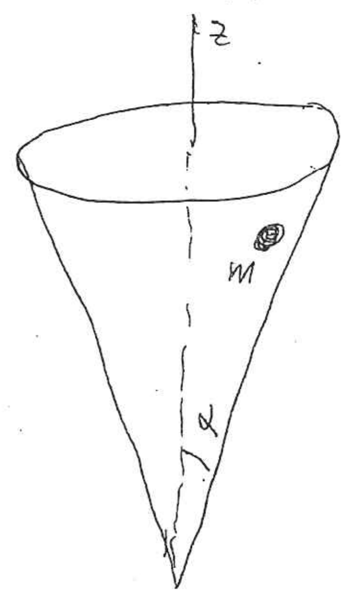
3. Interpret the ϕ equation in terms of the angular momentum λ, and use it to find an ODE in terms of only r(t).

4. State how this equation makes sense if the angular momentum = 0.

5. Find a value for r, r_0, in terms of λ, m, g, α, such that the particle can remain on a horizontal and circular path. What is the frequency of rotation in terms of r_0 and the parameters?

6. Suppose the particle is given a small radial kick, ie, $r(t) = r_0 + \varepsilon(t)$, where $|\varepsilon| << r_0$. Is the circular motion stable to small perturbations? If so, what is the oscillation frequency normalized to the rotation frequency?

Useful: In spherical coordinates, $\mathbf{dr} = dr\ \mathbf{r}\hat{} + r\ d\theta\ \theta\hat{} + r \sin\theta\ d\phi\ \phi\hat{}$.



Problem 4 (30 points)

We want to study free fall and Coriolis deflection. We will use equations developed in Taylor's text (Eqs 9.53 and Figure 9.15, attached). A mass m is released, from rest, from a height z=h, and x=y=0. The experiment is done at the equator (corresponding to $\theta = \pi/2$).

This problem has parameters Ω, g, and h. To simplify algebra, you may set g =1 and h = 1, corresponding to distance normalized to h and time normalized to $(h/g)^{1/2}$. You don't have to do this, however.

1. Write down Eqs 9.53 as they apply at the Equator.

2. Using the initial conditions supplied, find y(t). Also, by performing an integration, find [dz/dt](t) in terms of x(t).

3. Insert the above results into the equation for d^2x/dt^2. Specify at least 3 characteristics of the resulting equation which point you to the methods to be used to solve this ODE.

4. Solve completely for x(t) given the boundary conditions. This will be an exact solution.

5. We now assume that the free-fall time from height h is given by the inertial frame result, ie, $\tau^2 = 2h/g$. From 4., find the deflection in x(t), d, as evaluated according to d = $x(\tau)$. Your answer should be a two term expression for d given in terms of Ω, g, h, and τ.

6. Identify in your expression for d the dimensionless parameter $\varepsilon = \Omega\tau$, proportional to $h\Omega^2/g$. Assuming $\varepsilon << 1$, make an approximation in d to get a nonzero expression for (d/h) which is proportional to ε^α. What is the exponent α?

Maybe useful: $\cos(x) = 1 - x^2/2! + \ldots$ $\sin(x) = x - x^3/3! + \ldots$

<u>In general</u>

$\frac{d}{dx}\sinh x = \cosh x$

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where, as before, $m\mathbf{g}_o$ denotes the true force of the earth's gravity on the object. The centrifugal force is $m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$, (where $\mathbf{r}$ is the object's position relative to the center of the earth), but to an outstanding approximation we can replace $\mathbf{r}$ by $\mathbf{R}$ (the position on the earth's surface where the experiment is being conducted). Thus,

$$\mathbf{F}_{cf} = m(\boldsymbol{\Omega} \times \mathbf{R}) \times \boldsymbol{\Omega}.$$

Returning to the equation of motion (9.50), you will recognize that the sum of the first two terms on the right is just $m\mathbf{g}$, where $\mathbf{g}$ is the observed free-fall acceleration for an object released from rest at position $\mathbf{R}$, as introduced in (9.44). In other words, we can omit the term $\mathbf{F}_{cf}$ from (9.50), if we replace $\mathbf{g}_o$ by the observed $\mathbf{g}$ at the location of our experiment. If we substitute $2m\mathbf{v} \times \boldsymbol{\Omega}$ for $\mathbf{F}_{cor}$, the equation of motion becomes (after cancellation of a factor of m)

$$\ddot{\mathbf{r}} = \mathbf{g} + 2\dot{\mathbf{r}} \times \boldsymbol{\Omega}. \tag{9.51}$$

A simplifying feature of the equation (9.51) is that it does not involve the position $\mathbf{r}$ at all (only its derivatives $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$). This means the equation will not change if we make a change of origin (since a change of origin amounts to adding a constant to $\mathbf{r}$). Accordingly, I shall now choose my origin on the surface of the earth at the position $\mathbf{R}$, as shown in Figure 9.15. With this choice of axes, we can resolve the equation of motion into its three components. The components of $\dot{\mathbf{r}}$ and $\boldsymbol{\Omega}$ are

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$$

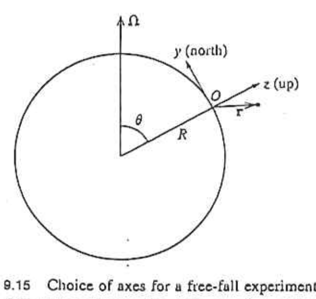


Figure 9.15 Choice of axes for a free-fall experiment. The origin O is on the earth's surface at the experiment's location (position $\mathbf{R}$ relative to the center of the earth). The z axis points vertically up (more precisely, in the direction of $-\mathbf{g}$, where $\mathbf{g}$ is the observed free-fall acceleration), the x and y axes are horizontal (that is, perpendicular to $\mathbf{g}$), with y pointing north, and x due east. The position of the falling object relative to O is $\mathbf{r}$.

and

$$\boldsymbol{\Omega} = (0, \Omega \sin\theta, \Omega \cos\theta).$$

Thus, those of $\dot{\mathbf{r}} \times \boldsymbol{\Omega}$ are

$$\dot{\mathbf{r}} \times \boldsymbol{\Omega} = (\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta,\ -\dot{x}\Omega\cos\theta,\ \dot{x}\Omega\sin\theta) \tag{9.52}$$

and the equation of motion (9.51) resolves into the following three equations:

$$\begin{aligned} \ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \\ \ddot{y} &= -2\Omega\dot{x}\cos\theta \\ \ddot{z} &= -g + 2\Omega\dot{x}\sin\theta. \end{aligned} \tag{9.53}$$

We shall solve these three equations by making a succession of approximations that depend on the smallness of Ω. First, because Ω is very small, we get a reasonable starting approximation if we ignore Ω entirely. In this approximation, the equations reduce to

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \text{and} \qquad \ddot{z} = -g, \tag{9.54}$$

which are the equations of free fall solved in every introductory physics course. If the object is dropped from rest at $x = y = 0$ and $z = h$, then the first two equations imply that $\dot{x}$, $\dot{y}$, x, and y all remain zero, while the last equation implies that $\dot{z} = -gt$ and $z = h - \frac{1}{2}gt^2$. Thus our approximate solution is

$$x = 0, \qquad y = 0, \qquad \text{and} \qquad z = h - \tfrac{1}{2}gt^2, \tag{9.55}$$

that is, the object falls vertically down with constant acceleration g. This approximation is sometimes called the *zeroth-order* approximation because it involves only the zeroth power of Ω (that is, it is independent of Ω). It is well known to be a very good approximation, but it shows none of the effects of the Coriolis force.

To get the next approximation, we argue as follows: The terms in (9.53) that involve Ω are all small. Thus, it will be safe to evaluate these terms using our zeroth-order approximation for x, y, and z. Substituting (9.55) into the right side of (9.53), we get

$$\ddot{x} = 2\Omega g t \sin\theta, \qquad \ddot{y} = 0, \qquad \text{and} \qquad \ddot{z} = -g. \tag{9.56}$$

The last two of these are exactly the same as in zeroth order, but the equation for x is new and is easily integrated twice to give

$$x = \tfrac{1}{3}\Omega g t^3 \sin\theta, \tag{9.57}$$

with y and z the same as in the zeroth approximation (9.55). This result is naturally called the *first-order* approximation (being good through the first power of Ω). We can repeat this process again to get a second-order approximation and so on, but the first-order is good enough for our purposes.

The striking thing about the solution (9.57) is that a freely falling object does not fall straight down. Instead the Coriolis force causes it to curve slightly to the east (positive x direction). To get an idea of the magnitude of the effect, consider an object

Minimize

$$S = \int_0^1 dx\, f(y, y')$$

$$f = y'^2 + y^3 y' + y^2$$

$\delta S = 0 \Rightarrow$

$$f_y = 3y^2 y' + 2y$$

$$f_{y'} = 2y' + y^3 \xrightarrow{d/dx} 2y'' + 3y^2 y'$$

$$\Leftrightarrow (d/dx) f_{y'} = f_y$$

$$\Rightarrow 2y'' = 2y$$

$$\boxed{y = \frac{\sinh x}{\sinh 1}}$$

$y(0) = 0$
$y(1) = 1$

E-L form

$$\frac{d}{dt}\left(\dot{x} + \varepsilon \sinh(\dot{x})\right) = -\frac{dU}{dx}$$

for $\varepsilon = 0$, try $\dot{x} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\dot{x}^2\right)$

for $\varepsilon \neq 0$, try $\dot{x} + \varepsilon \sinh(\dot{x})$

$$= \frac{\partial}{\partial \dot{x}}\left[\frac{\dot{x}^2}{2} + \varepsilon \cosh(\dot{x})\right]_x$$

Let $\boxed{\mathcal{L} = \frac{1}{2}\dot{x}^2 + \varepsilon \cosh \dot{x} - U(x)}$

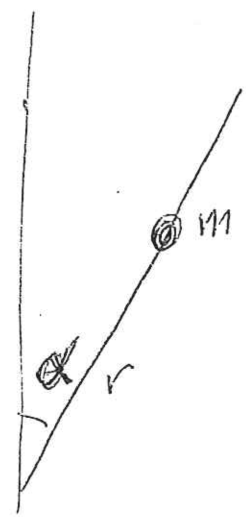
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} + \varepsilon \sinh \dot{x}, \quad \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \left(\frac{\partial \mathcal{L}}{\partial x}\right)}$$

$\therefore$ E-L form

Cone

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$



$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{r} + r\sin\alpha\,\dot{\phi}\,\hat{\phi}$$

for $\dot{\theta} = 0$

$$\boxed{\begin{aligned} T &= \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2} m r^2 \sin^2\alpha\,\dot{\phi}^2 \\ U &= mgz = mgr\cos\alpha \end{aligned}}$$

① $\boxed{\mathcal{L} = T - U}$

② $\dfrac{\partial \mathcal{L}}{\partial r} = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha$

$\dfrac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$ $\quad\boxed{\Rightarrow m\ddot{r} = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha}$

$\dfrac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\sin^2\alpha\,\dot{\phi}$

$\Rightarrow \boxed{mr^2\sin^2\alpha\,\dot{\phi} = l = \text{const}}$

(4) $\underline{\ell = 0} \Rightarrow \ddot{r} = -g\cos\alpha$

gravity, inclined plane

<u>circular motion</u>

(3) $$\ddot{r} = \frac{\ell^2 \sin^2\alpha}{m^2 r^3 \sin^4\alpha} - g\cos\alpha\ , \quad r(t)$$

(5) $$\ddot{r} = 0 \Rightarrow \boxed{\frac{\ell^2}{g\sin^2\alpha\cos\alpha\, m^2} = r_0^3}$$ circl motion

$$\Omega = \dot{\phi}$$

$$\Rightarrow \boxed{\Omega = \frac{\ell}{m r_0^2 \sin^2\alpha}}$$

(6) <u>small oscillations</u> $$\ddot{\tilde{r}} = \frac{-3\ell^2 \tilde{r}}{r_0^4 \sin^2\alpha}$$ stable

$$\Rightarrow \omega^2 = \frac{3\ell^2}{r_0^4 \sin^2\alpha}$$

$$\frac{\omega^2}{\Omega^2} = \frac{3\cancel{\ell^2}}{\cancel{r_0^4}\sin^2\alpha}\,\frac{m^2 \cancel{r_0^4}\, \overset{\sin^2\alpha}{\cancel{\sin^4\alpha}}}{\cancel{\ell^2}}$$

$$\left(\frac{\omega}{\Omega}\right)^2 = 3\sin^2\alpha \qquad \boxed{\frac{\omega}{\Omega} = \sqrt{3}\sin\alpha}$$

Coriolis

(1)
$$\boxed{\begin{aligned}\ddot{x} &= -(2\Omega)\dot{z}\\ \ddot{z} &= (2\Omega)\dot{x} - g\\ \ddot{y} &= 0\end{aligned}}$$

$z(0)=h$
$\vec{v}(0)=0$
$x(0)=0$
$y(0)=0$

(2)
$$\boxed{\begin{aligned}y &= 0\\ \dot{z} &= (2\Omega)x - gt\end{aligned}}$$

(3) $\Rightarrow$ $\boxed{\ddot{x} = -(2\Omega)^2 x + (2\Omega)gt}$

linear, 2nd order, inhomogeneous

(4) $x_p = \dfrac{gt}{2\Omega}$

$\Rightarrow x = \dfrac{gt}{2\Omega} - A\sin(2\Omega t)$; $x(0)=0$

$\Rightarrow \dot{x} = \dfrac{g}{2\Omega} - 2\Omega A\cos(2\Omega t)$

$\Rightarrow 0 = \dfrac{g}{2\Omega} - 2\Omega A \quad \Rightarrow A$

$$\Rightarrow \boxed{x(t) = \frac{g}{(2\Omega)^2}\left[t\,2\Omega - \sin(2\Omega t)\right]}$$

⑤ ~~let~~ $t = \tau, \quad \tau^2 = 2h/g$

$$\Rightarrow (2\Omega\tau)^2 = (2\Omega)^2 \frac{2h}{g} \ll 1$$

$$\boxed{d = x(\tau) = \frac{g}{(2\Omega)^2}\left[2\Omega\tau - \sin(2\Omega\tau)\right]}$$

$$(2\Omega\tau) \equiv 2\varepsilon$$

⑥ $\varepsilon \ll 1 \quad d \simeq \frac{g}{(2\Omega)^2}\left[2\varepsilon - 2\varepsilon + \frac{\varepsilon^3 8}{6}\right]$

$$\frac{d}{h} \simeq \frac{g\tau^2}{6h}\varepsilon 8 = \frac{2}{6}\varepsilon -$$

$$\Rightarrow \boxed{\frac{d}{h} \simeq \frac{2\varepsilon}{3}} \qquad \boxed{\varepsilon^2 \equiv 2h\Omega^2/g \ll 1}$$

$$\frac{2h}{g}\frac{g\tau^2}{4\Omega^2\tau^2}\frac{\varepsilon^3 8}{h\,3} \to \frac{2}{3}\varepsilon$$