Solutions

Phys410/F16/Hassam/Midterm 1
1 hr 30 min 1 sheet 2-sided

Box all important results

Problem 1 (20 points)

A projectile mass m is shot from the origin at speed v_0, at an angle α with respect to the horizontal x-axis. There is gravity downward. There is also a funny friction – the atmosphere is anisotropic so that the friction acts only in the x-direction. In particular, if v_x is positive, the friction force is $-bv_x$; but there is no friction in the y-direction. The projectile is shot from a cliff so it does not encounter ground for a long time. g is in the negative y direction.

(a) From differential equations, find explicit solutions for x(t) and y(t).

(b) Find the x-range of the projectile, defined as the x-distance traversed as $t \to \infty$. Express the answer in terms of v_0, α, m, b, g.

Problem 2 (30 points)

There are 2 masses m_1 and m_2 which attract each other by a central force, ie, the attractive force is along the distance vector joining the masses. The forces are equal and opposite, ie, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Let the mass positions be given by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. The angular momentum of m_1 is $\mathbf{L}_1 = m_1 \mathbf{r}_1 \times \mathbf{v}_1$, $\mathbf{v}_1 = d\mathbf{r}_1/dt$. Likewise, for m_2. The total angular momentum of the system is $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$.

(a) Prove whether or not **L** is a constant of the motion. Your proof must start with the line

dL/dt = ...

and must show all the important steps. (Can be done in 6-10 steps). The proof depends on two crucial steps (apart from Newton's Laws and action = reaction), one of which is mathematical and the other one mathematical and dependent on the properties of the interparticle force. You must clearly highlight these 2 steps.

(b) A grad student experiments theoretically with the forces. Instead of $\mathbf{F}_{12}$ as defined above, the student assumes that $\mathbf{F}_{12} = b\, \mathbf{r}_2 |\mathbf{r}_1 - \mathbf{r}_2|$, ie, the force on 1 due to 2 is proportional to the position vector of 2 and proportional to the distance between 1 and 2. (b is a constant.) In a symmetric manner, $\mathbf{F}_{21} = b\, \mathbf{r}_1 |\mathbf{r}_2 - \mathbf{r}_1|$.

1. (Yes/No only) Is the new force a central force (in general, as conventionally defined)?
2. (Yes/No only) Is the new interparticle force equal and opposite?
3. Prove whether or not the total angular momentum a constant of the motion?
(you do not have to repeat the entire proof as long as you can demonstrate a key step or two).

Problem 3 (30 points)

A force field is given as $\mathbf{F} = (-x,y)$, in 2D Cartesian coordinates. A particle mass m=1 is placed at the origin and given a kick so that its initial velocity (at t=0) is v_0 in the x-direction, ie, $\mathbf{v}(0) = (v_0,0)$.

(a) From differential equations, by explicit calculation starting from Newton's Laws $m\mathbf{a} = \mathbf{F}$, find x(t) and y(t).

(b) At some time T, the mass comes to a momentary stop. What is the earliest T? At what (x,y) location does this happen?

(c) We can explore some facets of the above problem by using Energy as a constant of the motion. Show which property of **F** allows you to do this. Given this fact, obtain the potential energy U(x,y). Write down the expression for the Energy constant in terms of U, etc, in general Cartesian coordinates.

(d) For the explicit solution found in (a), show that the energy at t=0 is equal to the energy at t=T, at the location where the mass comes to a momentary stop.

(e) Make a rough sketch of U(x,y) contours. (Use this to confirm for yourself your findings above.)

Problem 4 (20 points)

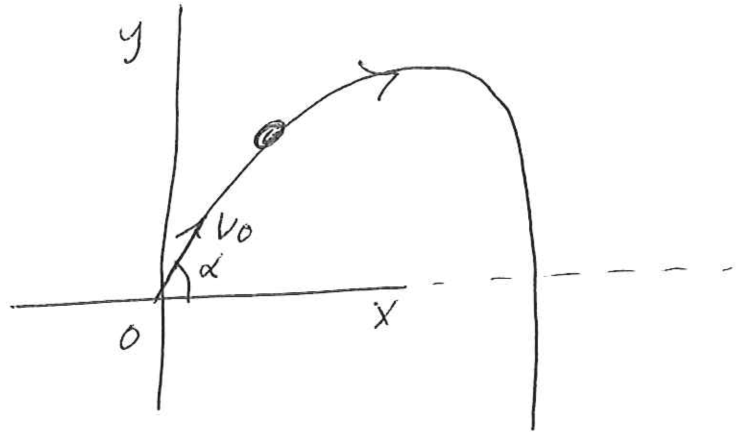
A particle mass m moving along the x-coordinate is subject to a force
$F(x) = -F_0 \sinh(\alpha x)$, α = constant.

(a) Find the equilibrium position of the particle.

(b) Show that the equation satisfied by x(t) for *small* oscillations of this mass about its equilibrium position is of the undamped harmonic oscillator type with some angular frequency ω_0. Define ω_0 in terms of F_0, m, α.

(c) The mass is subjected to an external driving force $f(t) = f_0 \cos(\omega t)$. Find an expression for the amplitude of small oscillations in response to the driving force if $\omega^2 = 0.99\,\omega_0^2$. Compare this with the amplitude of oscillations for when $\omega << \omega_0$, or, effectively, when $\omega = 0$. Assume friction is negligible.

Projectile



$$m\ddot{x} = -b\dot{x}$$

$$m\ddot{y} = -mg$$

$$\boxed{\begin{aligned}\ddot{x} &= -k\dot{x}\\ \ddot{y} &= -g\end{aligned}} \qquad k = b/m$$

$$\dot{x} = -kx + C \Rightarrow \dot{x} = -kx + v_0\cos\alpha$$

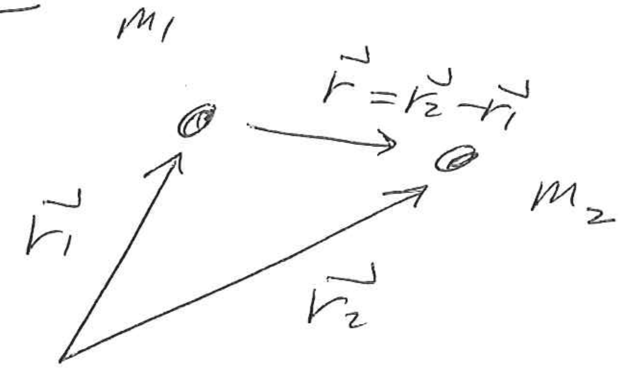
$$x_h = Ae^{-kt}, \quad x_p = \frac{v_0\cos\alpha}{k}$$

(a) $$\Rightarrow \boxed{x(t) = \frac{v_0\cos\alpha}{k}\left(1 - e^{-kt}\right)}$$

$$\dot{y} = v_0\sin\alpha - gt \qquad \boxed{y(t) = v_0\sin\alpha\, t - \frac{gt^2}{2}}$$

(b) $$\underline{t\to\infty} \quad \boxed{x(t) \longrightarrow \frac{v_0\cos\alpha}{k} \equiv \text{Range}}$$

Angular Momentum



$$\vec{L} = \sum_k m_k \vec{r_k} \times \vec{v_k} \,, \quad \vec{v_k} = \dot{\vec{r_k}}$$

$$(a) \quad \dot{\vec{L}} = \sum_k m_k \Big(\underbrace{\dot{\vec{r_k}} \times \dot{\vec{r_k}}}_{=0} + \vec{r_k} \times \ddot{\vec{r_k}} \Big) \leftarrow (\vec{r_k} \times \dot{\vec{r_k}})^{\cdot} = \text{product rule}$$

$$= \sum_k m_k \vec{r_k} \times (\vec{F_{k\ell}} / m_k)$$

$$= \vec{r_1} \times \vec{F_{12}} + \vec{r_2} \times \vec{F_{21}}$$

$$= \vec{r_1} \times \vec{F_{12}} - \vec{r_2} \times \vec{F_{12}}$$

$$= (\vec{r_1} - \vec{r_2}) \times \vec{F_{12}} \quad \leftarrow \text{Forces equal opposite}$$

$$= 0 \quad \leftarrow \text{because } \vec{r_1} - \vec{r_2} \parallel \vec{r} \parallel \vec{F_{12}}$$

central force

(b) $\vec{F}_{12} = b\vec{r}_2 |\vec{r}_1 - \vec{r}_2|$

$\vec{F}_{21} = b\vec{r}_1 |\vec{r}_1 - \vec{r}_2|$

(b1) No $\vec{F}_{12} \parallel \vec{r}_2$. Central $\vec{F}_{12}$ must be parallel to $(\vec{r}_1 - \vec{r}_2)$.

$\vec{r}_2 \times (\vec{r}_1 - \vec{r}_2) = \vec{r}_2 \times \vec{r}_1 \neq 0$ in general

(b2) No if $A = R$, $\vec{F}_{12} + \vec{F}_{21} = 0$

$\vec{F}_{12} + \vec{F}_{21} = b|r_1 - r_2| (\vec{r}_2 + \vec{r}_1) \neq 0$ in general

(b3) Based on the proof above,

Consider $\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$

$= (\vec{r}_1 \times \vec{r}_2) b|\,| + (\vec{r}_2 \times \vec{r}_1) b|\,|$

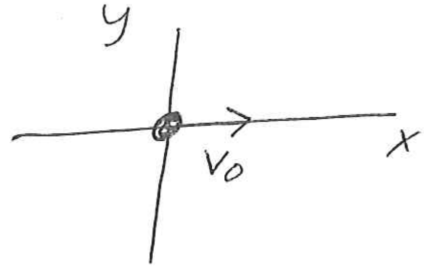
$= b|\vec{r}_1 - \vec{r}_2| (\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_1)$

$= 0$

$\therefore \boxed{\dot{\vec{L}} = 0}$ Yes

Fields

$\vec{F} = (-x, y)$



(a) $\ddot{x} = -x, \quad \ddot{y} = y$

$\Rightarrow \boxed{x(t) = V_0 \sin t}$ since $x(0) = 0$, $\dot{x}(0) = V_0$

$y \sim \left\{ \begin{matrix} e^{t} \\ e^{-t} \end{matrix} \right\} = Ae^{t} + Be^{-t}$

$y(0) = 0, \; \dot{y}(0) = 0$

$\Rightarrow \boxed{y(t) = 0}$

(b) $\dot{x}(t) = V_0 \cos t$

$\dot{x}(T) = 0 \Rightarrow \cos T = 0 \Rightarrow \boxed{T = \pi/2}$

and $\boxed{x(T) = V_0}$ stopping distance

(c) $\vec{\nabla} \times \vec{F} = \partial_x y - \partial_y(-x) = 0 \Rightarrow \vec{F} = -\vec{\nabla} U$

$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x,y) = \text{const}}$

$-dU = \vec{F} \cdot d\vec{r} = -x\,dx + y\,dy = d\left(-\frac{x^2}{2} + \frac{y^2}{2}\right)$

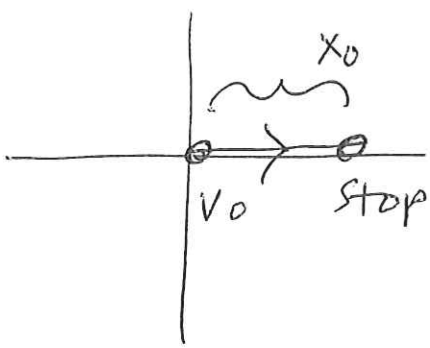
$\Rightarrow \boxed{U = \frac{1}{2}(x^2 - y^2)}$

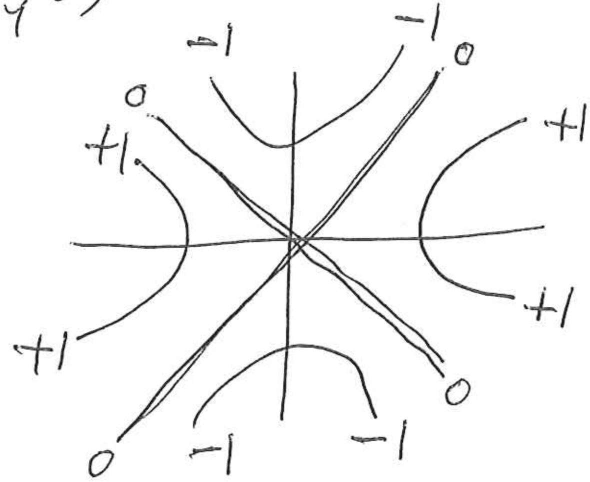
(d) $\mathcal{E}(t=0) = \frac{1}{2}v_0^2 + U(0,0) = \frac{1}{2}v_0^2$

$\mathcal{E}(t=T) = U(v_0,0) = \frac{1}{2}v_0^2$

<u>equal</u>

$U = \frac{1}{2}(x^2 - y^2)$





Mass goes "uphill" & momentarily stops cleanly, $\frac{1}{2}mv_0^2 = U(x_0, y=0)$

$= \frac{1}{2}x_0^2$

$\Rightarrow x_0 = v_0$

Oscillations

$$m\ddot{x} = F(x) = -F_0 \sinh(\alpha x)$$

(a) $\ddot{x} = 0 \Rightarrow \sinh(\alpha x) = 0 \Rightarrow \boxed{x=0}$

(b) Let x be small, and assume stays small w.r.t. $x = 0$

$$\Rightarrow \sinh(\alpha x) \simeq \sinh(0) + \alpha \cosh(0)\, x$$
$$= \alpha x$$

$$\Rightarrow m\ddot{x} \simeq -F_0 \alpha x$$

$\Rightarrow$ oscillations at $\boxed{\omega_0 = \sqrt{\frac{F_0 \alpha}{m}}}$

(c) $m\ddot{x} = -F_0 \alpha x + f_0 \cos(\omega t)$

$$\ddot{x} = -\omega_0^2 x + \frac{f_0}{m}\cos(\omega t)$$

Try $x(t) = A\cos(\omega t)$

$$\Rightarrow (\omega_0^2 - \omega^2) A = \frac{f_0}{m} \Rightarrow A(\omega)$$

$\omega^2 = 0.99\,\omega_0^2 \Rightarrow A = \frac{f_0/m}{\omega_0^2}\, 100 \quad \Rightarrow \times 100$

$\omega^2 \to 0 \quad \Rightarrow \quad A = \frac{f_0/m}{\omega_0^2}$