Final Take Home Solns

Phys410/F16

Modes

let $k=1$, $m=1$

$\kappa = \sqrt{2}$

$$U = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + \kappa(x_1+x_3)x_2\right)$$

$$T = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right)$$

$$\partial_1 U = x_1 + \frac{\kappa}{2}x_2$$

$$\partial_3 U = x_3 + \frac{\kappa}{2}x_2$$

$$\partial_2 U = x_2 + \frac{\kappa}{2}(x_1+x_3)$$

$$\Rightarrow \quad \ddot{x}_1 = -x_1 - \frac{\kappa}{2}x_2 \quad (1)$$

$$\ddot{x}_3 = -x_3 - \frac{\kappa}{2}x_2 \quad (2)$$

Note symmetry

$$\ddot{x}_2 = -x_2 - \frac{\kappa}{2}(x_1+x_3) \quad (3)$$

Try $e^{i\omega t}$ $\Rightarrow$

$\cancel{(1-\omega^2)}$

$$\boxed{\begin{aligned} (\omega^2-1)\,x_1 &= \frac{\kappa}{2}x_2 \quad (4)\\ (\omega^2-1)\,x_3 &= \frac{\kappa}{2}x_2 \quad (5)\\ (\omega^2-1)\,x_2 &= \frac{\kappa}{2}(x_1+x_3) \quad (6)\end{aligned}}$$

If $\omega^2 \neq 1$, plug in for X_1, X_3 using (4), (5) in (6)

$$\Rightarrow (\omega^2-1) X_2 = \left(\frac{K}{2}\right)^2 (1+1) \frac{X_2}{(\omega^2-1)}$$

$$\Rightarrow (\omega^2-1)^2 = \frac{2K^2}{4} = 1$$

$$\Rightarrow (\omega^2-1) = \pm 1$$

$$\Rightarrow \boxed{\omega^2 = 0}, \boxed{\omega^2 = 2}$$

$\checkmark$ $\boxed{\omega^2=0} \Rightarrow X_1 = -\frac{K}{2} X_2, \quad X_3 = -\frac{K}{2} X_2$

$$\Rightarrow \boxed{(1, -\sqrt{2}, 1)} \checkmark$$

$\checkmark$ $\boxed{\omega^2=2} \Rightarrow X_1 = \frac{K}{2} X_2, \quad X_3 = \frac{K}{2} X_2$

$$\Rightarrow \boxed{(1, \sqrt{2}, 1)} \checkmark$$

Suppose $\checkmark$ $\boxed{\omega^2=1} \Leftrightarrow X_2 = 0, \; X_1 = -X_3$

$$\Rightarrow \boxed{(1, 0, -1)} \checkmark$$

Note $\omega^2 = 0 \Rightarrow$ RHS of (1)-(3) $= 0$

$\Rightarrow$ No Forces on any mass $\Rightarrow$ No accels.

$\Rightarrow$ zero frequency

Diagonal $\quad m=1 \quad b=1$

$m_1 = m_2 = m_3 = 1 \qquad (1,0,1)$

$(1,1,-1)$

$(-1,1,0)$

$$\sum mxy = (10) + (11) + (-11) = 0$$

$$\sum mxz = (11) + (1-1) + (-10) = 0$$

$$\sum yz = (01) + (1-1) + (10) = -1$$

$$\sum m(y^2+z^2) = (0^2+1^2) + (1^2+1^2) + (1^2+0^2) = 4$$

$$\sum m(x^2+z^2) = (1^2+1^2) + (1^2+1^2) + (1^2+0^2) = 5$$

$$\sum m(x^2+y^2) = (1^2+0^2) + (1^2+1^2) + (1^2+1^2) = 5$$

$$\Rightarrow \quad I = mb^2 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & +1 \\ 0 & +1 & 5 \end{pmatrix}$$

Diagonal if $\quad \bar{4}\left[\bar{5}\bar{5} - 1\cdot 1\right] = 0 \quad \Rightarrow \bar{4} = 0, \ \bar{5} = 1, \ \bar{5} = -1$

$\Rightarrow \lambda = 4, \ \lambda = 4, \ \lambda = 6$

$\underline{\lambda = 6}$ $\Rightarrow$ $4\omega_1 = 0$, $\omega_2 = \omega_3$ $\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = 4}$ $\omega_2 = -\omega_3$ $\Rightarrow \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$ ortho

a arbitrary.

To make orthogonal, want $\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} b \\ 1 \\ -1 \end{pmatrix} = 0$

$$\Rightarrow ab + 1 + 1 = 0 \Rightarrow ab = -2$$

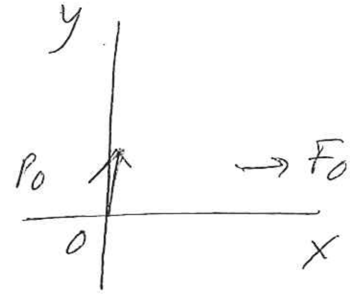
$a = 1$, $b = -2$ $\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix},$ AND $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

P axes ?

15.84 Special Rel

$P_0 = \gamma_0 m v_{y0}$

$v_{x0} = 0$



$$\frac{d}{dt}(\gamma m \vec{v}) = F_0 \hat{x} \qquad \gamma^2 = (1 - \vec{v}\cdot\vec{v})$$

Components

$$d_t(\gamma m v_x) = F_0 \Rightarrow \gamma m v_x = F_0 t$$

$$d_t(\gamma m v_y) = 0 \Rightarrow \gamma m v_y = \gamma_0 m v_{y0}$$

$$\Rightarrow \boxed{\gamma m \vec{v} = \begin{pmatrix} F_0 t \\ \gamma_0 m v_{y0} \end{pmatrix}} \quad (1)$$

Calculate $\vec{v}\cdot\vec{v}$ $\Rightarrow m^2 \gamma^2 v^2 = (F_0 t)^2 + \gamma_0^2 m^2 v_{y0}^2$

use $\gamma^2 v^2 = \gamma^2 - 1$

$$\Rightarrow \gamma^2 - 1 = \left(\frac{F_0}{m} t\right)^2 + \gamma_0^2 v_{y0}^2 \quad (2)$$

$$\Rightarrow \boxed{\gamma^2 = 1 + \left(\frac{F_0}{m} t\right)^2 + \gamma_0^2 v_{y0}^2 / c^2} \Rightarrow \gamma(t)$$

using in (1) $\Rightarrow \vec{v}(t)$

use $\gamma_0^2 = 1 + (\gamma_0 v_{y0})^2$

$$\Rightarrow \boxed{\gamma^2 = \gamma_0^2 + \left(\frac{F_0}{m} t\right)^2}$$

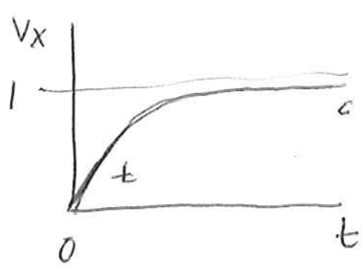
$$\Rightarrow \boxed{\vec{v}(t) = \frac{1}{\gamma}\begin{pmatrix} F_0 t/m \\ \sqrt{\gamma_0^2 - 1} \end{pmatrix}} \qquad \gamma(t)$$

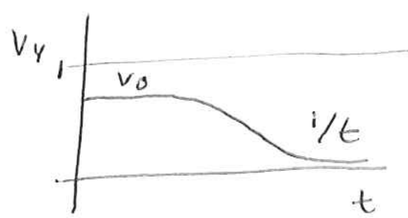
<u>given</u> $\frac{F_0}{m}$, v_{0y}^2,

$$\text{OR} \quad \boxed{\vec{v}(t) = \frac{1}{\gamma}\begin{pmatrix} F_0 t/m \\ \gamma_0 v_0 \end{pmatrix}}$$

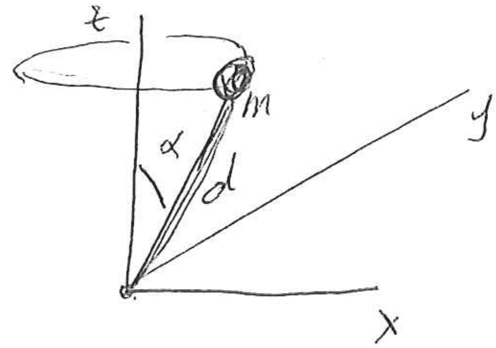
<u>$t \to 0$</u> $\quad \gamma \to \gamma_0 \quad \vec{v}(t) \to \begin{pmatrix} \frac{F_0}{m} t/\gamma_0 \\ v_0 \end{pmatrix}$

<u>$t \to \infty$</u> $\quad \gamma \to \left(\frac{F_0}{m}\right) t \quad \vec{v}(t) \to \begin{pmatrix} 1 \\ \gamma_0 v_0 / \left(\frac{F_0}{m}\right) t \end{pmatrix}$





Torque



(1) $\frac{d\vec{r}}{dt} = \vec{w} \times \vec{r}$

$\vec{r} = (x, y, z)$

$\vec{w} = \hat{z}\, w_0$

$$\Rightarrow \begin{aligned} \dot{x} &= -w_0 y \\ \dot{y} &= w_0 x \\ \dot{z} &= 0 \end{aligned} \qquad \begin{aligned} &\Rightarrow \ddot{x} = -w_0^2 x \\ &\Rightarrow x \sim \begin{Bmatrix} \cos w_0 t \\ \sin w_0 t \end{Bmatrix} \end{aligned}$$

$y(0) = 0$

$x(0) = d \sin\alpha$

$$\Rightarrow \boxed{\begin{aligned} x &= d \sin\alpha \cos(w_0 t) \\ y &= d \sin\alpha \sin(w_0 t) \end{aligned}}$$

$$\boxed{z = d\cos\alpha}$$

(2) $d\vec{L}/dt = \vec{\Gamma}$

$\vec{\omega} = \omega_0 \hat{z}$

$\vec{L} = \overset{\leftrightarrow}{I}_0 \vec{\omega}$

$$I = m\begin{pmatrix} y^2+z^2 & -xy & -xz \\ & x^2+z^2 & -yz \\ \text{sym} & & x^2+y^2 \end{pmatrix}$$

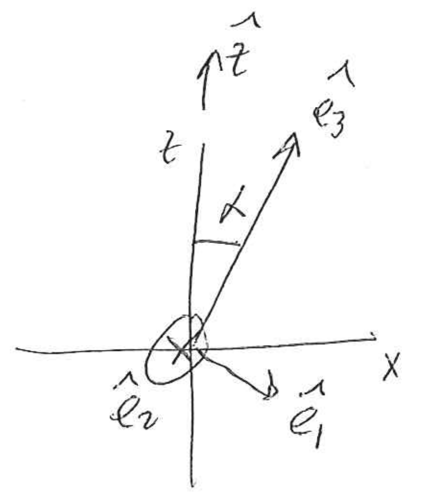
$$\overset{\leftrightarrow}{I} \cdot \vec{\omega} = m(-xz, -yz, x^2+y^2)\omega_0$$

$$= md^2\omega_0(-\sin\alpha\cos\alpha\cos\omega_0 t, -\sin\alpha\cos\alpha\sin\omega_0 t, \sin^2\alpha)$$

$$d\vec{L}/dt = md^2\omega_0^2(\sin\alpha\cos\alpha\sin\omega_0 t, -\sin\alpha\cos\alpha\cos\omega_0 t, 0)$$

$$\Rightarrow \boxed{\vec{\Gamma} = md^2\omega_0^2\sin\alpha\cos\alpha(\underset{\hat{x}}{\sin\omega_0 t}, \underset{\hat{y}}{-\cos\omega_0 t}, \underset{\hat{z}}{0})}$$

(3)



$$\vec{\omega} = \omega_0 \hat{z}$$

$$\vec{\omega} \cdot \hat{e}_3 = \omega_0 \cos\alpha$$

$$\vec{\omega} \cdot \hat{e}_2 = 0$$

$$\vec{\omega} \cdot \hat{e}_1 = -\sin\alpha\, \omega_0$$

In bodybased, $x = 0, y = 0, z = d$

$$\Rightarrow \overset{\leftrightarrow}{I} \rightarrow m \begin{pmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{diagonal}$$

$\Rightarrow$ $\hat{e}_k$'s are P-axes

(4) Now, in body based,

$$(\overset{\leftrightarrow}{I} \cdot \vec{\omega})^{\cdot} + \vec{\omega} \times (\overset{\leftrightarrow}{I} \cdot \vec{\omega}) = \vec{\Gamma}$$

But $\dot{\overset{\leftrightarrow}{I}} = 0$ and, here, $\dot{\vec{\omega}} = 0$.

$\Rightarrow$ Euler equations apply, & become

$$-(\lambda_2 - \lambda_3)\, \omega_2 \omega_3 = \Gamma_1$$
$$-(\lambda_3 - \lambda_1)\, \omega_3 \omega_1 = \Gamma_2$$
$$-(\lambda_1 - \lambda_2)\, \omega_1 \omega_2 = \Gamma_3$$

w.r.t. $\hat{e}_k$'s

But $\omega_2 = \vec{\omega} \cdot \hat{e}_2 = 0$

and $\lambda_3 - \lambda_1 = -d^2 m$

$\Rightarrow md^2 \omega_3 \omega_1 = \Gamma_2$

$$\Rightarrow \vec{\Gamma} = \hat{e}_2 \, md^2 \omega_0^2 (-\sin\alpha) \cos\alpha$$

$$\boxed{\vec{\Gamma} = -\hat{e}_2 \, md^2 \omega_0^2 \sin\alpha \cos\alpha}$$

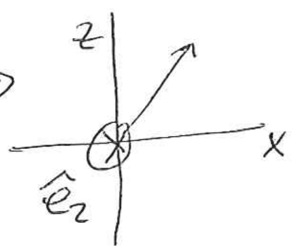
(5)

Compare to

$$\boxed{\vec{\Gamma} = md^2 \omega_0^2 \sin\alpha \cos\alpha \, (\hat{x} \sin\omega_0 t - \hat{y} \cos\omega_0 t)}$$

agree $\Leftrightarrow$ $\hat{e}_2 = -\hat{x} \sin\omega_0 t + \hat{y} \cos\omega_0 t$

$t = 0 \Rightarrow$



$\hat{e}_2 = \hat{y}$ agrees

$\hat{e}_2 = -\hat{x}$ agrees

$\omega_0 t = \pi/2 \Rightarrow$