Laws of Nature @ 1900

Newton:

$$m\frac{d\vec{v}}{dt} = \vec{F} \quad *$$

$$\frac{d\vec{x}}{dt} = \vec{v}$$

Maxwell:

(normalized units)

$$\vec{\nabla}\cdot\vec{E} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$

$$\vec{\nabla}\times\vec{B} = \partial_t \vec{E} + \vec{J}$$

ρ = charge density

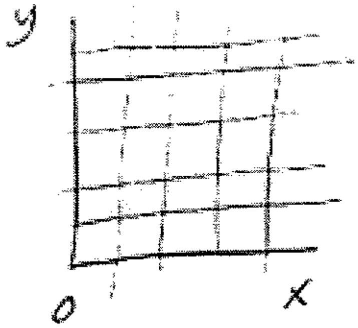
$\vec{J}$ = current density

* **Forces**

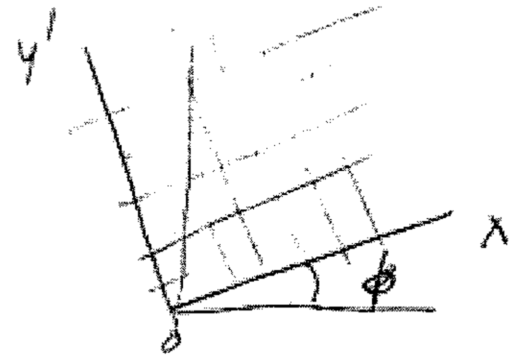
$$\vec{F} = GmM\vec{r}/r^2 \text{ , gravitational}$$

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B} \text{ , EM}$$

Coordinate systems and Rotations



Cartesian grid



rotated Cartesian grid

By geometrical methods,

$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$R \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Easy to see ($\phi \to -\phi$) that

$\boxed{\text{Clearly, } RR^T = 1}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = R^T \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{if} \begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$

Properties of Rotation Transformation, Index notation, etc.

- $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- $\Rightarrow \boxed{\begin{aligned} x'_k &= R_{kl}\, x_l \\ \Rightarrow\ x_k &= R^T_{kl}\, x'_l \end{aligned}}$

  Einstein summation convention: sum over repeated index

- $RR^T = 1 \Rightarrow R^T_{kl} R_{lm} = \delta_{km}$

  $\Rightarrow \boxed{R_{lk} R_{lm} = \delta_{km}}$

- Rotations preserve length

$$\text{length} \equiv x_1^2 + x_2^2 = x_k x_k$$

length preserved, i.e, $x_k' x_k' = x_l x_l$

Proof $\quad x_k' x_k' = R_{kl} x_l R_{km} x_m$

$$= R_{kl} R_{km} x_l x_m = \delta_{lm} x_l x_m$$

$$= x_l x_l .$$

The idea of covariance

In some coordinate frame (say x, y) (x_1, x_2) we discover the laws

$$m\mathring{\vec{v}} = \vec{F}, \quad \mathring{\vec{x}} = \vec{v}$$

where this means

$$m\mathring{v}_1 = F_1, \quad m\mathring{v}_2 = F_2 \qquad (1)$$
$$\mathring{x}_1 = v_1, \quad \mathring{x}_2 = v_2 \quad \text{etc}$$

with their specific meanings, set of operations for implementation, etc.

Since space is isotropic, observers in a rotated lab, using the coordinate (x', y') or (x_1', x_2'), ~~will~~ should get equations in terms of x_1' and x_2' that look exactly the same, ie,

$$m\mathring{v}_1' = F_1', \quad m\mathring{v}_2' = F_2' \qquad (2)$$
$$\mathring{x}_1' = v_1', \quad \mathring{x}_2' = v_2'$$

where the specific meanings of the terms and the various sets of operations for implementation are identical.

But observers in the (x_1, x_2) lab know the transformation from $(x_1, x_2) \to (x_1', x_2')$ [or they think they know it.]. Thus, they can start with the laws they found, set ①, and apply to this set the transformations

$$\bar{\bar{x}}_k = R_{\ell k} x_\ell' \quad \text{——} ③$$

and deduce the eqns that the rotated lab observers should find. Thus, they insert ③ into ①. Starting with

$$m\ddot{x}_1 = F_1, \quad m\ddot{x}_2 = F_2 \quad ④$$

They find, from (3), $\ddot{x}_k = R_{\ell k}\,\ddot{x}'_\ell$, L 6

which results in (4) becoming

$$\ddot{x}_1'\cos\phi - \ddot{x}_2'\sin\phi = F_1/m$$
$$\ddot{x}_1'\sin\phi + \ddot{x}_2'\cos\phi = F_2/m \qquad (5)$$

By appropriately adding and subtracting the 2 equations in (5), they find equations for x_1' and x_2' to be

$$\ddot{x}_1' = F_1\cos\phi + F_2\sin\phi$$
$$\ddot{x}_2' = -F_1\sin\phi + F_y\cos\phi \qquad (6)$$

They conclude from this the following:

space is isotropic $\Rightarrow$ RHS of (6) must look like $\begin{pmatrix} F_1' \\ F_2' \end{pmatrix}$ $\Rightarrow$ $\begin{pmatrix} F_1' \\ F_2' \end{pmatrix}$ are related to $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ according to

$$\begin{pmatrix} F_1' \\ F_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \qquad (7)$$

But this is just the transformation rule for x_k.

Thus, they conclude, there are entities called "vectors" that are different in each coordinate system but related ~~according~~ in the same way that the coordinate transformation is related.

That is, if

$$x_k' = R_{kl} x_l \quad (8)$$

takes us from (x_1, x_2) to (x_1', x_2'), then the forces (F_1, F_2) is a vector and is related to F_1', F_2' in the same way, ie

$$F_k' = R_{kl} F_l . \quad (9)$$

The laws of nature in the X_k frame are

$$m \ddot{x}_k = F_k , \quad (10)$$

Now that we know that F_k is a vector, i.e. it transforms the way x_k does, as above, (8) + (9), then (10) is manifestly covariant since an application of $R_{\ell k}$ from the left

$$m R_{\ell k} \ddot{x}_k = R_{\ell k} F_k$$

immediately ~~directly~~ results in the covariant equation in x_k'

$$m \ddot{x}_k' = F_k'.$$

We conclude the following:

1. if there is a symmetry of nature, equations obtained in 2 "frames" ~~places~~ of that symmetry must be covariant (ie, look the same).
2. There are things called vectors that transform from one symmetry frame to another in the same way that the coordinates transform.
3. Using 2 and demanding 1 $\Rightarrow$

laws of nature must be relations between vectors (or, more generally, tensors, as discussed later).

New definition of "vector"

Notice that, viewed as described above, vectors take on a new definition. A vector is any set of quantities that transforms according to the law of coordinate transformation.

For example, suppose (A_1, A_2) is a vector, i.e., $A'_k = R_{k\ell} A_\ell$. Then, in 2D (x_1, x_2) space, it can be shown that $(-A_2, A_1)$ is also a vector (with R as defined above). See the homework for a proof of this. However, by similar reasoning, (A_2, A_1) is not a vector.

$\vec{\nabla}$ is a vector

The gradient operator can be shown to be a vector, using the new definition.

Proof Let a function φ ~~...~~ be a function of x_k, $\varphi = \varphi(x_k)$. x_k' and x_k are related $x_k' = R_{k\ell} x_\ell$. Thus $\varphi = \varphi(x_\ell')$ also.

Now, is $\left(\frac{\partial}{\partial x_1}\varphi, \frac{\partial \varphi}{\partial x_2}\right)$ a vector?

Well, $\frac{\partial \varphi}{\partial x_1}(x_1', x_2') = \frac{\partial x_1'}{\partial x_1}\frac{\partial \varphi}{\partial x_1'} + \frac{\partial x_2'}{\partial x_2}\frac{\partial \varphi}{\partial x_2'}$

But $x_k' = R_{k\ell} x_\ell \Rightarrow \frac{\partial x_1'}{\partial x_1} = R_{11}, \frac{\partial x_2'}{\partial x_2} = R_{21}$

$$\Rightarrow \frac{\partial \varphi}{\partial x_1} = R_{11}\frac{\partial \varphi}{\partial x_1'} + R_{21}\frac{\partial \varphi}{\partial x_2'}$$

likewise,

$$\frac{\partial \varphi}{\partial x_2} = R_{12}\frac{\partial \varphi}{\partial x_1'} + R_{22}\frac{\partial \varphi}{\partial x_2'}$$

$$\Rightarrow \begin{pmatrix} \dfrac{\partial \varphi}{\partial x_1} \\ \dfrac{\partial \varphi}{\partial x_2} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \varphi}{\partial x_1'} \\ \dfrac{\partial \varphi}{\partial x_2'} \end{pmatrix}$$

But this is just, analogous to $x_k = R_{\ell k} x_\ell'$.

Thus, $\left(\dfrac{\partial \varphi}{\partial x_1}, \dfrac{\partial \varphi}{\partial x_2}\right)$ is a vector.

<u>QED</u>

Incidentally, this proof is greatly compactified by using the Einstein summation convention. Also use $\partial/\partial x_1 \to \partial_1$, or $\partial/\partial x_k \to \partial_k$.

Then, $\varphi = \varphi(x_k)$, $x_k' = R_{k\ell} x_\ell$

$\Rightarrow \partial_\ell' \varphi = \dfrac{\partial x_k}{\partial x_\ell'} \partial_k \varphi$. But $x_k = R_{\ell k} x_\ell'$

$\Rightarrow \dfrac{\partial x_k}{\partial x_\ell'} = R_{\ell k}$. $\Rightarrow \partial_\ell' \varphi = R_{\ell k} \partial_k \varphi$

$\Rightarrow \partial_\ell' = R_{\ell k} \partial_k \Rightarrow \partial_\ell$ is a vector.

It follows that laws of nature may have $\vec{\nabla}\varphi$ as part of them.

Scalars

A scalar is a quantity that does not change when the coordinates change. Thus, suppose $\varphi = \varphi(x_1, x_2)$. Using $x_k = R_{\ell k} x'_\ell$, we have

$$\varphi = \varphi(R_{11}x_1' + R_{21}x_2', R_{12}x_1' + R_{22}x_2')$$

which will give the same value.

[This is not true for vectors.]

Examples
- $\varphi = x_1$ is a scalar is the same. since $\varphi = R_{11}x_1' + R_{21}x_2'$

- The "length" of a vector $x_1^2 + x_2^2$ is a scalar since $L^2 = x_1^2 + x_2^2 = x_k x_k$

$$= R_{\ell k} x_\ell' R_{mk} x_m' = \delta_{\ell m} x_\ell' x_m' = x_m' x_m'$$
$$= x_1'^2 + x_2'^2 .$$

- A "scalar product" between 2 vectors is a scalar. Suppose A_k, B_k are vectors. $\Rightarrow A_k' B_k' = R_{k\ell} A_\ell R_{km} B_m = A_m B_m$.