Phys410/F16 Homework Problems

From *Classical Mechanics, by Taylor*

Homework to be turned in in hardcopy on due date, in class.

Problem Set 1 due 09/08/16
Ch 1: 6, 9, 17, 23, 28, 31, 36, 39
Ch 2: 5 (solve explicitly for v(t), using the method Taylor uses)
Ch 2: repeat 5, using separation of variables. Note that ln(x) is defined only for x > 0.
Use Integral[dx/x] = ln(x), and exp[ln(x)] = x.
Ch 2: 8 (use separation of variables, as done in class)
Ch 2: 27 (look up any integrals)

Problem Set 2 due 09/15/16
Ch 3: 3, 5, 11, 18, 36
Ch 4: 2, 8 (use polar coordinates), 12, 13 (a) and (c)
4.1H (a) Consider the 2D field **F** = (y,x). Is this conservative? Do line integrals along 2 paths C_1 and C_2, both from (0,0) to (1,1), where C_1 is along the line y=x, and C_2 is along the line $y=x^2$.
4.1H (b) Repeat part (a) but use **F** = (-y,x).

Problem Set 3 due 09/22/16
Ch 4: 21 (to find the potential, just try U proportional to 1/r, where r is the radius)
Ch 4: 23
Ch 4: 28 (note the kick is to the right, so pick the correct sign of dx/dt. You may do the integral as suggested by Taylor, or simply look it up)
Ch 4: 34, 36
Ch 5: 1, 2, 5, 9, 10, 23, 35
5.1H: Solve the equation for x(t) if $d^3x/dt^3 - d^2x/dt^2 + dx/dt - x = 0$ and x(0)=0, (dx/dt)(0)=-1, $(d^2x/dt^2)(0)=-2$.

Problem Set 4 due 09/29/16
5.2H: In class, we solved for the solutions of the HO equation (5.28) perturbatively, when beta is small. Notes are posted for the perturbation expansion correct to 1^{st} order. Now, continue this expansion to 2^{nd} order. Find the correction to omega and discuss how the solution is modified (does the damping rate change? Or the oscillation frequency? Both? Change how?) Compare your answer to the exact solution for omega by expanding out the exact solution to the appropriate order.
Ch 5: 42 (look up the definition of Q), 43
Ch 6: 1, 3, 9, 10, 16

Problem Set 5 due 10/06/16
Ch 6: 18 (pick a convenient point for the 1st endpoint and thus fix the constant)
Ch 7: 1, 3, 8, 10 (the Lagrangian can be constructed directly in non-Cartesian coordinates by intuiting the kinetic energy in those coordinates. However, as a general rule, it helps to start with the kinetic energy in Cartesian coordinates and then transform to the non-Cartesian by using the transformation of coordinates).

Problem Set 6 due 10/13/16
Ch 7: 15, 21, 22, 29, 36

Problem Set 7 due 10/20/16
7.1H: A mass moves in a harmonic oscillator central force $\mathbf{F} = -k\mathbf{r}$. We will investigate this motion in detail, along the lines of Chapter 8 in Taylor. The problem is attached below.

Problem Set 8 due 10/27/16
Ch 8: 1, 7(a) and (b), 10 (for "describe the motion", comment on the oscillation frequencies, in particular in the limits α is very small or very large.), 12, 13 (12 and 13 were done in class, so this is a review).

Problem Set 9 due 11/03/16
Ch 9: 2, 9, 12, 16, 17, 18 (gravity is assumed; assume also that $x(0) = x_0$, all other initial conditions zero), 25

Problem Set 10 due 11/10/16
Ch 11*: 4, 14 (this is a good problem on which to practice how to normalize; solution is pre-posted for reference), 19, 27, 32
*Algebra in this chapter is facilitated by normalizing quantities and therefore non-dimensionalizing the equations, as discussed in class and in Taylor. You are not required to do this but encouraged, for any number of problems as you wish. Problem 14 is good for practice, may want to start with this. I have pre-posted the solution.

Problem Set 11 due 11/17/16
Ch 9: 26 (note that in the lowest order, the mass has only $\mathbf{g}$ acting on it, as in the calculation of p. 353. However, there is also an initial velocity, $\mathbf{v}(0)$: the effects of the initial velocity are also to be considered as part of the lowest order solution.)

Problem Set 12 due 12/01/16

Ch 10: 2 (read about the breakdown of T between CM and relative to CM, or fixed point, Eq. 10.18 and prior), 3, 10, 13, 15, 22, 23

Problem Set 13 not graded, but material will be assumed for Final Exam

Also, subject to update: more problems will be added to this set, with notification.

Ch 10: 35, 36, 44 (to solve the final equations, use a new time coordinate defined by $d\tau = \omega_3(t)dt$, ie, $\tau(t) = \int_0^t \omega_3(t')dt'$. For simplicity, you may assume that $\omega_3(0)=0$.)

13.1H (attached, see below).

Ch 15: 79, 80 (in class, we obtained the relativistic Newton's Equation for motion along the direction of **F**. The generalization to the transverse directions can be made. The general equation is $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{p} = \gamma m\mathbf{v}$, $\gamma^2 = (1 - \mathbf{v}.\mathbf{v}/c^2)$. Use this for problem 79, and use 79 in 80. Note also that $(\gamma v)^2 = (\gamma\mathbf{v}).(\gamma\mathbf{v})$.)

<u>Problem 13.1H</u> In this problem, we review some properties associated with Principal Axes of a rigid body (RB). We first summarize the properties and then step thru the proofs.

Summary

For any RB, there exist three, orthogonal, Principal axes which have the following properties if physics is conducted w.r.t. to a coordinate system based on the P-axes: (1) the **I** tensor is diagonal; (2) if $\boldsymbol{\omega}$ // P-axes then **L** is also // $\boldsymbol{\omega}$, and vice-versa. Recall that $\mathbf{L} = \mathbf{I}.\boldsymbol{\omega}$.

To check all this, we step thru as below.

1. First, lets define P-axes as coordinate axes such that if $\boldsymbol{\omega}$ is // to a P-axis, then **L** // $\boldsymbol{\omega}$. Assume also there is a theorem which says three P-axes always exist and P-axes are orthogonal (or can be made so).

2. As per the definition, to find the P-axes, we solve the equation $\mathbf{I}.\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$ (i.e., L // $\boldsymbol{\omega}$) for all possible $\boldsymbol{\omega}^{(n)}$. We solve this in an arbitrary Cartesian system. If we can find such $\boldsymbol{\omega}^{(n)}$, we have the P-axes. The equation $\mathbf{I}.\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$ is an eigenvalue equation. There is a theorem (related to the above and more precise) which says that an *n x n* real, symmetric matrix has *n* real eigenvalues, and *n* eigenvectors which are orthogonal. Since **I** is symmetric, we are assured we will find 3 P-axes, $\boldsymbol{\omega}^{(n)}$, and associated $\lambda^{(n)}$.

3. Now assume we have found three $\boldsymbol{\omega}^{(n)}$. We now return to the RB and this time around we work in coordinates based on the P-axes. Note that in this coordinate system, the $\boldsymbol{\omega}^{(n)}$ are simply (1,0,0), (0,1,0), and (0,0,1). Also, **I** will look different. Let **I** be a general matrix with elements I_{xx}, I_{xy}, I_{xz}, etc, w.r.t. the P-axes. Since $\mathbf{I}.\boldsymbol{\omega}^{(n)} = \lambda^{(n)}\boldsymbol{\omega}^{(n)}$, show, by considering each $\boldsymbol{\omega}^{(n)}$ in turn, that **I** must be diagonal, made up from the $\lambda^{(n)}$'s, and that the $\lambda^{(n)}$'s are just the moments of inertia corresponding to the respective axes.

4. Using the fact that **I** is diagonal, prove the property mentioned in the Summary, ie, $\boldsymbol{\omega}$ // P-axes ⇔ **L** // $\boldsymbol{\omega}$. Your proof should work both ways. (The left to right proof is straightforward; the R to L one is not: make sure write down all 3 equations and solve them simultaneously.)

Homework 7.1H: **Motion in harmonic oscillator (HO) central force**

Motion in harmonic oscillator (HO) central force

Consider a single mass m in a central force field **F** = -k**r**, with respect to a fixed origin. Here **r**(t) is the radius vector, **r** = (x,y,z), in Cartesian. In this problem, we want to understand the various orbits of the mass about the HO central force.

1. Check that **F** is conservative.

2. Find the potential U(**r**).

3. It follows that the energy is conserved. Write down the expression for this. Note that **v**(t) = d**r**/dt.

4. **F** is a central force (in addition to being conservative). Therefore, show that the angular momentum **L** is a constant, where **L** = m **r** x **v**. Your proof must use 2 important vector identities.

5. If **L** is constant, prove that **r**(t) must stay on a 2-dimensional plane. State the important vector identity needed to prove this. Thus, assume that **r** = x(t)**x^** + y(t)**y^**.

6. Polar coordinates are clearly preferable. Refer to Taylor Sec 1.7 and check the following (you do not have to derive these – just collect and review them): **r**(t) = r(t)**r^**(t), **v** = **r^** dr/dt + **ϕ^** rdϕ/dt. Also, obtain the expression for d^2**r**/dt^2.

7. So, write down the 2 polar components of m**a** = **F**. From the **ϕ^** component, show that the angular moment magnitude, L, is also a constant of the motion (so far we only used the direction of **L**).

8. Eliminate in the radial equation all the ϕ terms by using the angular momentum equation from 7. Thus, find a 2nd order nonlinear ODE for r(t). This equation cannot be solved in an easy way.

9. But, we haven't used energy conservation yet. Write down the energy constant using polar coordinates (use 6 above). Again eliminate the ϕ terms using the constancy of L, from 7. Check that this energy constant contains terms only in dr/dt and r.

10. As a check on energy conservation, start with your energy constant in 9 and differentiate E in t. This should be zero, ie, dE/dt = 0. Show that this results in the 2nd order ODE for r(t) in 8 above.

11. Take a square root in the equation from 9 and thus obtain a 1st order ODE for $r(t)$. (there will be a +/- sign from the square root). Check that this equation is separable and therefore solvable in principle.

12. Define an effective potential starting from your energy equation in 9. Let $U_{eff}(r) = U(r)$ (of the central force) + angular momentum term in r. Make a sketch of $U_{eff}(r)$ vs r. Draw a horizontal line for E on this sketch. Since $(m/2)(dr/dt)^2 > 0$, observe that we must have $E > U_{eff}(r)$ for a solution to exist. Thus, show that $r(t)$ must be bounded from above and below according to $r_{min} < r(t) < r_{max}$.

13. If $U_{eff}(r)$ is at its minimum point, and E is adjusted so that $E = U_{min}$, observe that $r(t)$ must be a constant, corresponding to circular orbit. Find r_0, the radius corresponding to U_{min}. Insert this value of r in the angular momentum equation for $d\phi/dt$ found in 7 above to show that the angular frequency of the circular orbit, $d\phi/dt$, is a constant, ω_0, and given in terms of L, m, and r_0.

14. Using your knowledge of centripetal force, applied to the $-k\mathbf{r}$ force, find the frequency of circular orbit. Compare this to ω_0 found above. You will have to eliminate L from your expression using the angular momentum equation found in 7.

15. We now want to perform small oscillations of the mass *about the circular orbit*. To do this, start from the 2nd order ODE for $r(t)$ found in 8. Let $r(t) = r_0 + s(t)$. Substitute this into the ODE and Taylor expand the RHS for small s. Use the definition of r_0 to rewrite the RHS, so its only proportional to s. Thus, obtain a 2nd order ODE for $s(t)$. Solve this in general and find ω, the frequency of small oscillations about r_0. Compare this frequency with ω_0. Can you sketch the circular orbit + small oscillations added on?