Notes to accompany

Ch 4 – Part II

updated: 09/14/16

WE Thm for conservative $\vec{F}$

$\Rightarrow$ Energy conservation

- if $W = \int_A^B d\vec{r} \cdot \vec{F}$

then WE Thm $\Rightarrow$ $K = K_0 + W$

where $K = \frac{1}{2} m v^2$

convention

- if $\vec{F}$ is conservative, $\Rightarrow$ $\vec{F} = -\vec{\nabla} U$

$\Rightarrow$ $K_B = K_A - \int_A^B d\vec{r} \cdot \vec{\nabla} U$ indep of path

$\Rightarrow$ $K_B = K_A - \int_A^B dU,$ because $d\vec{r} \cdot \vec{\nabla} U \equiv dU$

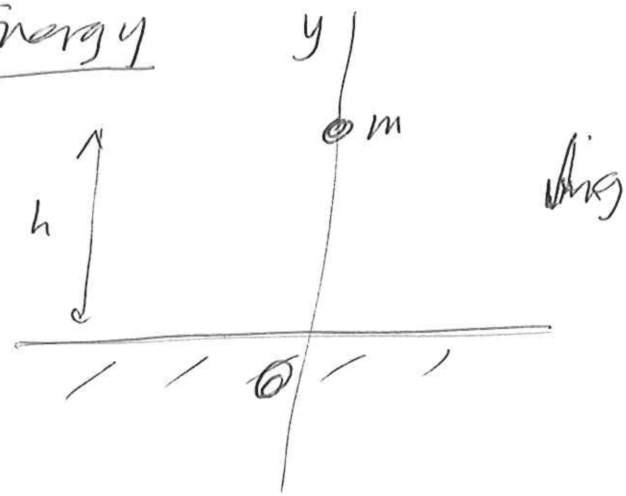
$= K_A - U_B + U_A$

$\Rightarrow$ $K_B + U_B = K_A + U_A$

$$\Rightarrow \boxed{K + U \equiv \mathcal{E} = \text{const}}$$

conservation of sum Kinetic + Potential Energies

Free fall in $\vec{g}$ by Energy



Find K at ground
if $K_h = 0$ @ h.

$$U = mgy$$

$$\Rightarrow K_h + mgh = K_0 + mg(0)$$

$$\Rightarrow \boxed{K_0 = mgh} \Rightarrow \boxed{v_0^2 = 2gh}$$

Direct Solution
(ie, not new)

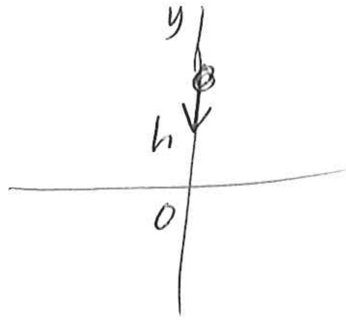
$$v_y = -gt$$
$$y = h - \frac{1}{2} g t^2$$

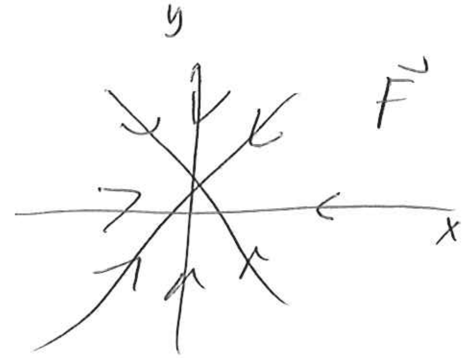
$$y = 0 \Rightarrow t^2 = 2h/g$$

$$\Rightarrow v_y(0)^2 = (gt)^2 = g^2 \frac{2h}{g}$$

$$\Rightarrow \boxed{v_0^2 = 2gh}$$

Free Fall in $\vec{F} = -k\vec{r}$





$$U = \frac{k}{2}(x^2 + y^2), \quad \vec{F} = -\vec{\nabla} U = -k\vec{r}$$

- Find K_0 if $K_h = 0$ and fall is directly along y axis

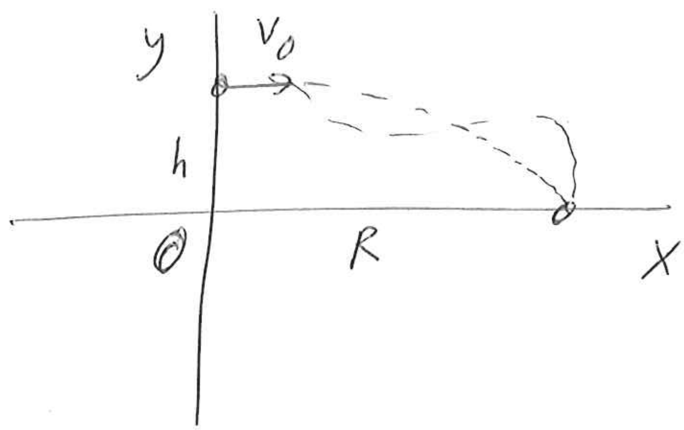
$$K + U = \text{const}$$

$$\Rightarrow \quad U_h = K_0 + U_0$$

$$U_h = \frac{1}{2}kh^2 \quad U_0 = 0$$

$$\Rightarrow \quad \boxed{K_0 = \frac{1}{2}kh^2}$$

"Projectile" in different $\vec{F}$ fields



- Given: mass starts at $(0,h)$ with $\vec{v}_0 = \hat{x} v_0$,
  ends at $(R,0)$
- Find $K(R,0)$ for $\vec{F} = -\vec{\nabla} U$
  for 3 cases: $U_1 = mgy$, $U_2 = \frac{1}{2}k(x^2+y^2)$, $U_3 = kxy$
- In all cases, $\frac{1}{2}mv_0^2 + U(0,h) = K_{final} + U(R,0)$

Case 1 $\quad K_f = K_0 + mgh$

Case 2 $\quad K_f = K_0 + \frac{1}{2}kh^2 - \frac{1}{2}kR^2 = K_0 + \frac{1}{2}k(h^2 - R^2)$

only possible if $K_f \geq 0$. Not always "accessible".

Case 3 $\quad K_f = K_0 + k(xy)_{(0,h)} - k(xy)_{(R,0)} = K_0$.

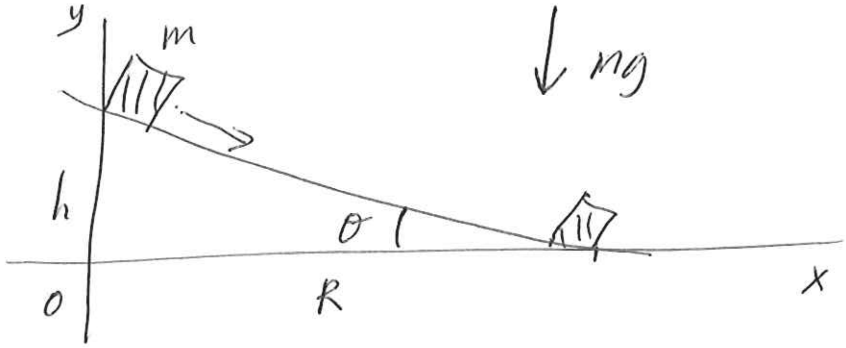
same "height"! no change

Constrained Motion

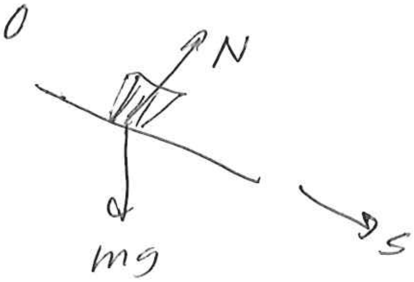
W15

2D - Constrained motion in one-dimensional system

Ex ①



- Mass slides down incline, no friction
- Mass stays on incline, $N > 0$
- N does no work



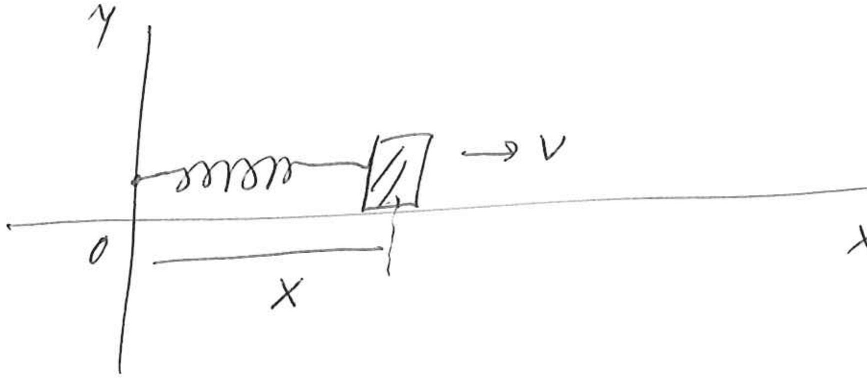
$\Rightarrow$ potential energy only from $m\vec{g}$

- If $v(t=0) = 0$ find v at bottom.
  Note v along incline
- only single dimension, s, along incline.
- First, $N = mg\cos\theta$
- $K + U_g = \text{const} \Rightarrow mgh = K_{final} = \frac{1}{2} m v_f^2$

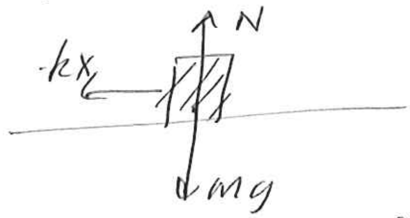
$$\Rightarrow \boxed{v_f^2 = 2gh}$$

- Compare this to Taylor Problem 4.8, where N can go to zero.

2D constrained motion, another example



- spring force $F = -kX$
- other forces



N and mg do NO work
$\Rightarrow$ only spring force

- $F = -kx$. Is F conservative?
Yes, if its one-dimensional and depends only on that dimension (but not on $V(t)$ or t)
- If conservative, then $F(x) = -dU/dx$

$$\Rightarrow \quad U = -\int F\,dx = +k\int x\,dx = \frac{1}{2}kx^2 + C$$

Let $\boxed{U = \frac{1}{2}kx^2}$

- Find amount of extension if $V(0) = V_0$

$$K + U = \text{const} \quad \Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}kx_f^2$$

$$\Rightarrow \boxed{x_f^2 = mv_0^2/k}$$

W 10/7

Solution of $x(t)$ in 1-D problems

for conservative F

- $$m\ddot{x} = F(x)$$

$$\vec{\nabla} \times F(x)\hat{x} = 0 \Rightarrow F = -dU/dx$$

$$\Rightarrow \text{can find } U(x)$$

- Energy is a const of motion

$$\Rightarrow \quad \mathcal{E} = \frac{1}{2} m \dot{x}^2 + U(x) = \text{const}$$

$$\text{ie,} \quad \frac{1}{2} m \dot{x}^2 + U(x) = \underbrace{\frac{1}{2} m \dot{x}^2(0) + U(x(0))}_{\text{call this } \mathcal{E} \text{ from i.c.}}$$

$$\Rightarrow \quad \dot{x}^2 = \frac{2}{m}\left[\mathcal{E} - U(x)\right]$$

known

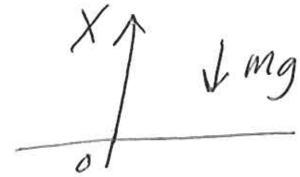
$$\Rightarrow \quad \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}} \left[\mathcal{E} - U(x)\right]^{1/2}$$

use separation variables

WB

$$\frac{dx}{[\mathcal{E}-U(x)]^{1/2}} = \pm\sqrt{\frac{2}{m}}\,dt$$

2 signs



Let $U = mgx$

$$v(t=0)=0, \quad x(0)=0$$

$$\Rightarrow \mathcal{E} = \frac{1}{2}mv^2\Big|_0 + mgx\Big|_0 = 0$$

$dx/dt\big|_0 < 0$

so pick negative sign

$$\int_0^x \frac{dx}{(-mgx)^{1/2}} = -\sqrt{\frac{2}{m}}\int_0^t dt$$

$$-2(-x)^{1/2} = -\sqrt{2g}\,t$$

$$(-x)^{1/2} = \sqrt{\frac{g}{2}}\,t$$

$$-x = \underline{g}\,t^2$$

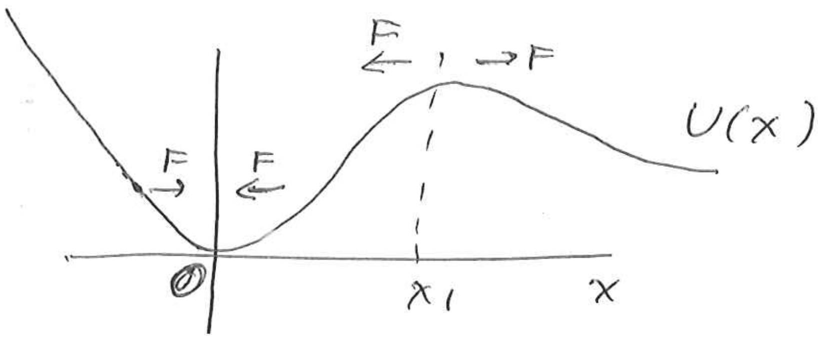
$$\boxed{x = -\frac{1}{2}gt^2}$$

One-Dim potentials

$$m\ddot{x} = F(x)$$

$$m\ddot{x} = -dU/dx$$

$$\mathcal{E} = \frac{1}{2}mv^2 + U(x) = \text{const}$$



Equil $\quad dU/dx = 0 \Rightarrow x = 0, \; x = x_1$

stable $\qquad$ unstable

$U'' > 0 \Rightarrow$ stable, $\quad U'' < 0 \Rightarrow$ unst

Small osc $\quad x = x_0 + \tilde{x} \qquad {}^{\prime\prime}O$

$$U = U(x_0 + \tilde{x}) = U_0 + U_0'\tilde{x} + U_0''\frac{\tilde{x}^2}{2}$$

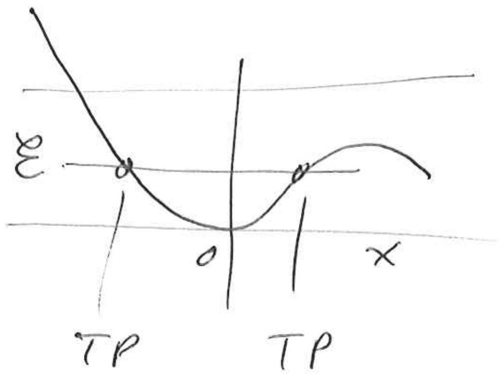
$$\Rightarrow U' = U_0''\tilde{x}$$

$$\boxed{m\ddot{\tilde{x}} = -U_0''\tilde{x}} \quad H.O.$$

$U_0'' > 0$ stable $\qquad U_0'' < 0 \Rightarrow e^{\alpha t}$ UNST

Turning points

$\mathcal{E} = \text{const} \Rightarrow$ ball on hill



$$\frac{1}{2} m \dot{x}^2 = \mathcal{E} - U(x)$$

$$\Leftrightarrow \mathcal{E} > U(x)$$

TP = turning points