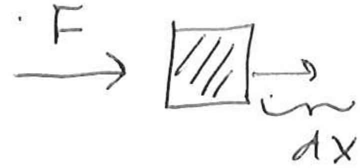
Notes to accompany

Ch 4 - Part I

Conservative forces
and Energy

updated: 09/14/16

Work

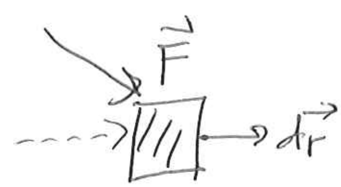


- When a Force pushes a mass, work is done on mass

$$dW = F\,dx$$

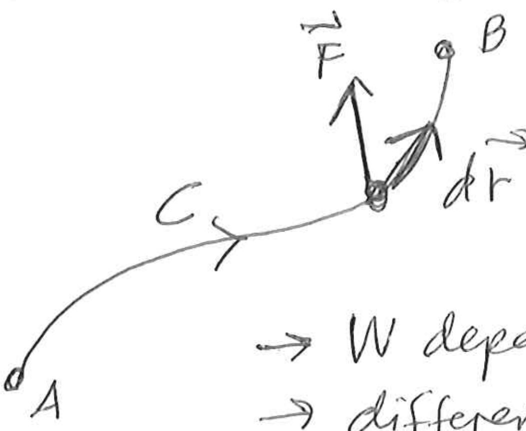
$$W(A \to B) = \int_A^B F\,dx$$

- $\vec{F}$ must be the component in the direction of $d\vec{r}$, so



$$dW = \vec{F} \cdot d\vec{r}$$

- In 2-D, 3-D, total work done is a line integral along the path C



$$dW = \vec{F} \cdot d\vec{r}$$

$$W(A \to B)_C = {}_C\int_A^B \vec{F} \cdot d\vec{r}$$

→ W depends on $\vec{F}$, A, B, C

→ different C ⇒ different W

→ $\vec{F}$ can be $\vec{F}(\vec{x}, \vec{v}, t)$

Work - Energy Theorem

If work is done on mass M by net force $\vec{F}$, W shows up in KE of M

$$dW = F\,dx$$

$$\text{But } F = M\frac{dv}{dt}$$

$$\Rightarrow \quad dW = M\frac{dv}{dt}dx = M\frac{dv}{dt}\underbrace{\frac{dx}{dt}}_{d\cancel{x}(t)}dt = M\,dv\,v$$

$$dv = \frac{dv}{dt}dt$$

$$\text{But } dv\,v = d(v^2/2)$$

$$\Rightarrow \quad dW = d\left(\tfrac{1}{2}Mv^2\right)$$

$$\Rightarrow \quad K = K_0 + W \qquad\qquad K \equiv \tfrac{1}{2}Mv^2$$

WE Thm in 3D

$$dW = \vec{F}\cdot d\vec{r} = M\frac{d\vec{v}}{dt}\cdot d\vec{r} = M\,d\vec{v}\cdot\vec{v}$$

$$\text{But } d(v^2) = d(\vec{v}\cdot\vec{v}) = d\vec{v}\cdot(\vec{v}) + \vec{v}\cdot d\vec{v} = 2\vec{v}\cdot d\vec{v}$$

$$\Rightarrow \boxed{dW = dK} \qquad K = \tfrac{1}{2}Mv^2$$

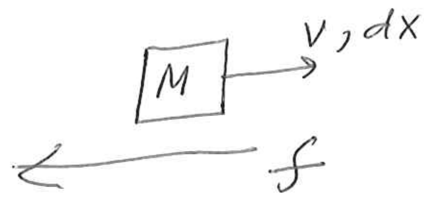
Example 1 — Mass under gravity force

$$K = K_0 + \int_0^h mg\, dy$$

mg = constant

$$\boxed{K(h) = K_0 + mgh}$$

Example 2 — Slow down under friction



$$K = K_0 + \int_0^{\ell} (-f)\, dx$$

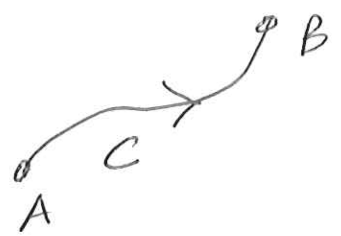
$$\boxed{K(\ell) = K_0 - \int_0^{\ell} dx\, f}$$

given $f \Rightarrow K$ decreases from K_0
as opposing force acts

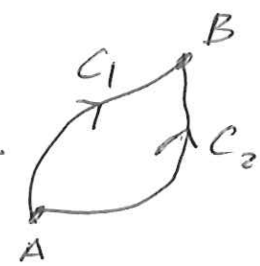
Conservative Forces

- Conservative Forces are very important in physics. Many forces are conservative.

- Definition: $\vec{F}$ is conservative if the line integral ${}_{C}\int_A^B d\vec{r}\cdot\vec{F}$ does not depend on C.

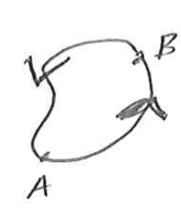
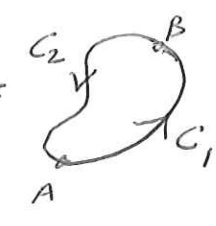


i.e., ${}_{C_1}\int_A^B d\vec{r}\cdot\vec{F} = {}_{C_2}\int_A^B d\vec{r}\cdot\vec{F}$ for any two C_1, C_2



- Corollary: If $\int_{C_1} = \int_{C_2}$ for any C's,

$\Rightarrow \oint_C d\vec{r}\cdot\vec{F} = 0$ for closed C

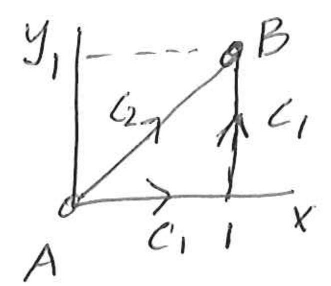
Proof: $\oint_C =$  $=$  $=$  $= 0$

$$\oint_C = \int_{C_1} + \int_{C_2} = \int_{C_1} - \int_{-C_2} = 0 \Rightarrow \int_{C_1} = \int_{-C_2}$$

Examples Conservative and Non-C Forces

(by line integral test)

Let $\vec{F} = (F_x, F_y)$ and C_1, C_2 given as



$A = (0,0)$
$B = (1,1)$

For C_2,
$y = x$

<u>Case 1</u> $\vec{F} = (y, x)$ $\quad \vec{F} \cdot d\vec{r} = dx\, y + dy\, x$

$$\int_{C_1} = \int_{\rightarrow} + \int_{\uparrow} = \int_{x=0}^{x=1} dx\, \phi + \int_{y=0}^{y=1} dy\, 1 = 1$$

$$\int_{C_2} = \int (dx\, y + dy\, x) = \int_{y=x} (dx\, x + dx\, x) = 2\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\therefore \boxed{\int_{C_1} = \int_{C_2}}$$ Possibly Conservative

<u>Case 2</u> $\vec{F} = (y, -x)$ $\quad \vec{F} \cdot d\vec{r} = dx\, y - dy\, x$

$$\int_{C_1} = -\int_0^1 dy\, 1 = -1$$

$$\boxed{\int_{C_1} \neq \int_{C_2}}$$

$$\int_{C_2} = \int_0^1 (dx\, x - dx\, x) = 0$$

Not conservative

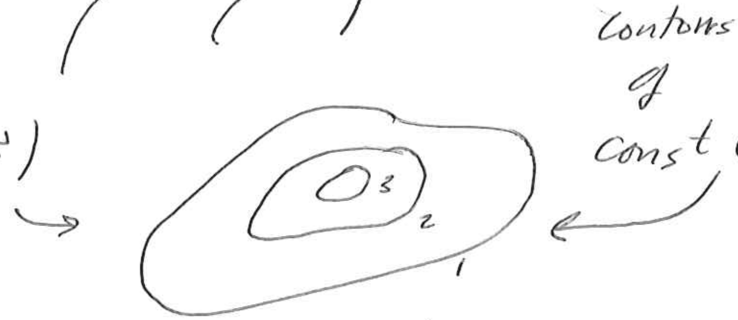
Math needed from Phys 274

W6

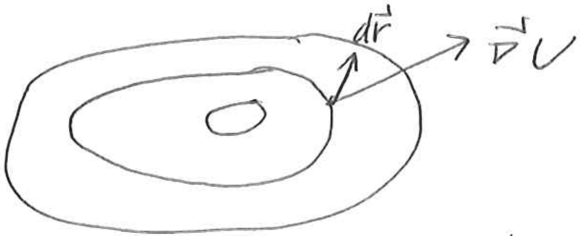
- Perform line integrals ${}_C\int_A^B d\vec{r}\cdot\vec{F}(\vec{r})$

- Vector Fields $\vec{F}(\vec{r})$

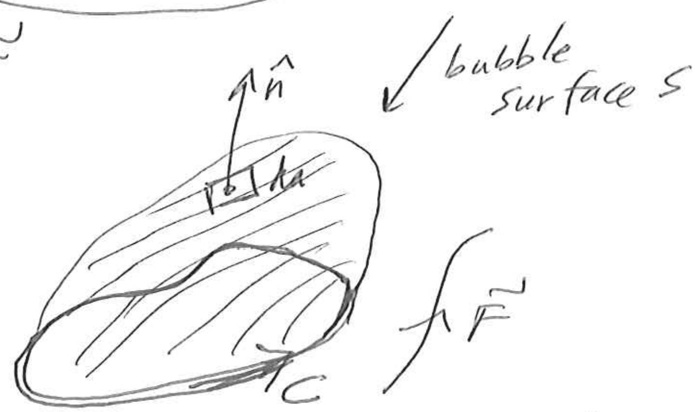
- Scalar field $U(\vec{r})$



- Understand $\vec{\nabla}U$



- Understand $dU = d\vec{r}\cdot\vec{\nabla}U$
  - change in U for $d\vec{r}$



- Perform $\vec{\nabla}\times\vec{F}$
- Stokes Theorem

  $$\oint_C d\vec{r}\cdot\vec{F} = \int_S da\,\hat{n}\cdot\vec{\nabla}\times\vec{F}$$, right hand rule

  for given C, all bubbles S allowed

- Cartesian, polar, cylindrical, spherical coordinate systems

$\oint_C \vec{F}\cdot d\vec{r}$, $\vec{F} = \vec{\nabla} U$, and $\vec{\nabla}\times\vec{F}$ are related   W7

The below boxes are $\Longleftrightarrow$ true

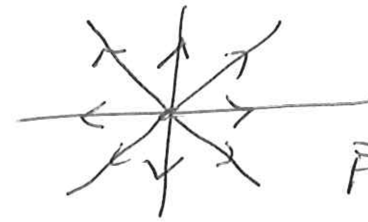
$$\begin{tikzcd}
 & \boxed{\oint_C \vec{F}\circ d\vec{r} = 0,\ \text{all } C} \arrow[dl, bend right, "5"'] \arrow[dr, bend left, "1"] & \\
\boxed{\vec{F} = \vec{\nabla} U} \arrow[ur, "2"] \arrow[rr, bend right, "3"'] & & \boxed{\vec{\nabla}\times\vec{F} = 0} \arrow[ul, "4"'] \arrow[ll, "6"']
\end{tikzcd}$$

1. by Stokes' Theorem
2. by the fact $d\vec{r}\circ\vec{\nabla} U = dU$
3. by direct plug in
4. by Stokes'
5. $\int_{\vec{r}}^{\vec{r}+d\vec{r}} \vec{F}\circ d\vec{r} \Rightarrow \vec{F}\cdot d\vec{r} = dU$. But $dU = d\vec{r}\cdot\vec{\nabla} U$
   $\Rightarrow \vec{F} = \vec{\nabla} U$
6. use 4 and 5

wf

Some Conservative $\vec{F}$'s and how to find U

Ex (1) $\vec{F} = (x, y) \Rightarrow$



$\vec{F}$ field

$$\vec{\nabla} \times \vec{F} \to \partial_x F_y - \partial_y F_x$$

$$= \partial_x y - \partial_y x = 0 \Rightarrow \text{conservative}$$

let $\vec{F} = -\vec{\nabla} U$ (convention)

$$-dU = d\vec{r} \cdot \vec{F} = dx\, x + dy\, y$$

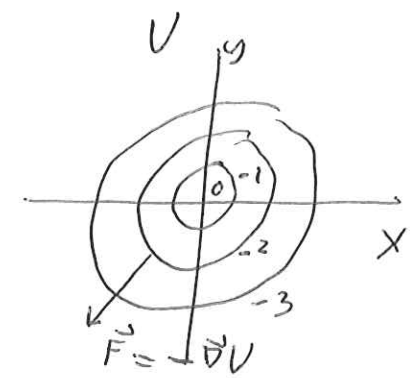
$$\Rightarrow -U = -\int dU = \int dx\, x + \int dy\, y + \text{const}$$

integrable since all y's

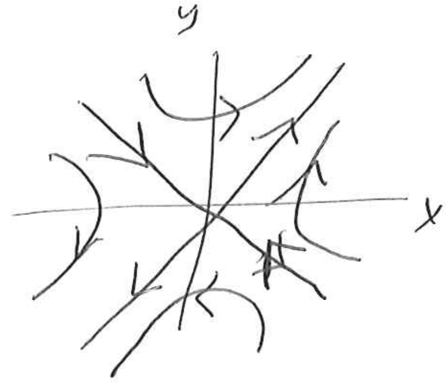
$$\Rightarrow \boxed{U(x,y) = -\left(\frac{x^2}{2} + \frac{y^2}{2}\right) + \text{const}}$$

Check $-\vec{\nabla} U = -(\partial_x U, \partial_y U)$

$$= (x, y) \quad \checkmark$$



Ex ② $\vec{F} = (y, x)$



$\vec{\nabla} \times \vec{F} \to \partial_x x - \partial_y y = 0$

$\Rightarrow \vec{F} = -\vec{\nabla} U$

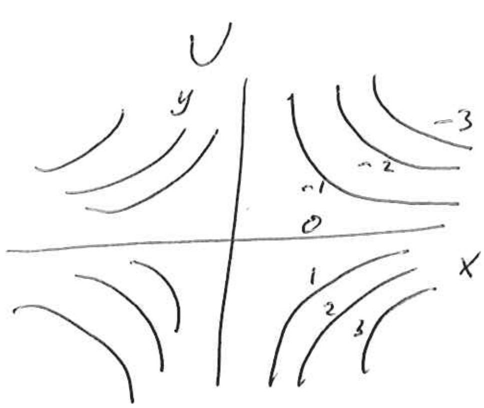
$-dU = +\int d\vec{r} \cdot \vec{F} = \int dx\, y + \int dy\, x$

not directly integrable since x, y both change

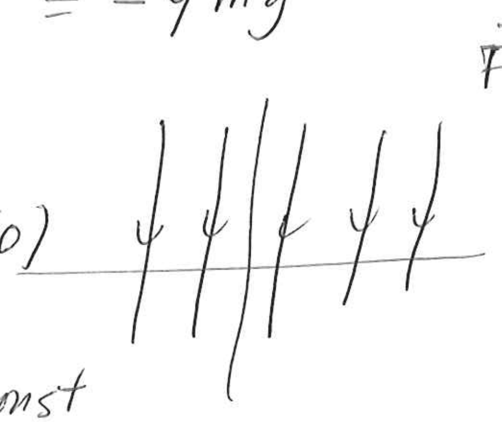
Can show $U = -xy$

Check : $-\vec{\nabla} U = -(\partial_x U, \partial_y U)$

$= (y, x)$ ✓



Ex (3) $\vec{F} = m\vec{g} = -\hat{y}\, mg = (0, -mg)$



$\vec{\nabla} \times \vec{F} = \partial_x(-mg) - \partial_y(0)$

$= 0$ $\quad mg = \text{const}$

$\Rightarrow \vec{F} = -\vec{\nabla} U$

$-\int dU = \int \vec{F} \cdot d\vec{r} = -\int mg\, dy = -mgy + \text{const}$

$\Rightarrow \boxed{U = mgy}$