Motion in harmonic oscillator (HO) central force

Consider a single mass m in a central force field $\mathbf{F} = -k\mathbf{r}$, with respect to a fixed origin. Here $\mathbf{r}(t)$ is the radius vector, $\mathbf{r} = (x,y,z)$, in Cartesian. In this problem, we want to understand the various orbits of the mass about the HO central force.

1. Check that $\mathbf{F}$ is conservative.

2. Find the potential $U(\mathbf{r})$.

3. It follows that the energy is conserved. Write down the expression for this. Note that $\mathbf{v}(t) = d\mathbf{r}/dt$.

4. $\mathbf{F}$ is a central force (in addition to being conservative). Therefore, show that the angular momentum $\mathbf{L}$ is a constant, where $\mathbf{L} = m\, \mathbf{r} \times \mathbf{v}$. Your proof must use 2 important vector identities.

5. If $\mathbf{L}$ is constant, prove that $\mathbf{r}(t)$ must stay on a 2-dimensional plane. State the important vector identity needed to prove this. Thus, assume that $\mathbf{r} = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$.

6. Polar coordinates are clearly preferable. Refer to Taylor Sec 1.7 and check the following (you do not have to derive these – just collect and review them): $\mathbf{r}(t) = r(t)\hat{\mathbf{r}}(t)$, $\mathbf{v} = \hat{\mathbf{r}}\, dr/dt + \hat{\boldsymbol{\phi}}\, r d\phi/dt$. Also, obtain the expression for $d^2\mathbf{r}/dt^2$.

7. So, write down the 2 polar components of $m\mathbf{a} = \mathbf{F}$. From the $\hat{\boldsymbol{\phi}}$ component, show that the angular moment magnitude, L, is also a constant of the motion (so far we only used the direction of $\mathbf{L}$).

8. Eliminate in the radial equation all the ϕ terms by using the angular momentum equation from 7. Thus, find a 2nd order nonlinear ODE for r(t). This equation cannot be solved in an easy way.

9. But, we haven't used energy conservation yet. Write down the energy constant using polar coordinates (use 6 above). Again eliminate the ϕ terms using the constancy of L, from 7. Check that this energy constant contains terms only in dr/dt and r.

10. As a check on energy conservation, start with your energy constant in 9 and differentiate E in t. This should be zero, ie, $dE/dt = 0$. Show that this results in the 2nd order ODE for r(t) in 8 above.

11. Take a square root in the equation from 9 and thus obtain a 1st order ODE for r(t). (there will be a +/- sign from the square root). Check that this equation is separable and therefore solvable in principle.

12. Define an effective potential starting from your energy equation in 9. Let $U_{eff}(r) = U(r)$ (of the central force) + angular momentum term in r. Make a sketch of $U_{eff}(r)$ vs r. Draw a horizontal line for E on this sketch. Since $(m/2)(dr/dt)^2 > 0$, observe that we must have $E > U_{eff}(r)$ for a solution to exist. Thus, show that r(t) must be bounded from above and below according to $r_{min} < r(t) < r_{max}$.

13. If $U_{eff}(r)$ is at its minimum point, and E is adjusted so that $E = U_{min}$, observe that r(t) must be a constant, corresponding to circular orbit. Find r_0, the radius corresponding to U_{min}. Insert this value of r in the angular momentum equation for $d\phi/dt$ found in 7 above to show that the angular frequency of the circular orbit, $d\phi/dt$, is a constant, ω_0, and given in terms of L, m, and r_0.

14. Using your knowledge of centripetal force, applied to the $-k\mathbf{r}$ force, find the frequency of circular orbit. Compare this to ω_0 found above. You will have to eliminate L from your expression using the angular momentum equation found in 7.

15. We now want to perform small oscillations of the mass *about the circular orbit*. To do this, start from the 2nd order ODE for r(t) found in 8. Let $r(t) = r_0 + s(t)$. Substitute this into the ODE and Taylor expand the RHS for small s. Use the definition of r_0 to rewrite the RHS, so its only proportional to s. Thus, obtain a 2nd order ODE for s(t). Solve this in general and find ω, the frequency of small oscillations about r_0. Compare this frequency with ω_0. Can you sketch the circular orbit + small oscillations added on?