

Problem 1 (20 points)

Find the equation of the path joining the origin (0,0) to the point (1,1) in the x-y plane that makes the integral $\int dx (y'^2 + y^3 y' + y'^2)$ an extremum, where $y' = dy/dx$.

\uparrow y^2 typo

Problem 2 (20 points)

A new physics equation is proposed to update Newton's Equation. The new equation takes the form

$$m (d/dt)[x' + \epsilon \sinh(x')] = F(x), \quad \text{where } F(x) = -dU/dx, \text{ and } x' = dx/dt.$$

If this equation is to be taken seriously, one must be able to recast it into standard Euler-Lagrange form, $(d/dt)(\partial L/\partial x') = (\partial L/\partial x)$, thus ensuring the existence of a variational principle. Here, $L = L(x, x')$. Lagrange was able to accomplish this for $\epsilon = 0$ by introducing the idea that x could be treated as an independent variable thus allowing derivatives of the form $(\partial f/\partial x')_x$ to replace any functions of x' . Here, in general, $f = f(x, x')$.

X
typo

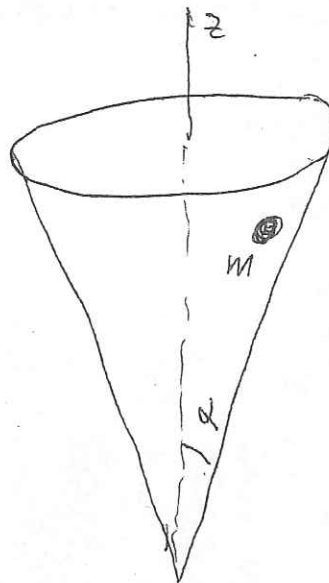
Using Lagrange's approach, investigate if you can recast the above equation with constant ϵ into E-L form. If so, define the new $L(x, x')$. For partial credit, you may present the $\epsilon = 0$ formulation.

Problem 3 (30 points)

A particle is constrained to move on the surface of a circular cone. The cone axis is the vertical z -axis, the vertex is at the origin (pointing down), and the half angle is α . There is gravity g pointing down. Mass = m .

1. Write down the Lagrangian, L , in terms of spherical polar coordinates (r, ϕ) .
2. Find two equations of motion.
3. Interpret the ϕ equation in terms of the angular momentum λ , and use it to find an ODE in terms of only $r(t)$.
4. State how this equation makes sense if the angular momentum = 0.
5. Find a value for r , r_0 , in terms of λ , m , g , α , such that the particle can remain on a horizontal and circular path. What is the frequency of rotation in terms of r_0 and the parameters?
6. Suppose the particle is given a small radial kick, ie, $r(t) = r_0 + \epsilon(t)$, where $|\epsilon| \ll r_0$. Is the circular motion stable to small perturbations? If so, what is the oscillation frequency normalized to the rotation frequency?

Useful: In spherical coordinates, $dr = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$.



Problem 4 (30 points)

We want to study free fall and Coriolis deflection. We will use equations developed in Taylor's text (Eqs 9.53 and Figure 9.15, attached). A mass m is released, from rest, from a height $z=h$, and $x=y=0$. The experiment is done at the equator (corresponding to $\theta = \pi/2$).

This problem has parameters Ω , g , and h . To simplify algebra, you may set $g=1$ and $h=1$, corresponding to distance normalized to h and time normalized to $(h/g)^{1/2}$. You don't have to do this, however.

1. Write down Eqs 9.53 as they apply at the Equator.
2. Using the initial conditions supplied, find $y(t)$. Also, by performing an integration, find $[dz/dt](t)$ in terms of $x(t)$.
3. Insert the above results into the equation for d^2x/dt^2 . Specify at least 3 characteristics of the resulting equation which point you to the methods to be used to solve this ODE.
4. Solve completely for $x(t)$ given the boundary conditions. This will be an exact solution.
5. We now assume that the free-fall time from height h is given by the inertial frame result, ie, $\tau^2 = 2h/g$. From 4., find the deflection in $x(t)$, d , as evaluated according to $d = x(\tau)$. Your answer should be a two term expression for d given in terms of Ω , g , h , and τ .
6. Identify in your expression for d the dimensionless parameter $\epsilon = \Omega\tau$, proportional to $h\Omega^2/g$. Assuming $\epsilon \ll 1$, make an approximation in d to get a nonzero expression for (d/h) which is proportional to ϵ^α . What is the exponent α ?

Maybe useful: $\cos(x) = 1 - x^2/2! + \dots$ $\sin(x) = x - x^3/3! + \dots$

In general

$$\frac{d}{dx} \sinh x = \cosh x$$

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where, as before, $m\mathbf{g}_0$ denotes the true force of the earth's gravity on the object. The centrifugal force is $m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$, (where \mathbf{r} is the object's position relative to the center of the earth), but to an outstanding approximation we can replace \mathbf{r} by \mathbf{R} (the position on the earth's surface where the experiment is being conducted). Thus,

$$\mathbf{F}_{cf} = m(\boldsymbol{\Omega} \times \mathbf{R}) \times \boldsymbol{\Omega}.$$

Returning to the equation of motion (9.50), you will recognize that the sum of the first two terms on the right is just $m\mathbf{g}$, where \mathbf{g} is the observed free-fall acceleration for an object released from rest at position \mathbf{R} , as introduced in (9.44). In other words, we can omit the term \mathbf{F}_{cf} from (9.50), if we replace \mathbf{g}_0 by the observed \mathbf{g} at the location of our experiment. If we substitute $2m\mathbf{v} \times \boldsymbol{\Omega}$ for \mathbf{F}_{cor} , the equation of motion becomes (after cancellation of a factor of m)

$$\ddot{\mathbf{r}} = \mathbf{g} + 2\dot{\mathbf{r}} \times \boldsymbol{\Omega}. \tag{9.51}$$

A simplifying feature of the equation (9.51) is that it does not involve the position \mathbf{r} at all (only its derivatives $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$). This means the equation will not change if we make a change of origin (since a change of origin amounts to adding a constant to \mathbf{r}). Accordingly, I shall now choose my origin on the surface of the earth at the position \mathbf{R} , as shown in Figure 9.15. With this choice of axes, we can resolve the equation of motion into its three components. The components of $\dot{\mathbf{r}}$ and $\boldsymbol{\Omega}$ are

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$$

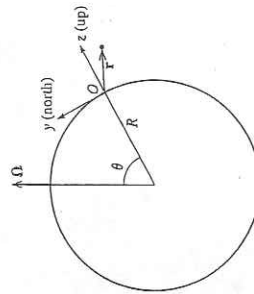


Figure 9.15 Choice of axes for a free-fall experiment. The origin O is on the earth's surface at the experiment's location (position \mathbf{R} relative to the center of the earth). The z axis points vertically up (more precisely, in the direction of $-\mathbf{g}$, where \mathbf{g} is the observed free-fall acceleration), the x and y axes are horizontal (that is, perpendicular to \mathbf{g}), with y pointing north, and x due east. The position of the falling object relative to O is \mathbf{r} .

and

$$\boldsymbol{\Omega} = (0, \Omega \sin \theta, \Omega \cos \theta).$$

Thus, those of $\dot{\mathbf{r}} \times \boldsymbol{\Omega}$ are

$$\dot{x} \times \boldsymbol{\Omega} = (\dot{y}\Omega \cos \theta - \dot{z}\Omega \sin \theta, -\dot{x}\Omega \cos \theta, \dot{x}\Omega \sin \theta) \tag{9.52}$$

and the equation of motion (9.51) resolves into the following three equations:

$$\begin{aligned} \ddot{x} &= 2\Omega(\dot{y} \cos \theta - \dot{z} \sin \theta) \\ \ddot{y} &= -2\Omega\dot{x} \cos \theta \\ \ddot{z} &= -g + 2\Omega\dot{x} \sin \theta. \end{aligned} \tag{9.53}$$

We shall solve these three equations by making a succession of approximations that depend on the smallness of Ω . First, because Ω is very small, we get a reasonable starting approximation if we ignore Ω entirely. In this approximation, the equations reduce to

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \text{and} \quad \ddot{z} = -g. \tag{9.54}$$

which are the equations of free fall solved in every introductory physics course. If the object is dropped from rest at $x = y = 0$ and $z = h$, then the first two equations imply that $\dot{x}, \dot{y}, x,$ and y all remain zero, while the last equation implies that $\dot{z} = -gt$ and $z = h - \frac{1}{2}gt^2$. Thus our approximate solution is

$$x = 0, \quad y = 0, \quad \text{and} \quad z = h - \frac{1}{2}gt^2, \tag{9.55}$$

that is, the object falls vertically down with constant acceleration g . This approximation is sometimes called the *zeroth-order* approximation because it involves only the zeroth power of Ω (that is, it is independent of Ω). It is well known to be a very good approximation, but it shows none of the effects of the Coriolis force.

To get the next approximation, we argue as follows: The terms in (9.53) that involve Ω are all small. Thus, it will be safe to evaluate these terms using our zeroth-order approximation for $x, y,$ and z . Substituting (9.55) into the right side of (9.53), we get

$$\ddot{x} = 2\Omega gt \sin \theta, \quad \ddot{y} = 0, \quad \text{and} \quad \ddot{z} = -g. \tag{9.56}$$

The last two of these are exactly the same as in zeroth order, but the equation for x is new and is easily integrated twice to give

$$x = \frac{1}{3}\Omega g t^3 \sin \theta, \tag{9.57}$$

with y and z the same as in the zeroth approximation (9.55). This result is naturally called the *first-order* approximation (being good through the first power of Ω). We can repeat this process again to get a second-order approximation and so on, but the first-order is good enough for our purposes.

The striking thing about the solution (9.57) is that a freely falling object does not fall straight down. Instead the Coriolis force causes it to curve slightly to the east (positive x direction). To get an idea of the magnitude of the effect, consider an object