Perturbative solution of quadratic for "small" β

- $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

  $e^{-i\omega t} \Rightarrow (\omega_0^2 - \omega^2) - 2i\beta\omega = 0$ — (1)

  $\{\omega_0, \beta\}$ given, Find ω

- Suppose β is "small" (to be defined)

  Expect $\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)} + \dots$

  where* $|\omega^{(1)}| \ll \omega^{(0)}$, $|\omega^{(2)}| \ll |\omega^{(1)}|$, etc

- Lets keep only upto $\omega^{(1)}$

  $\Rightarrow \quad \omega \simeq \omega^{(0)} + \omega^{(1)}$

  $\omega^2 \simeq \omega^{(0)2} + 2\omega^{(0)}\omega^{(1)}$

  Insert into (1)

$$\Rightarrow \quad \omega_0^2 - \omega^{(0)2} - \underbrace{2\omega^{(0)}\omega^{(1)}}_{\text{small}} - \underbrace{2i\beta\omega^{(0)}}_{\text{small}} - \underbrace{2i\beta\omega^{(1)}}_{\text{smaller}} \simeq 0$$

* to be checked

- <u>Lowest order</u> : discard all small terms

$$\Rightarrow \quad \omega_0^2 - \omega^{(0)2} = 0$$

$$\Rightarrow \quad \boxed{\omega^{(0)} = \pm \omega_0}$$

- <u>1st order</u> : Keep small terms but not "smaller"

$$\Rightarrow^{*} \quad -2\omega^{(0)}\omega^{(1)} - 2i\beta\,\omega^{(0)} = 0$$

$$\Rightarrow \quad \boxed{\omega^{(1)} = -i\beta}$$

- <u>Solution correct to 1st order</u>

$$\omega \simeq \pm \omega_0 - i\beta \quad \text{— (2)}$$

$$e^{-i\omega t} \rightarrow e^{\mp i\omega_0 t}\, e^{-\beta t}$$

$$\Rightarrow \quad \boxed{x(t) \simeq \left[B_1 \cos(\omega_0 t) + B_2 \sin(\omega_0 t)\right] e^{-\beta t}}$$

* <u>Note</u> ω_0^2 and $\omega^{(0)2}$ "used up" in lowest order

- Note that there are 2 solns, ie, 2 solns for ω

- Note that we could continue the perturbation to higher order,* using corresponding methods

- <u>Meaning of "small"!</u>

- We made the assumption $|\omega^{(1)}| \ll |\omega^{(0)}|$

- for this to be self-consistent with our findings, we must demand, from (2),

$$\beta \ll \omega_0 .$$

* To do this, must use $(y_0 + y_1 + y_2 + y_3 \cdots)^2$

$$= \underbrace{y_0^2}_{\text{zeroth}} + \underbrace{2y_0 y_1}_{\text{1st}} + \underbrace{y_1^2}_{\text{2nd}} + \underbrace{2y_0 y_2}_{\text{2nd}} + \underbrace{2y_0 y_3}_{\text{3d}} + \underbrace{2y_1 y_2}_{\text{3d}} + \cdots$$

2nd order solution

$$(\omega_0^2 - \omega^2) - 2i\beta\omega = 0$$

Let $\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)}$

$$\Rightarrow \omega^2 = (\omega^{(0)} + \omega^{(1)} + \omega^{(2)} \ldots)^2$$

$$\simeq \underbrace{\omega^{(0)2}}_{\text{zero order}} + \underbrace{2\omega^{(0)}\omega^{(1)}}_{\text{1st}} + \underbrace{\omega^{(1)2}}_{\text{2nd}} + \underbrace{2\omega^{(0)}\omega^{(2)}}_{\text{2nd}} + \underbrace{\cdots}_{\text{3rd \& more}}$$

$$-2i\beta\omega \rightarrow \underbrace{-2i\beta(\omega^{(0)}}_{\text{1st order}} + \underbrace{\omega^{(1)}}_{\text{2nd}} + \cdots)$$

Therefore,

zero order: $\boxed{\omega_0^2 - \omega^{(0)2} = 0}$

1st order: $-2\omega^{(0)}\omega^{(1)} - 2i\beta\omega^{(0)} = 0$

$$\Rightarrow \boxed{\omega^{(1)} = -i\beta}$$

2nd order: $-\omega^{(1)2} \cancel{+} 2\omega^{(0)}\omega^{(2)} - 2i\beta\omega^{(1)} = 0$

$$\Rightarrow \omega^{(0)}\omega^{(2)} + \frac{\omega^{(1)2}}{2} + \cancel{2}i\beta\omega^{(1)} = 0$$

Plug in $\omega^{(1)}$: $\omega^{(0)}\omega^{(2)} + \frac{\omega^{(1)}}{2}\underbrace{\left(\omega^{(1)} + 2i\beta\right)}_{i\beta} = 0$

$$\omega^{(0)}\omega^{(2)} = -\frac{1}{2}\beta^2$$

$$\omega^{(2)} = \frac{-\beta^2/2}{(\pm\omega_0)}$$

$$\Rightarrow \quad \omega \simeq \pm\omega_0 - i\beta - \frac{\beta^2}{2(\pm)\omega_0} \simeq (\pm)\omega_0 - (\pm)\frac{\beta^2}{2\omega_0} - i\beta$$

$$\omega \simeq (\pm)\omega_0\left(1 - \frac{\beta^2}{2\omega_0^2}\right) - i\beta$$

<u>Solutions</u> $\quad e^{-\beta t}\begin{Bmatrix} e^{-i\omega_0(1 - \beta^2/2\omega_0^2)t} \\ e^{+i\omega_0(1 - \beta^2/2\omega_0^2)t} \end{Bmatrix}$

shift in frequency

<u>Check with quadratic</u>

$$\omega^2 + 2i\beta\omega - \omega_0^2 = 0$$

$$\omega = -i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

$$= -i\beta \pm \omega_0\left(1 - \beta^2/\omega_0^2\right)^{1/2}$$

$$\simeq -i\beta \pm \omega_0\left(1 - \tfrac{1}{2}\beta^2/\omega_0^2\right)$$

using binomial