Symmetry of Inertial Frames

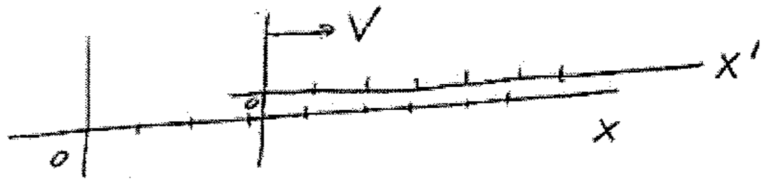
- Instincts developed while playing pool in the lab are very well suited to playing pool on a train moving at constant, smooth speed V. We elevate this to the principle that inertial frames (IF) are equivalent $\Rightarrow$ laws of nature must be covariant in two IF's.

- We know how to test this, based on our experience with rotations. Namely, write down the equations in the lab, find the coordinate transformation from lab $\rightarrow$ train, apply this to the lab equations, then watch for covariance. If there is no covariance, our laws are

not correct — OR IF symmetry is
not a valid principle.

- The "obvious" relation between (x, t) lab coordinates and (x', t') train coordinates are the Galilean transformations (GT)

$$x' = x - Vt$$
$$t' = t$$



- Apply this to the 1-D Newtons' equation

$$m\ddot{x} = F \qquad (1) \qquad \ddot{x} = \frac{d^2x}{dt^2}$$

well, $x = x' + Vt' \Rightarrow \frac{dx}{dt} = \frac{dx'}{dt} + V$,

$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2}$. Since $t' = t$, $dt' = dt$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{d^2x'}{dt'^2} \qquad (2)$$

Using (2) in (1)

$$m\frac{d^2x'}{dt'^2} = F \qquad (3)$$

(3) and (1) $\Rightarrow$ covariance if $F' = F$.

This is reasonable (i.e., F is a scalar as far as GT goes).

- <u>What about Maxwell?</u>

Maxwell equations are also laws of nature. One consequence of Maxwell equations, in 1-D, is the wave equation

$$\boxed{\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}}. \qquad (4)$$

The solution to this yields waves propagating at c.

We apply the GT to (4).

Note, $\frac{\partial \varphi}{\partial t}(x',t') = \frac{\partial x'}{\partial t}\frac{\partial \varphi}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial \varphi}{\partial t'}$

Using the GT, $\frac{\partial x'}{\partial t} = -V$, $\frac{\partial t'}{\partial t} = 1$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'},}$$

likewise $\boxed{\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x'},}$ (5)

Using these in (4), we get

$$\boxed{\left(\frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}\right)^2 E_y = c^2\frac{\partial^2}{\partial x'^2}E_y} \quad (6)$$

(4) and (6) are clearly <u>NOT</u> covariant

In fact (6) $\Rightarrow$ waves propagating at $c+V$ or $c-V$. Thus, laser beams in moving trains should take longer or shorter to traverse corresponding equal distances between the train & the lab.

Conclude

- ME are not covariant under GT, Newton's eqns are.

- But ~~MW~~ electrodynamic phenomena do not change when magnets move into coils or vice-versa (Einstein's intuition). Likewise, there seems to be no preferred frame where lasers propagate at c, with speeds $\neq c$ in any other frame (eg, Michelson-Morley expt, stellar abberation, etc)

$\Rightarrow$ {GT is incorrect
ME is incorrect
Newton is incorrect
IF symmetry is incorrect}

Some combination of these possibilities.

Suppose ME are OK and IF symmetry is OK

$\Rightarrow$ wave equation must be covariant in $(x,t) \to (x',t')$

$\Rightarrow$ GT must be incorrect.

- We insist on this to see what the new covariant transformation must be if GT is not the correct one.

- let the new transformation be

$$x_0' = a x_0 + b x_1 \qquad (7)$$
$$x_1' = b x_0 + a x_1$$

where $x_0 = ct$, $x_1 = x$, (a, b) are constants
The relation is linear since space and time are homogeneous. There is a symmetry in the constants since the wave equation (4) exhibits symmetry between x_0, x_1

ie, (4) becomes

$$\frac{\partial^2 E_y}{\partial X_0^2} = \frac{\partial^2 E_y}{\partial x_1^2} \, . \qquad (8)$$

Now, using (7), we find

$$\frac{\partial \varphi}{\partial X_0} = a \frac{\partial \varphi}{\partial X_0'} + b \frac{\partial \varphi}{\partial X_1'}$$

$$\frac{\partial \varphi}{\partial X_1} = b \frac{\partial \varphi}{\partial X_0'} + a \frac{\partial \varphi}{\partial X_1'}$$

Insert into (8) $\Longrightarrow$

$$(a\partial_0' + b\partial_1')^2 E_y = (b\partial_0' + a\partial_1')^2 E_y$$

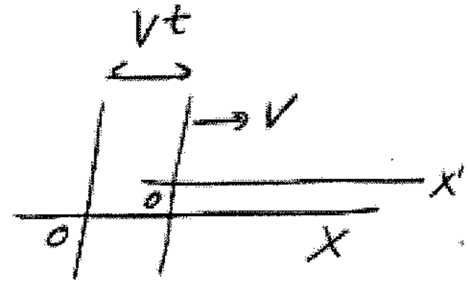
$$\Rightarrow (a^2-b^2)\partial_0'^2 E_y = (a^2-b^2)\partial_1'^2 E_y$$

But covariance $\Longleftrightarrow$

$$\partial_0'^2 E_y = \partial_1'^2 E_y$$

Thus, $a^2 - b^2 = 1$

But from (7), and the picture



we must have $b = -\frac{V}{c} a$ if $x = Vt$ at the origin of $x' = 0$.

$$\Longrightarrow \quad a^2(1 - V^2/c^2) = 1 \quad \text{and} \quad b = -\frac{V}{c} a$$

$\Rightarrow$ Lorentz transformation

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

where $\boxed{\gamma = (1-\beta^2)^{-1/2}, \quad \beta \equiv v/c.}$

$$\Rightarrow \quad \boxed{\begin{aligned} x_\mu' &= \Lambda_{\mu\nu} x_\nu \\ \Lambda &= \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \end{aligned}}$$

Clearly, the inverse transfmn is for $\beta \to -\beta$, <u>viz</u>

$$\boxed{\begin{aligned} x_\mu &= \Lambda_{\mu\nu}' x_\nu' \\ \text{where} \quad \Lambda' &= \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \end{aligned}}$$

OR $\Lambda'(\beta) = \Lambda(-\beta)$.