# Other PDE's

- So far, we have solved Laplace's equation. In 2-D, this is

$$\nabla^2\varphi \to \partial_x^2\varphi + \partial_y^2\varphi = 0$$

- There are two other important PDE's in physics, the Schroedinger Equation, and the wave equation.

These are

Schroedinger

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r},t)\psi \qquad V(\vec{r},t) \text{ given}, \quad \psi = \psi(\vec{r},t)}$$

and

Wave

$$\boxed{\frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi} \quad \boxed{\psi(\vec{r},t)}$$

- Note that both these PDE's are linear in $\Psi$, like Laplace's. Pretty much, all the methods we applied to Laplace's eqn will apply here. In particular, these are PDE's in $\vec{r}$ and $t$; thus, both boundary conditions and initial conditions will be given.

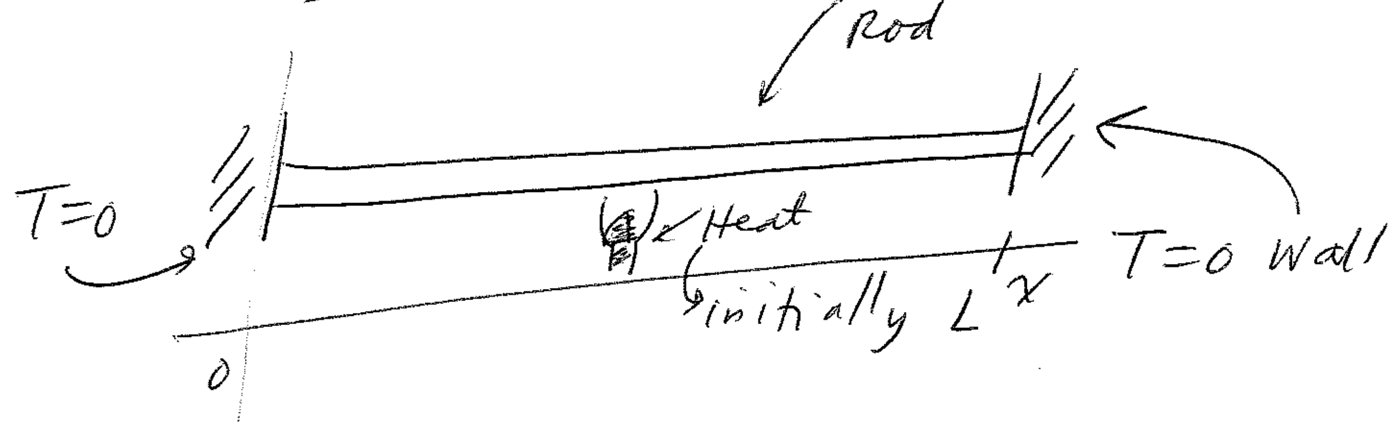
- The Heat Conduction equation

$$\boxed{\partial T/\partial t = \kappa \nabla^2 T \,,\ T(\vec{r},t)}$$

is structurally identical to the SE. We will also look at this.

- Each of these PDE's has associated characteristic physics. We will investigate these — methods + physics — by example.

# Heat Conduction Eqn




$T(x,t)$ satisfies

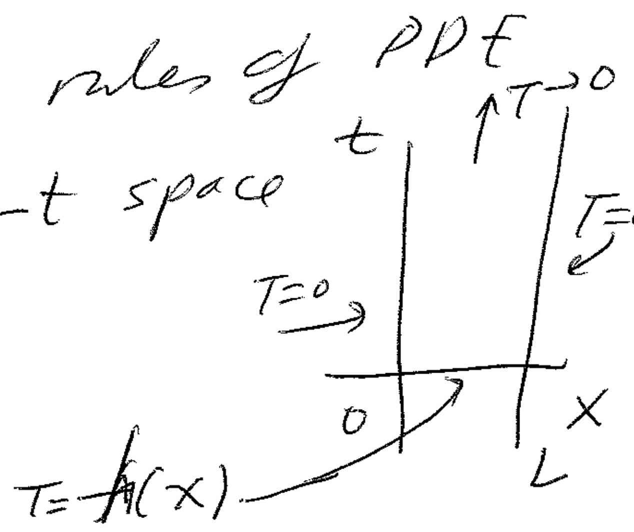
$$\boxed{\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}}$$

$K$ ← Heat conductivity

$$\boxed{T(x,t)\ ; \quad \begin{array}{l} T(0,t)=0 \\ T(L,t)=0 \end{array}}$$

$$\boxed{\text{Suppose, initially, } T(x,0) = h(x)}$$

- To solve, note this is a linear operation and all the rules of PDE solving apply in x–t space




- ⇒ execute all the steps

let $T(x,t) = f(x)g(t)$ H2

Sep var $\Rightarrow \dfrac{1}{g}\dfrac{dg}{dt} = \dfrac{1}{f}\dfrac{d^2f}{dx^2}$

$\boxed{\text{use } K=1 \quad L=1}$

oscillatory in $x \Rightarrow$ let $\dfrac{d^2f}{dx^2} = -k^2 f$

$\Rightarrow f \sim \left\{ \begin{matrix} \sin kx \\ \cos kx \end{matrix} \right\}$

$\Rightarrow \dfrac{dg}{dt} = -k^2 g \Rightarrow g = \left\{ e^{-k^2 t} \right\}$

$f(0) = 0 \Rightarrow$ use $\sin kx$

$f(1) = 0 \Rightarrow$ quantize $k$, $\Rightarrow k \to \boxed{k_n = n\pi}$

$$\Rightarrow \boxed{T(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-(n\pi)^2 t}}$$

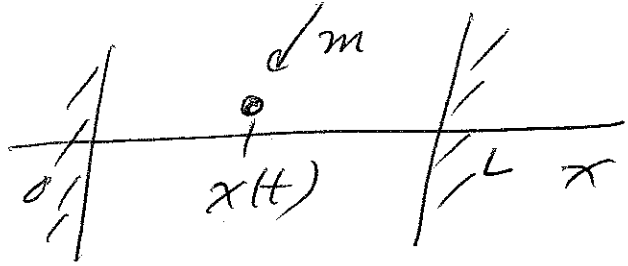
$\Leftrightarrow h(x) = \sum_{n=1} A_n \sin(n\pi x)$

$\Rightarrow A_n$ from Fourier's trick

$\Rightarrow$ solution.

# Schroedinger Equation

- Consider a particle confined between two walls.




- 1-D problem, No Forces
- classically, particle bounces from wall to wall, forever, @ initial speed $V_0$.
- QM'ly, m satisfies Schroedingers' eqn for $\psi(x,t)$, where $|\psi|^2(x,t)dx$ is the probability that the particle will be found in $dx$ at $(x,t)$.

~~[illegible]~~ The SE is given as

$$\boxed{i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}}$$

(one-dimensional; for <u>no</u> forces; else there is an extra term)

- We let $\hbar/m = 1$ and $L = 1$.

$\hbar/m$ has dimensions of $[L]^2/[T]$.

$$\Rightarrow \boxed{i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi}$$

$\psi(x,t)$ $\quad$ $\left.\begin{array}{l}\psi(0,t) = 0 \\ \psi(1,t) = 0\end{array}\right\}$ No penetration beyond walls assumed.

let $\psi(x,0) = h(x)$

- Note this PDE is structurally similar to Heat Conduction eqn; except the "$i$", iota, $\Rightarrow$ complex function $\psi(x,t)$.

- Linear PDE, + BC, + IC

$\Rightarrow$ same procedures as before

$\Rightarrow$ let $\psi(x,t) = f(x)\,g(t)$

$$\Rightarrow \quad \frac{i}{g}\frac{dg}{dt} = -\frac{1}{2f}\frac{d^2f}{dx^2}$$

- we want oscillations in $x$,

$\Rightarrow$ let $\frac{1}{f}\frac{d^2f}{dx^2} = -k^2$ ← sep const

$\Rightarrow \frac{1}{g}\frac{dg}{dt} = -\frac{ik^2}{2}$

- $\Rightarrow f \sim \begin{Bmatrix}\sin kx \\ \cos kx\end{Bmatrix}, \; g \sim \begin{Bmatrix} e^{-\frac{ik^2}{2}t} \end{Bmatrix}$

- $f(0) = 0, \; f(1) = 0$

$\Rightarrow f \to \sin(k_n x), \quad k_n \to n\pi$

$$\Rightarrow \boxed{\Psi(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-\frac{i}{2}(n\pi)^2 t}}$$

$$\text{i.c.} \Rightarrow h(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$\Rightarrow A_n$

$\Rightarrow \Psi(x,t)$

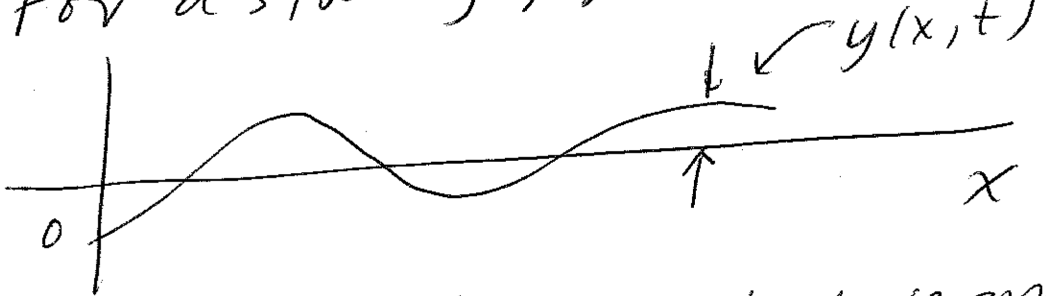
- Can find $|\Psi|^2$ by $|\Psi|^2 = \Psi\Psi^*$

# The Wave Equation

- The wave equation is a PDE that applies to strings under tension or to Electromagnetic waves

- In 1-D, the wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, for $y(x,t)$.

For a string, $y(x,t)$ is the displacement

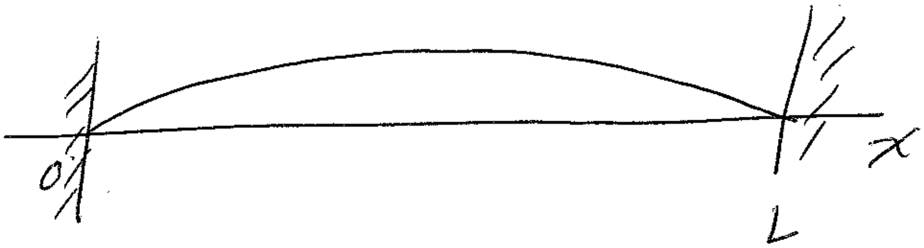



and $c$ is the speed of propagation; for a string, $c^2 = T/\mu$, $T=$ tension, $\mu =$ mass/unit length.

For EM waves, $c =$ speed of light

## Guitar string

Consider first the guitar string




Here, $y(0,t) = 0$, $y(L,t) = 0$.

Also, suppose the two initial conditions

$$\left.\begin{aligned} y(x,0) &= h(x) \\ (\partial y/\partial t)(x,0) &= 0 \end{aligned}\right\} \text{ plucked static string at } t=0$$

Proceed to solve the PDE as usual.

- $y(x,t) = f(x)\,g(t)$

$$\Rightarrow \frac{1}{g}\frac{d^2 g}{dt^2} = \frac{c^2}{f}\frac{d^2 f}{dx^2}$$

- We want $f(x)$ oscillatory, $\Rightarrow$

let $\dfrac{d^2 f}{dx^2} = -k^2 f$, separation constant and $\frac{d^2 g}{dt^2} = -(kc)^2 g$

$$\Rightarrow y(x,t) \sim \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} \begin{Bmatrix} \sin(kct) \\ \cos(kct) \end{Bmatrix}$$

- All 4 combinations are ok.

But we want $y(0,t)=0 \Rightarrow$ pick $\sin(kx)$, Also want $\sin(kL)=0$

$$\Rightarrow kL = n\pi \Rightarrow \boxed{k \to k_n = \frac{n\pi}{L}}$$

$$\Rightarrow y(x,t) \sim \sin(kx)\sin(kct), \quad \sin(kx)\cos(kct)$$

$$\Rightarrow \partial_t y(x,t) \sim kc\sin(kx)\cos(kct), \quad -\sin(kx)\sin(kct)$$

But $\partial_t y(x,0) = 0 \Rightarrow$ eliminate the first combo

$$\Rightarrow \text{Let } \boxed{y(x,t) = \sum_{n=1}^{\infty} a_n \sin(k_n x)\cos(k_n ct)}$$

Apply $y(x,0) = h(x) = \sum a_n \sin(k_n x)$

$\Rightarrow a_n \Rightarrow$ Solution

# Travelling waves

- Travelling waves are also solutions to the wave eqn. (depending on boundary conditions)

- Suppose we have an infinite string ⇒ can form various other combos from the earlier set, eg,

$$y \sim \begin{Bmatrix} \cos[k(x-ct)] \\ \sin[k(x-ct)] \end{Bmatrix} \begin{Bmatrix} \cos[k(x+ct)] \\ \sin[k(x+ct)] \end{Bmatrix}$$

or

$$y \sim \begin{Bmatrix} e^{ik(x-ct)} \\ e^{-ik(x-ct)} \end{Bmatrix} \begin{Bmatrix} e^{ik(x+ct)} \\ e^{-ik(x+ct)} \end{Bmatrix}$$

- $k(x-ct)$ combos propagate rightward
- $k(x+ct)$ " " leftward

W5

For example, consider $\lambda$ = given

$$y(x,0) = a \sin\left[\frac{2\pi x}{\lambda}\right]$$

$$(\partial y/\partial t)(x,0) = -akc\cos\left[\frac{2\pi}{\lambda}x\right]$$

Then, from the previous combos, try the sine combos for $y \Rightarrow$

$$y(x,t) = A\sin[k(x-ct)] + B\sin[k(x+ct)]$$

$$\Rightarrow \partial_t y(x,t) = -Akc\cos[k(x-ct)] + Bkc\cos[k(x+ct)]$$

$$\Leftarrow) \quad a\sin\left[\frac{2\pi x}{\lambda}\right] \equiv (A+B)\sin(kx)$$

$$\Longleftarrow \quad k = 2\pi/\lambda, \quad A+B = a$$

AND,

$$-akc\cos\left[\frac{2\pi x}{\lambda}\right] = (-A+B)\cos[kx]$$

$$\Rightarrow kc(-A+B) = -akc$$

From the circled eqns, $B=0$, $A=a$.

$$\Rightarrow \boxed{y(x,t) = a\sin\left[\frac{2\pi}{\lambda}(x-ct)\right]}$$

propagates rightward