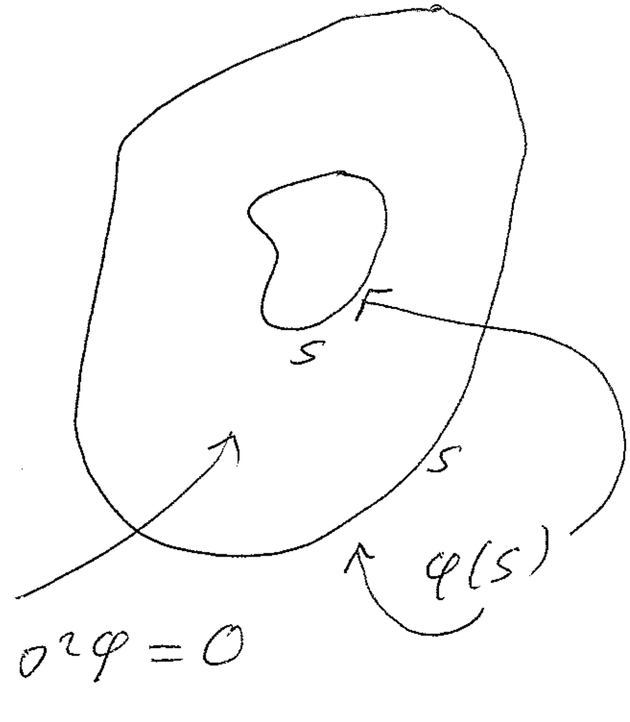
Boundary value problems for Laplace's Equation, $\nabla^2\varphi = 0$

- The general boundary value problem for Laplace was stated earlier, i.e.,

Find $\varphi(\vec{r})$ inside enclosed volume if $\nabla^2\varphi = 0$ in the volume and $\varphi = \varphi(S)$ = given on enclosing surface S



- We stated the Uniqueness Theorem earlier: A solution $\varphi(\vec{r})$ satisfying $\nabla^2\varphi = 0$ with $\varphi(S)$ = as given is the solution.

- For high degree of Symmetry, Laplace's equation can be reduced to an ODE immediately and integrated easily (see earlier notes).

For 2D or 3D problems (one or none symmetry directions), the solution is more involved. This is the focus of this section.

- We will do some 2D Boundary Value problems, by example. In general, there are 8–10 specific steps that need to be followed to solve such problems. We will list these steps.

11 Steps to Solve BV problems

① Write down the equation and Boundary Values

② Pick right coordinate system.
Write down PDE in these coords.

③ Use symmetries to eliminate some partials.

④ Separate variables

⑤ Introduce separation constant
$\Rightarrow$ PDE $\longrightarrow$ ODE's

⑥ Pick sign and form of separation constant for convenience. Sign depends on oscillatory vs evanescent behavior of solution. Solve ODE's.

⑦ Eliminate some combinations based on boundary conditions, no blow-ups, and any periodic solutions.

(8) Quantize the separation constant based on boundary conditions in the "oscillatory" direction.

(9) Make linear combination, with arbitrary coeffs, using all remaining solutions.

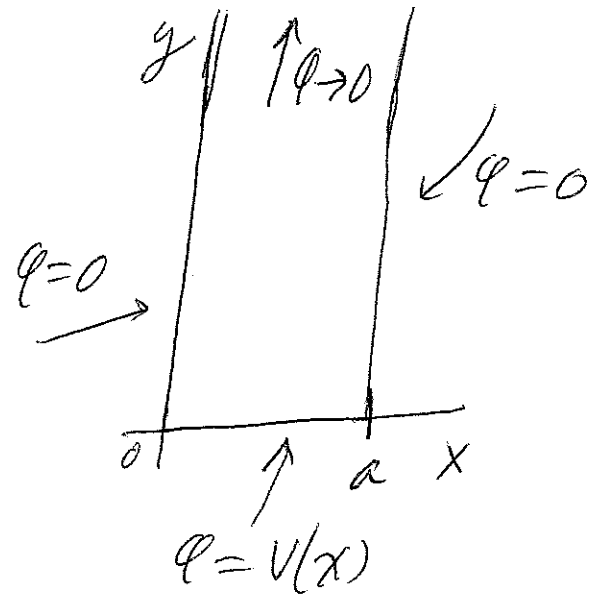
(10) Apply any remaining B.C.'s.

(11) Find the arbitrary coeffs using Fourier's trick.

Example ①

① $\nabla^2\varphi = 0$

$\varphi(s)$ as shown →



② Use Cartesian ⇒

$$\partial_x^2\varphi + \partial_y^2\varphi + \partial_z^2\varphi = 0$$

③ Clearly $\partial_z = 0$ ⇒

$$\boxed{\partial_x^2\varphi + \partial_y^2\varphi = 0\ , \quad \varphi(x,y)} \quad (1)$$

④ Let $\varphi(x,y) = f(x)g(y)$

plug in (and divide by φ) ⇒

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{fn(x)} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{fn(y)} = 0 \quad — (2)$$

* true $\forall x, y$.

(2) is completely separated ⇒ each

⑤ Separate fn is a constant, i.e.,

$$\frac{1}{f}\frac{d^2f}{dx^2} = C, \quad \frac{1}{g}\frac{d^2g}{dy^2} = -C.$$

B6

(6) We want oscillation in x (there are $\geq$ zeroes in x).

Since $\frac{d^2f}{dx^2} = Cf$ gives oscillatory solutions for $C<0$ and evanescent solutions for $C>0$ $\Rightarrow$ Pick $C<0$. Also, let $C=-k^2$, more convenient form $\Rightarrow$

$$\boxed{\frac{d^2f}{dx^2} = -k^2 f, \quad \frac{d^2g}{dy^2} = k^2 g} \quad (3)$$

$$\Rightarrow f(x) \sim \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} \quad g(y) \sim \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix}$$

(7) $\Rightarrow$ 4 possible combos for fg.
But we want $\varphi \to 0$ as $y \to \infty$
$\Rightarrow$ pick only e^{-ky}, $k>0$.
We also want $\varphi = 0$ for $x = 0$
$\Rightarrow$ pick only $\sin(kx)$.

Thus, only the combo

$$\varphi(x,y) \sim \{\sin kx\}\{e^{-ky}\},$$

$k>0$, survives

(8) But $\varphi(x=a) = 0$.

Thus, $\sin(ka) = 0 \Rightarrow$

$ka = n\pi$

$$\boxed{\begin{aligned} &n = 1,2,3,4\ldots \\ \Rightarrow\ &k_n = n\pi/a \end{aligned}} \quad (4)$$

$\leftarrow$ quantitation

(9) Thus, most general solution is

$$\boxed{\varphi(x,y) = \sum_{n=1}^{\infty} A_n e^{-k_n y} \sin(k_n x)} \quad (5)$$

(10) But $\varphi(x,0) = V(x) = $ given

$$\Leftrightarrow \boxed{V(x) = \sum_{n=1}^{\infty} A_n \sin(k_n y)} \quad (6)$$

$\uparrow$ arb constants

(11) But the sines are a complete set — a Fourier series.

$\Rightarrow$ use Fourier's trick to find A_n

$$\Rightarrow \boxed{A_n = \frac{\int_0^a dx\, V(x) \sin(k_n x)}{\int_0^a dx \sin^2(k_n x)}} \qquad (7)$$

$\therefore$ Solution is as in (5')

$$\boxed{\begin{aligned} \varphi(x,y) &= \sum_{n=1}^{\infty} A_n e^{-k_n y} \sin(k_n x) \\ k_n &= n\pi/a \end{aligned}} \qquad (8)$$

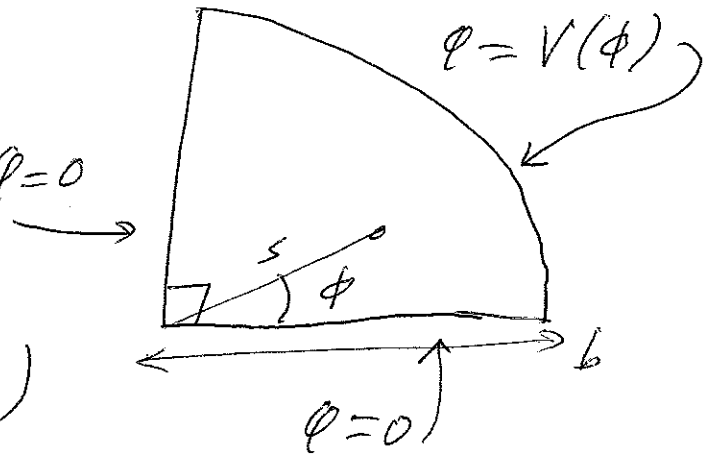
with A_n given by (7)

Example ② (Polar coords)

① $\nabla^2 \varphi = 0$

$\varphi(s)$ as shown

(quadrant of cylinder)



② Use cylindrical coords, $\{s, \phi, z\} \Rightarrow$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \varphi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

③ clearly, $\partial/\partial z = 0 \Rightarrow$

$$\boxed{\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \varphi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 \varphi}{\partial \phi^2} = 0} \quad (\text{T})$$

④ Let $\varphi(s,\phi) = f(s)\,g(\phi)$

$$\times s^2 \Rightarrow \quad \underbrace{\frac{s}{f}\frac{d}{ds}\left(s\frac{df}{ds}\right)}_{fn(s)} + \underbrace{\frac{1}{g}\frac{d^2 g}{d\phi^2}}_{f(\phi)} = 0$$

⑤ $\Rightarrow \quad s\frac{d}{ds}\left(s\frac{df}{ds}\right) = Cf, \quad \frac{d^2 g}{d\phi^2} = -Cg$

(6) Need oscillations in ϕ

$\Rightarrow$ let $c = k^2$

$$\Rightarrow \quad s\frac{d}{ds}\left(s\frac{df}{ds}\right) = k^2 f, \quad \frac{d^2 g}{d\phi^2} = -k^2 g$$

$$\Rightarrow \quad g(\phi) \sim \begin{Bmatrix} \sin k\phi \\ \cos k\phi \end{Bmatrix}$$

equidimensional in s $\Rightarrow$ $f \sim s^\alpha$

$$\Leftrightarrow \quad \alpha^2 = k^2 \Rightarrow \alpha = \pm k$$

$$\Rightarrow \quad f(s) \sim \begin{Bmatrix} s^k \\ s^{-k} \end{Bmatrix}$$

(7) Want $g(\phi = 0) = 0$ $\Rightarrow$ eliminate $\cos(k\phi)$

Want φ to be well behaved

$\Rightarrow$ eliminate s^{-k}, for $k \geq 0$

Thus, use the combo

$$\varphi(s, \phi) \sim s^k \sin(k\phi)$$

$$k > 0$$

(8) But $\varphi(\phi = \pi/2) = 0$

$\Rightarrow \sin(k\pi/2) = 0$

$\Leftrightarrow k\pi/2 = n\pi, \quad n = 1, 2, \ldots$

$\Rightarrow \boxed{k_n = 2n, \quad n = 1, 2, 3, \ldots}$

quantization

(9) Thus, most generally

$$\boxed{\varphi(s, \phi) = \sum_{n=1}^{\infty} A_n s^{2n} \sin(2n\phi)} \quad (10)$$

(10) But $\varphi(b, \phi) = V(\phi)$

$$\Leftrightarrow V(\phi) = \sum_{n=1}^{\infty} \underbrace{A_n b^{2n}}_{\text{coefficient } B_n} \sin(2n\phi)$$

(11) Fourier's trick $\Rightarrow$

$$B_n = \frac{\int_0^{\pi/2} d\phi\, V(\phi) \sin(2n\phi)}{\int_0^{\pi/2} d\phi\, \sin^2(2n\phi)} \Rightarrow B_n \Rightarrow A_n$$

Solution

$$\boxed{\varphi(s,\phi) = \sum_{n=1}^{\infty} B_n \left(\frac{s}{b}\right)^{2n} \sin(2n\phi)}$$

$$B_n = \frac{\int_0^{\pi/2} d\phi \; V(\phi) \sin(2n\phi)}{\int_0^{\pi/2} d\phi \; \sin^2(2n\phi)}$$