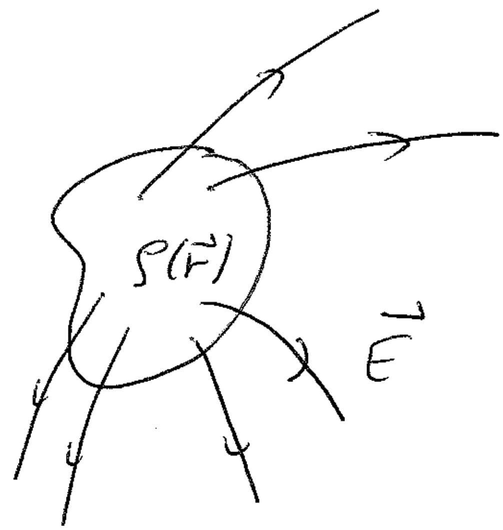
Field eqns for Electrostatic (& Gravitational) Fields

$\rho(\vec{r})$ = given charge density



$\vec{E}(\vec{r})$ = electric field

(1) $\boxed{\vec{\nabla} \cdot \vec{E} = \rho}$ Field equation (normalized)

$\rho \Rightarrow$ faucets in $\vec{E} \Rightarrow \vec{E}$

We also know $\vec{E}$ is conservative

ie, (2) $\boxed{\vec{E} = -\vec{\nabla}\varphi}$

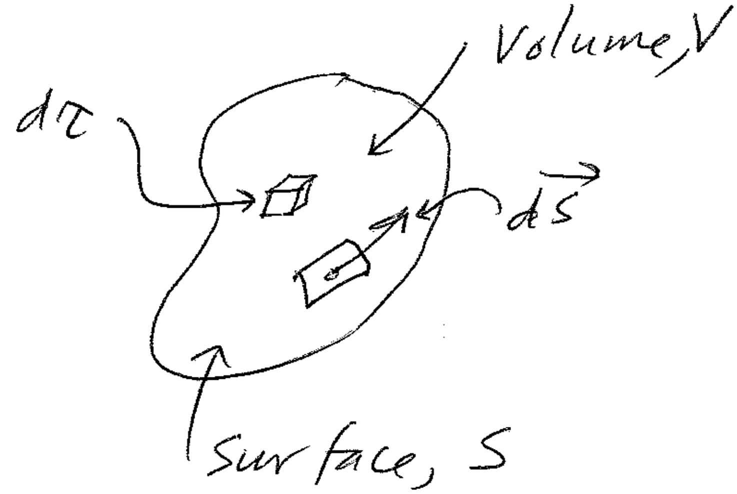
Substitute (2) $\rightarrow$ (1) $\Rightarrow$ $\boxed{\nabla^2\varphi = -\rho}$ (3)

Thus, $\rho \Rightarrow \varphi$

For Gravitational fields

$\rho(\vec{r})$ = mass density, $\vec{E} \rightarrow \vec{F}$ = grav. field; $\vec{F} = -\vec{\nabla}\varphi_G$. SAME eqns (normalized)

Enclosed Charge Theorem



Eqn ① can be rewritten as

$$\boxed{\oint_S \vec{E}\cdot d\vec{s} = (\text{Charge enclosed in volume } V) \equiv \int_V d\tau\, \rho}$$

<u>Proof</u> $\quad \vec{\nabla}\cdot\vec{E} = \rho \quad$ — ①

Gauss Theorem: $\int_V \vec{\nabla}\cdot\vec{E}\, d\tau \equiv \oint_S d\vec{s}\cdot\vec{E}$

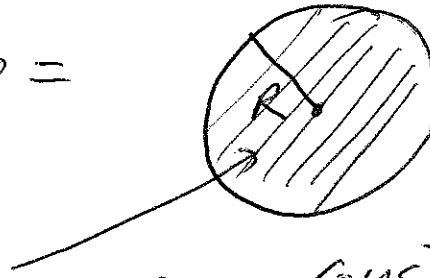
Apply $\int_V d\tau$ to both sides of ①

$$\Rightarrow \oint_S \vec{E}\cdot d\vec{s} = \int_V d\tau\, \rho .$$

QED

Enclosed Charge Theorem gives quick results for $\vec{E}$ if $\rho(\vec{r})$ is highly symmetric

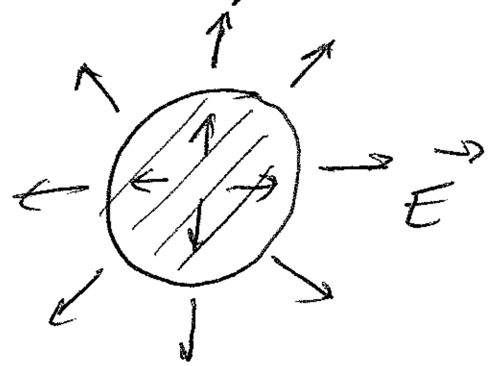
Example suppose $\rho =$



$\rho = \rho_0 =$ constant

To find $\vec{E}$

- First, $\vec{E}$ must be central, by symmetry and "Faucet physics".



- ie, $\vec{E}$ must point along $\vec{r}$ and $|\vec{E}|$ must be const for given radius r.

- Making above assumption, the LHS of the theorem becomes easy to evaluate if the chosen surface is spherical, ie, consider

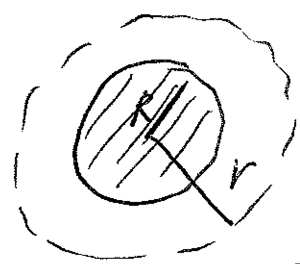
then $\oint_S \vec{E} \cdot d\vec{S} = \int |\vec{E}| dS$

$= |\vec{E}| \int dS = |\vec{E}| 4\pi r^2 = E 4\pi r^2$

- For the same surface, RHS = charge enclosed in 

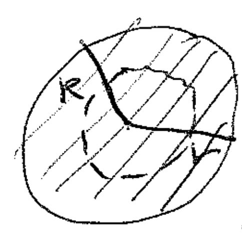
- Apply these to  $\rho = \rho_0$ problem

- There are two possible surfaces 

First, $r > R \Rightarrow$ 

then, LHS = RHS

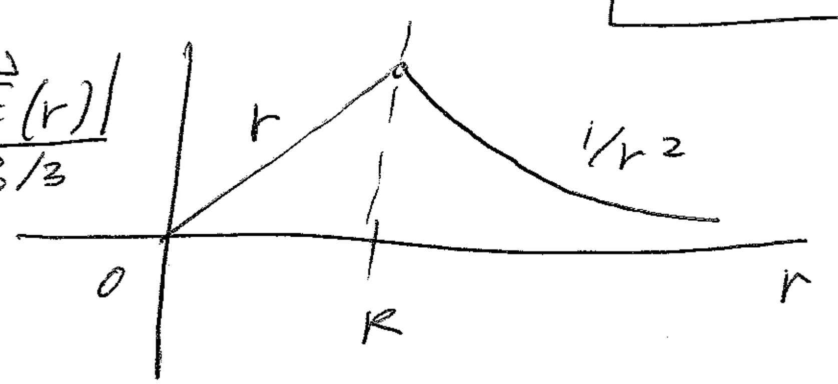
$$\Rightarrow E_{out} 4\pi r^2 = \rho_0 \frac{4}{3}\pi R^3 \Rightarrow \boxed{E_{out} = \frac{\rho_0 R^3}{3r^2}}$$

Next, $r < R \Rightarrow$ 

LHS = RHS

$$\Rightarrow E_{in} 4\pi r^2 = \rho_0 \frac{4}{3}\pi r^3 \Rightarrow \boxed{E_{in} = \frac{\rho_0}{3} r}$$

So $\dfrac{|\vec{E}(r)|}{\rho_0/3}$



φ can be found from $\vec{E}$

eg, $\vec{E}_{in} = \frac{\rho_0}{3} \vec{r}$

if $\vec{E}_{in} = -\vec{\nabla} \varphi_{in}$

$\Rightarrow -\vec{\nabla} \varphi_{in} = \frac{\rho_0}{3} \vec{r}$

$\Rightarrow \frac{\partial \varphi_{in}}{\partial x} = -\frac{\rho_0}{3} x$, etc

Cartesian coords

as shown earlier $\Rightarrow \boxed{\varphi_{in} = \frac{-\rho_0}{6} r^2} + \text{const}$

In Spherical coords: $\frac{\partial \varphi_{in}}{\partial r} = \frac{-\rho_0}{3} r$

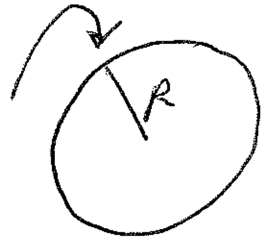
$\Rightarrow \boxed{\varphi_{in} = -\frac{\rho_0}{6} r^2 + \text{const}}$

Likewise, $\vec{E}_{out} = \frac{\rho_0}{3} \frac{R^3}{r^2}$

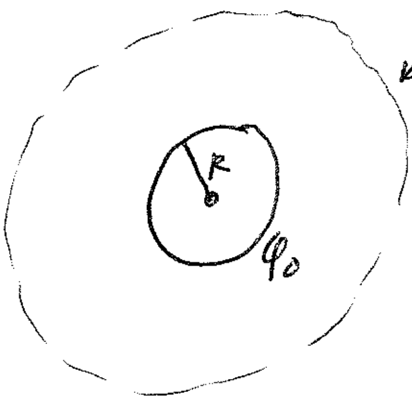
$\Rightarrow \frac{\partial \varphi_{out}}{\partial r} = -\frac{\rho_0}{3} \frac{R^3}{r^2}$

$\Rightarrow \boxed{\varphi_{out} = +\frac{\rho_0}{3} \frac{R^3}{r}}$ Note $\varphi \to 0$ as $r \to \infty$

Solution of $\varphi(\vec{r})$ as Boundary Value Problem for high symmetry situation

- Suppose $\varphi = \text{constant} = \varphi_0$ on 

and $\varphi \to 0$ as $|\vec{r}| \to \infty$

$\Rightarrow$ 

$\nabla^2 \varphi = 0$

(there is no charge for $r > R$)

- Use Spherical coordinates

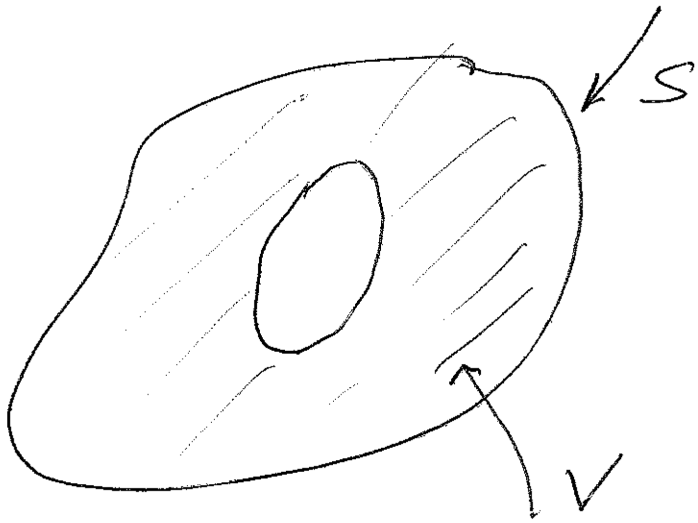
$$\Rightarrow \nabla^2 \to \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

But Symmetry $\Rightarrow \partial/\partial\theta = 0$ & $\partial/\partial\phi = 0$

$$\therefore \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\varphi}{dr}\right) = 0 \Rightarrow r^2 \frac{d\varphi}{dr} = C$$

$$\Rightarrow \varphi = -\frac{C}{r} + D \,. \quad \begin{array}{l} \varphi(r \to \infty) \to 0 \Rightarrow D = 0 \\ \varphi(r = R) = \varphi_0 \Rightarrow \boxed{\varphi = +\varphi_0\left(\frac{R}{r}\right)} \end{array}$$

General Boundary Value problem for $\varphi(\vec{r})$ $(\rho = 0)$



- $\nabla^2 \varphi = 0$

$\varphi(\text{Surface}) = \text{Given}$

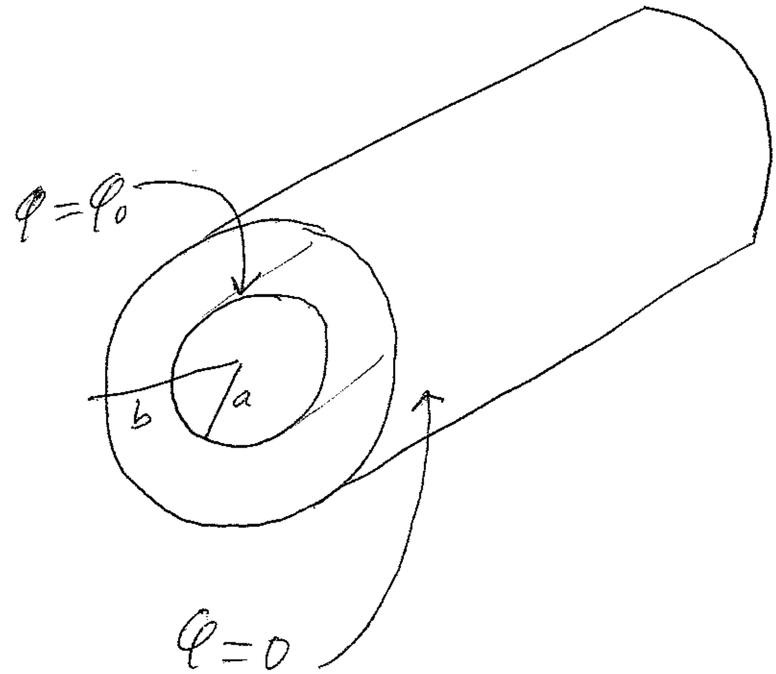
(both surfaces)

$\Rightarrow$ Solve $\nabla^2 \varphi = 0$ inside the volume and adjust φ to satisfy BC

$\Rightarrow$ $\varphi = \varphi(\vec{r})$ inside

- Uniqueness Theorem: if a solution exists $\ni \nabla^2 \varphi = 0$ and $\varphi = \varphi(S)$ on boundary, then it is *the* solution. (no proof)

High Symm BV Plm in Cyl Coords



- $\nabla^2\varphi = 0$, Find φ in between cyls.
- cylindrical $\Rightarrow$ $\nabla^2 \to \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$
- symmetry $\Rightarrow$ $\frac{\partial}{\partial\phi} = 0$, $\frac{\partial}{\partial z} = 0$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = 0$$

$\Rightarrow$ $r\,d\varphi/dr = C$ $\Rightarrow$ $\varphi = C\ln r + D$

$$\left.\begin{aligned}\varphi(a) &= \varphi_0 = C\ln a + D\\ \varphi(b) &= 0 = C\ln b + D\end{aligned}\right\rangle \Rightarrow C \text{ and } D$$

$$\Rightarrow \boxed{\varphi(r) = \varphi_0\frac{\ln(r/b)}{\ln(a/b)}}$$