# Vector Spaces, Operators, EigenProblems, and Complete sets

First, begin by illustrating with 2-component vectors and their space.

- <u>Vectors</u> $\quad x = \begin{pmatrix} a \\ b \end{pmatrix}$ ; $\{a, b\}$ complex #s

- <u>Space</u> $\quad x \subset V, \; y \subset V$

  $\Rightarrow \; x + y \subset V \quad \Rightarrow$ vector space

  For complex scalar $\lambda$, and vector $x$,

  $\lambda x \subset V$

- <u>Inner product</u>

$$(x, y) \equiv (a \; b)^* \begin{pmatrix} c \\ d \end{pmatrix} \equiv a^* c + b^* d = \text{scalar}$$

<u>Note</u> $\quad (x, y)^* = (y, x)$

$$(x_1, x_2) = 0 \Rightarrow x_1, x_2 \text{ "orthogonal".}$$

$$(x, x) = a^* a + b^* b = |a|^2 + |b|^2 = \text{"norm".}$$

- **Operators**

$$Ax = y. \quad e.g.\ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$$

- **Eigen Problem**

Suppose $A\varphi = \lambda\varphi$.

Does $\varphi$ exist for given $A$?

In general, $\varphi$ exist for special $\lambda$'s

Thus, $\boxed{A\varphi_n = \lambda_n \varphi_n}$ where $n = 1, 2, 3, \cdots$

$\{\varphi_n\} \rightarrow$ eigenvectors

$\{\lambda_n\} \rightarrow$ eigenvalues

**Example** $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$A\varphi = \lambda\varphi \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -(1+\lambda) \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1+\lambda) = 0$$

$$\Rightarrow \lambda = \pm 1$$

$$\lambda = +1 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \varphi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1 \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow \varphi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \{\varphi_+, \varphi_-\}$ are e'vectors

$\{+1, -1\}$ are e'values.

## • Self-Adjoint (Hermitean) Operators

Suppose $A \ni (y, Ax) = (Ay, x)$

$\Rightarrow$ A is special, called self-adjoint.

So, what restrictions are put on $A$? $= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$

Try it: $x = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow Ax = \begin{pmatrix} \alpha_1 a + \alpha_2 b \\ \alpha_3 a + \alpha_4 b \end{pmatrix}$

$y = \begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow (y, Ax) = \alpha_1 ac^* + \alpha_2 bc^* + \alpha_3 ad^* + \alpha_4 bd^*$

Likewise, $(Ay, x) = \alpha_1^* ac^* + \alpha_2^* ad^* + \alpha_3^* bc^* + \alpha_4^* bd^*$

$\therefore (y, Ax) = (Ay, x) \Rightarrow ac^*(\alpha_1 - \alpha_1^*) + bd^*(\alpha_4 - \alpha_4^*)$
$+ bc^*(\alpha_2 - \alpha_3^*) + ad^*(\alpha_3 - \alpha_2^*) = 0$

But this is true $\forall \{x, y\}$, i.e., $\forall (a, b, c, d)$.

Thus, e.g., $b = d = 0$, $a \neq 0$, $c \neq 0 \Rightarrow \alpha_1 = \alpha_1^*$.

Likewise, $\alpha_4 = \alpha_4^*$, $\alpha_2 = \alpha_3^* \Rightarrow \alpha_1 =$ real, $\alpha_4 =$ real, $\alpha_2 = \alpha_3^*$

$\Rightarrow$ A must have the form $\boxed{A = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}, \ \alpha \text{ real}, \ \beta \text{ real}}$ SA

- Theorem for H operators, their eigenvectors and eigenvalues

Theorem: H hermitean (or self-adjoint)

and $H\varphi_n = \lambda_n \varphi_n \Rightarrow \left\{ \begin{array}{l} \lambda_n \text{ are real} \\ \varphi_n \text{ are orthogonal} \end{array} \right\}$

(true for non-degenerate $\lambda_n$; but can be extended to degenerate $\lambda_n$)

Proof: (assume non-degenerate $\lambda_n$)

$$H\varphi_n = \lambda_n \varphi_n$$

$$\Rightarrow (\varphi_m, H\varphi_n) = \lambda_n (\varphi_m, \varphi_n)$$

complex conj $\Rightarrow (\varphi_m, H\varphi_n)^* = \lambda_n^* (\varphi_m, \varphi_n)^*$

$$\Rightarrow (H\varphi_n, \varphi_m) = \lambda_n^* (\varphi_n, \varphi_m)$$

interchange $m \leftrightarrow n$ $\Rightarrow (H\varphi_m, \varphi_n) = \lambda_m^* (\varphi_m, \varphi_n)$

H hermitean $\Rightarrow (\varphi_m, H\varphi_n) = \lambda_m^* (\varphi_m, \varphi_n)$

Compare 1st & last $\Rightarrow (\lambda_n - \lambda_m^*)(\varphi_m, \varphi_n) = 0$

Now $n \neq m \Rightarrow \lambda_n \neq \lambda_m^* =$

$n = m \Rightarrow (\varphi_m, \varphi_n) = (\varphi_m, \varphi_m) > 0$

$\Rightarrow \lambda_n = \lambda_n^* \Rightarrow \boxed{\lambda_n \text{ real}}$

$n \neq m \Rightarrow \lambda_n \neq \lambda_m \Rightarrow \boxed{(\varphi_m, \varphi_n) = 0}_{m \neq n}$

QED

- Completeness Theorem

(No Proof)

The eigenvectors $\{\varphi_n\}$ of a Hermitean operator $H$ form a complete set, i.e.,

$$H\varphi_n = \lambda_n \varphi_n \Rightarrow X = \sum_n a_n \varphi_n, \ \forall X \in V$$

Example $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is hermitean

$$H\varphi_n = \lambda_n \varphi_n \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow \lambda_\pm = \pm 1$ and $\varphi_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

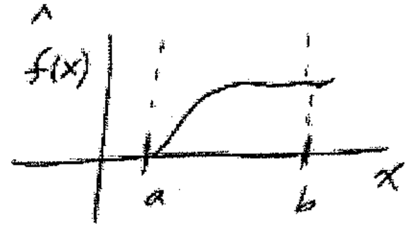
as before

$$\Rightarrow X = \sum_n c_n \varphi_n \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

## Extension of Completeness Theorem to Infinite-dimensional function space (Hilbert Space)

We now consider a space with vectors which are functions of $x$, as follows.
(Compare with the 2-component space earlier)
in $[a,b]$

- **Vectors** $f(x)$ is a vector.




Further
$$\int_a^b dx\, |f(x)|^2 < \infty$$

and $f(a) = 0$ OR $f'(a) = 0$
$f(b) = 0$ OR $f'(b) = 0$

OR $f$ is periodic in $[a,b]$, ie, $f(b) = f(a)$, $f'(b) = f'(a)$

- **Space**
for any $f(x)$, $g(x)$ in this space,
clearly $f+g$ is also in the space.

Thus, the space of square-integrable fns in $[a,b]$ satisfying certain b.c.'s is a vector space.

**Note** $\lambda$, complex scalar, $\Rightarrow \lambda f$ also in space.

- Inner Product $(f,g) \equiv \int_a^b dx\, f^*(x)\, g(x)$ $\forall x$

Note $(f,g)^* = (g,f)$

[Note: Inner Product can be generalized to include a "weight fn" – do this later]

- Operators
  example: $A = \frac{d}{dx}$ is an operator
  $$\mathbf{A} f = \frac{df}{dx} = \text{another function}$$

- Eigen Problem

Suppose $\boxed{A = \frac{d^2}{dx^2}}$

let $A\varphi = \lambda \varphi$

Suppose $f(0) = 0$, $f(b) = 0$

$\Rightarrow$ $\frac{d^2\varphi}{dx^2} = \lambda \varphi$, $\varphi(0) = 0$, $\varphi(b) = 0$

$$\Rightarrow \boxed{\varphi_n = \sin(n\pi x/b), \; \lambda_n = -\left(\frac{n\pi}{b}\right)^2}$$

$n = 0, 1, 2, 3, \cdots$

- Hermitean Operators

$H$ is hermitean $\Leftrightarrow (f, Hg) = (Hf, g)$

ie, $\int_a^b dx\, f^* Hg = \int_a^b dx\, (Hf)^* g$ .

We may now demand this to obtain conditions on a general, differential operator. This is possible. But here we will treat only specific simple examples, done later.

[Note: $A = d/dx$ is not Hermitean for periodic b.c.]

- Theorem for H operators

This holds. i.e., $H$ hermitean,

$$\& \ H\varphi_n = \lambda_n \varphi_n \Rightarrow \left\{ \begin{array}{l} \lambda_n \text{ are real} \\ \varphi_n \text{ are orthogonal} \end{array} \right\}$$

- Completeness Theorem

Also holds.* i.e., for any $f(x)$ in the space,

$$f(x) = \sum_n c_n \varphi_n(x)$$

where $\varphi_n(x)$ are the eigenvectors of a given $H$.

* Completeness is defined as "in the mean", to be discussed.

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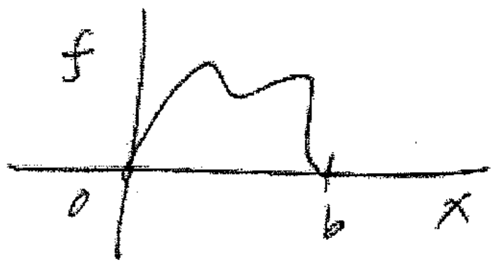
# Example of a complete set from a Hermitean operator in Hilbert space

- Space of $f(x)$ on $[0, b]$ with $f(0)=0$, $f(b)=0$




- Consider $H = d^2/dx^2$.

Clearly, $H$ is Hermitean since

$$(f, Hg) \equiv \int_0^b dx\, f^* \frac{d^2 g}{dx^2}$$

$$= \underbrace{\left[f^* \frac{dg}{dx}\right]_0^b}_{=0} - \int_0^b dx\, \frac{df^*}{dx}\frac{dg}{dx}$$

$$= \underbrace{-\left[\frac{df^*}{dx} g\right]_0^b}_{=0} + \int_0^b dx\, \frac{d^2 f^*}{dx^2} g$$

$$= (Hf, g)$$

- Do the <u>eigenproblem</u> for $H$

$$H\varphi = \lambda\varphi \;; \quad \varphi(0)=0,\ \varphi(b)=0$$

$$\Rightarrow \frac{d^2\varphi}{dx^2} = -k^2\varphi \,, \quad \lambda = -k^2 \text{ for convenience}$$

$$\Rightarrow \boxed{\begin{array}{l} \varphi_n(x) = \sin(k_n x)\,, \quad k_n = n\pi/b \\ \qquad \lambda_n = -k_n^2 \end{array}}$$

Note: $\lambda_n$ are <u>real</u>

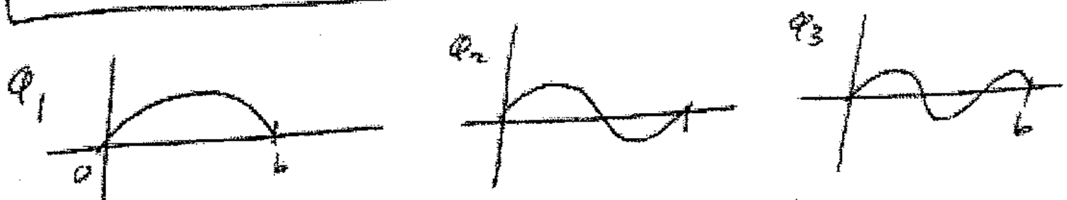
$\varphi_n$ are <u>orthogonal</u> (as expected)

since $\int_0^b dx \sin(k_n x)\sin(k_m x) = 0, \; n \neq m$

$\Rightarrow \{\varphi_n\} = \{\sin(k_n x)\}$ form a complete set

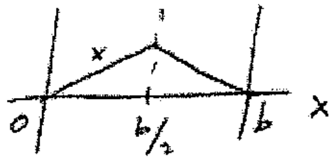
ie, <u>any</u> $f(x)$

$$\boxed{f(x) = \sum_n c_n \sin(k_n x)} \quad k_n = \frac{n\pi}{b}$$



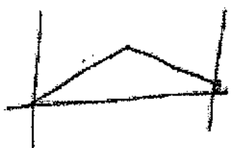

This is a Fourier Series. Other series can be found from other H's.

## Specific examples

(E1) Suppose $f(x) =$ 


and $f(x) = \sum_{n=1}^{\infty} c_n \sin(k_n x)$, $k_n = n\pi/b$

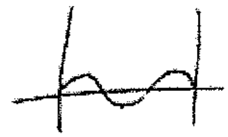
We can see how this will work, i.e.,



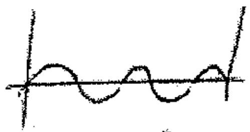
$f$

$=$ 

$c_1 \varphi_1$

$+$ 

$c_3 \varphi_3$

$+$  $+ \cdots$

$c_5 \varphi_5$

i.e.,  $\backsim$ , etc

Note: only odd n's survive.

This is because $f$ is symmetric about $b/2$, so, e.g., 

$\varphi_2$

will not survive

Calculate $c_n$ explicitly:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(k_n x)$$

$$\Rightarrow \left[\sin(k_m x), f\right] = \sum_n c_n \left[\sin(k_n x), \sin(k_m x)\right]$$

$$= \sum_n c_n \delta_{mn} \left[\sin(k_n x), \sin(k_n x)\right]$$

$$= c_m \left[\sin(k_m x), \sin(k_m x)\right]$$

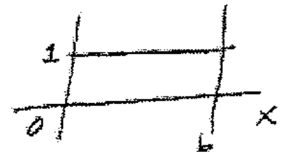
$$\Rightarrow \int_0^b dx \sin(k_m x) f(x) = c_m \int_0^b dx \sin^2(k_m x)$$

$$\text{RHS} = c_m \frac{b}{2}$$

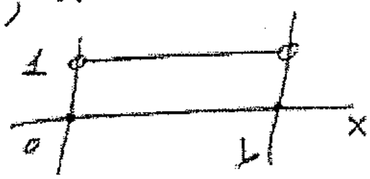
$$\text{LHS} = 2\int_0^{b/2} dx \sin(k_m x)\, x \quad \text{, by symmetry}$$

$$\Rightarrow \boxed{c_m = \frac{4}{b} \int_0^{b/2} dx\, x \sin(k_m x)}$$

which gives $c_m \Rightarrow$ series.

E② Suppose $f(x) = 1 =$ 


We want to expand this in $\varphi_n = \sin(k_n x)$.
But $f(x) = 1$ is <u>NOT</u> in the vector space whose e-fns. are $\varphi_n$ as above.
This is because, for that space, $g(0) = 0$ and $g(b) = 0$ $\forall\, g(x)$.

To circumvent this, we define a new function $\bar{f}(x) =$ 


which is just $f(x)$ except for 2 points at 0 and b where $\bar{f}(0) = 0$ & $\bar{f}(b) = 0$.
Now, $\bar{f}$ is in the space — the space allows piecewise continuous fns. Further, the completeness theorem holds for piecewise-continuous fns.

Thus, we have

$$\bar{f}(x) = \sum c_n \sin(k_n x)$$

$$\Rightarrow \int_0^b dx\, \bar{f}(x) \sin(k_n x) = c_n \frac{b}{2}$$

But LHS $\longrightarrow \int_0^b dx \sin(k_n x)$

$$= -\left[\frac{\cos(k_n x)}{k_n}\right]_0^b = \frac{2}{k_n}, \quad n \text{ odd}$$

$$\Rightarrow \quad c_n = 4/n\pi, \quad n \text{ odd}$$

$$\Rightarrow \boxed{f(x) \doteq \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{b}\right)}$$

this means completeness "in the mean."