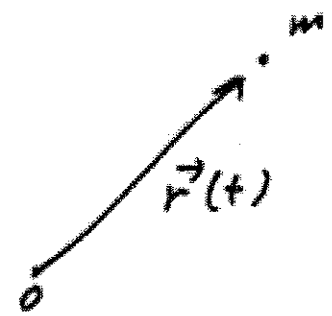
Other Examples of 3D Motion in Fields $\vec{F}$

<u>Given</u> $\vec{F}(\vec{x}, \vec{v}, t)$



$$\boxed{\begin{aligned} m\frac{d\vec{v}}{dt} &= \vec{F} \\ \frac{d\vec{r}}{dt} &= \vec{v} \end{aligned}}$$

Find $\vec{r}(t)$

<u>Example ①</u>

$$\vec{F} = r^2 \vec{r} \qquad \text{(normalized units)}$$

$\Rightarrow \quad \vec{L} = m\vec{r} \times \vec{v} = \text{const.}$

$\Rightarrow \quad U$ can be obtained since $\vec{\partial} \times \vec{F} = 0$

e.g., $(\vec{\partial} \times \vec{F})_z = \partial_x F_y - \partial_y F_x$. $\quad F_x = x(x^2+y^2)$
$\quad F_y = y(x^2+y^2)$

$$\Rightarrow \partial_x F_y - \partial_y F_x = 2xy - 2yx = 0$$

By inspection, $U = -r^4/4$
since $-\vec{\nabla} U = +r^3 \hat{r} \;\Rightarrow\; \vec{F} = -\vec{\nabla} U$.

Thus, $\mathcal{E} = \frac{1}{2} m v^2 + U = \text{constant}$

From $\vec{L}$ and $\mathcal{E}$ constancy, we have the following:

(1) motion is confined to a plane
   → let this be the x-y plane
   → use polar coordinates

(2) $|\vec{L}| = \text{constant}$; use $\vec{r} = r\hat{r}$
$$\Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\Rightarrow |\vec{L}| = |m\vec{r} \times \vec{v}| = m r^2 \dot{\theta} = \text{const}$$

$$\Rightarrow \boxed{m r^2 \dot{\theta} = L_0} \quad (1.1)$$

(3) $\mathcal{E} = \text{constant}$ and $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$
and $U = -r^4/4$,

$$\Rightarrow \boxed{\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{r^4}{4} = \mathcal{E}} \quad (1.2)$$

From (1.1) and (1.2), can solve for $r(t)$ and $\theta(t)$.

Example ②

Suppose $\vec{F} = \vec{r}\, f(\vec{r})$

$\Longrightarrow$ $\vec{L} = \text{const}$

$\Longrightarrow$ motion in 2-D plane, say (x, y)

Suppose $f(\vec{r}) = f(x, y, z)$

$$\longrightarrow 1 + \epsilon x^2$$

where $x = r\cos\theta$ in polar coordinates

$$\Longrightarrow \quad \vec{F} = \vec{r}\,[1 + \epsilon r^2 \cos^2\theta] \quad \text{in polar coordinates}$$

$$= \vec{r}\,[1 + \epsilon x^2]$$

Is $\vec{F}$ conservative?

Try $(\vec{\nabla} \times \vec{F})_z = \partial_x F_y - \partial_y F_x$

$$F_x = x\,[1 + \epsilon x^2], \quad F_y = y\,[1 + \epsilon x^2]$$

$$\Longrightarrow \partial_x F_y - \partial_y F_x = 2\epsilon xy - 0 = 2\epsilon xy \neq 0$$

$\Longrightarrow$ $\vec{F} \neq -\vec{\nabla} U$

and $\mathcal{E}$ cannot be defined.

$\therefore$ Can't use $\mathcal{E}$ constancy.

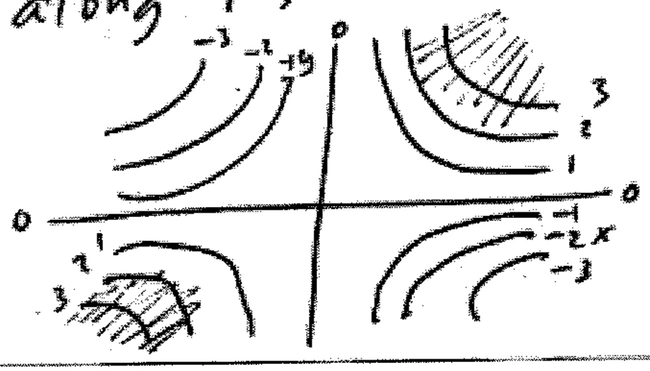
Thus, for this example, we have motion in 2D x-y plane, $|\vec{L}|$ = constant can be used, but E, as defined by $E = \frac{1}{2}mv^2 + U$, is not constant — in fact, "U" is not even defined.

As it turns out, further progress can be made – but we will not pursue this here.

Example ③

Suppose $\vec{F} = -\vec{\nabla} U$, where $U = xy$, x, y are Cartesian coordinates. Assume that motion is confined to x-y plane, for simplicity.

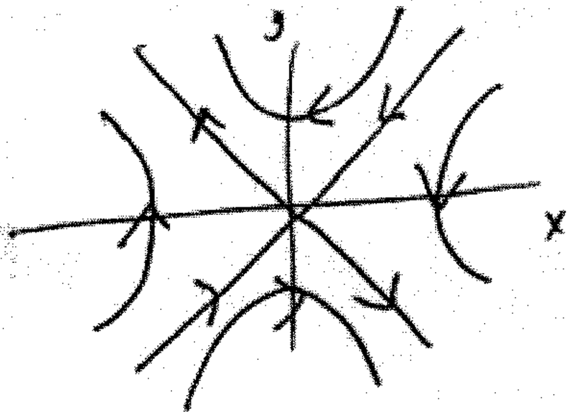
Now, $\vec{F}$ is not central, ie, not along $\vec{r}$. Note that contours of U are



$U(x,y) = xy$

shaded ⇒ high

$\Rightarrow$ $-\vec{\nabla} U$ looks like



This is the field $\vec{F}$. It is clearly not central.

$\Rightarrow$ $|\vec{L}|$ is not a constant.

But $\vec{F}$ is conservative and given as $-\vec{\nabla} U$

$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2} m v^2 + xy = \text{const}}$$

Thus, $\mathcal{E}$ constancy can be used.

Let us write down the equations for the particle using Cartesian coordinates

[note: "natural" coordinates are ~~the~~ hyperbolic, but we will not use these here.]

Thus, let $\vec{r}(t) = \hat{x}\, x(t) + \hat{y}\, y(t)$

$$\Rightarrow \vec{v}(t) = \dot{\vec{r}} = \hat{x}\dot{x} + \hat{y}\dot{y}$$

$m=1$

$$\Rightarrow \boxed{\ddot{x} = -y, \quad \ddot{y} = -x} \longrightarrow \text{equations of motion}$$

Coupled equations. To uncouple, differentiate twice more & substitute $\Rightarrow$

$$\boxed{\ddddot{x} = +x} \Rightarrow 4^{\text{th}} \text{ order ODE}$$

linear
const coeffs

$\Rightarrow$ try $x(t) = e^{\alpha t}$

$$\Leftrightarrow \alpha^4 = 1 \Rightarrow \alpha^2 = \pm 1 \Rightarrow \alpha = (\pm 1, \pm i)$$

$$\Rightarrow x(t) \sim \left\{ \begin{matrix} e^{\pm t} \\ e^{\pm it} \end{matrix} \right\} \sim \left\{ \begin{matrix} \sinh t \\ \cosh t \\ \sin t \\ \cos t \end{matrix} \right\}$$

Clearly, given 4 i.c.'s, we can solve this system. We have not used $\mathcal{E}$ constancy yet - but we could if we wanted to.

Note: We may check constancy of $\mathcal{E}$ by direct differentiation: since $\mathcal{E} = (\dot{x}^2 + \dot{y}^2)/2 + xy \Rightarrow$
$\dot{\mathcal{E}} = \dot{x}\ddot{x} + \dot{y}\ddot{y} + x\dot{y} + \dot{x}y = -y\dot{x} - x\dot{y} + x\dot{y} + \dot{x}y$, (using eqns of motion), $= 0$, i.e. $\dot{\mathcal{E}} = 0 \Rightarrow \mathcal{E} = \text{const}$.

Example ④ – The Lorentz Force

Two fundamental forces of nature are the gravitational force and the electromagnetic force. Newtonian gravitational forces are derivable from a potential. The EM force is more complicated. The EM force is given in terms of two vector fields: $\vec{E}(\vec{x},t)$, the electric field, & $\vec{B}(\vec{x},t)$, the magnetic field. In terms of $\vec{E}$ & $\vec{B}$, the force on a particle of charge q is

$$\vec{F}(\vec{x},\vec{v},t) = q\vec{E}(\vec{x},t) + q\vec{v}\times\vec{B}(\vec{x},t)$$

Note that this part depends on $(\vec{x},t)$.
But the 2nd part depends on $(\vec{x},t)$ <u>AND</u> $\vec{v}$.
This is the first time we are encountering a Force that depends on $\vec{v}$.

from which $\boxed{\dot{v}_x = v_y ,\ \dot{v}_y = -v_x ,\ \dot{v}_z = 0}$

$\Rightarrow$ immediately $v_z = \text{const.}$

Thus, along z, the charge moves at constant speed, the value at $t=0$.

Along $x-y$, however, we differentiate again to uncouple the eqns, viz

$$\ddot{v}_x = \dot{v}_y = -v_x$$

$$\Rightarrow \boxed{\ddot{v}_x = -v_x}$$

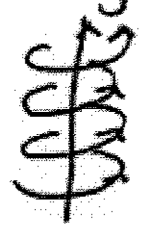
This is easily solved. Suppose $v_x(0) = 0$, $v_y(0) = v_0$.

$$\Rightarrow \boxed{v_x = v_0 \sin t}$$

and using $v_y = \dot{v}_x \Rightarrow \boxed{v_y = v_0 \cos t}$.

This corresponds to circular motion in the x-y plane, with frequency $= 1$.

$\Rightarrow$ total motion is helical



What about constants of the motion?

Clearly, $\vec{L} = m\vec{r}\times\vec{v} \neq$ const.

What about $\mathcal{E}$?

Suppose $m\dot{\vec{v}} = q\vec{E} + q\vec{v}\times\vec{B}$

Then $m\,\vec{v}\cdot\dfrac{d\vec{v}}{dt} = q\,\vec{v}\cdot\vec{E}$

$\Rightarrow \dfrac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\,\vec{v}\cdot\vec{E}.$

Note the following:

(a) energy is not a constant unless

$\vec{v}\cdot\vec{E} = 0$ or $\vec{E}$ is ~~[illegible]~~ conservative.

if $\vec{E} = -\vec{\nabla}\varphi \Rightarrow \vec{v}\cdot\vec{E} = -d\varphi/dt$, as usual

$\Rightarrow \dfrac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dfrac{d}{dt}(q\varphi)$

$\Rightarrow \boxed{\frac{1}{2}mv^2 + q\varphi = \mathcal{E} = \text{const}}$ if $\vec{E} = -\vec{\nabla}\varphi$

(b) $\vec{B}$ has dropped out! Thus, if $\vec{E}=0$ but $\vec{B}\neq 0$, then $\frac{1}{2}mv^2$ is still a constant. This can be checked for the foregoing simple example.