Constants of the Motion

In solving Newton's equations, we found that the solution was facilitated if constants of the motion were identified.
In the previous example, we found two:

① $\vec{L} = m\vec{r}\times\vec{v}$

② $E = \frac{1}{2}mv^2 + U(\vec{r})$.

How general are these?

Suppose $m\frac{d\vec{v}}{dt} = \vec{F}$, $\frac{d\vec{r}}{dt} = \vec{v}$; $m = \text{const}$.

Then, $\vec{L} = m\vec{r}\times\vec{v}$ and

$$\frac{d\vec{L}}{dt} = m\dot{\vec{r}}\times\vec{v} + m\vec{r}\times\dot{\vec{v}}$$
$$= m\,\vec{v}\times\vec{v} + \vec{r}\times\vec{F}$$

$$\Rightarrow \dot{\vec{L}} = \vec{r}\times\vec{F}$$

Clearly, $\vec{r}\times\vec{F} = 0 \Rightarrow \dot{\vec{L}} = 0 \Rightarrow \vec{L} = \text{const}$.

$\therefore$ $\boxed{\vec{L} = \vec{L}_0 = m\vec{r}\times\vec{v} \text{ if } \vec{F} \text{ is in the } \vec{r} \text{ direction (central force)}}$

Now, try $\mathcal{E}$.

$$m\frac{d\vec{v}}{dt} = \vec{F}; \text{ dot both sides with } \vec{v}$$

$$\Rightarrow \quad m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\vec{F}$$

$$\text{But } \frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\frac{d\vec{v}}{dt}\cdot\vec{v}$$

$$\Rightarrow \quad \frac{1}{2}m\frac{dv^2}{dt} = \vec{v}\cdot\vec{F}$$

Suppose $\vec{F} = -\vec{\nabla}U$ [Not always possible]

$$\Rightarrow \quad \vec{v}\cdot\vec{F} = -\vec{v}\cdot\vec{\nabla}U = -\frac{d\vec{r}}{dt}\cdot\vec{\nabla}U$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\frac{d\vec{r}\cdot\vec{\nabla}U}{dt}$$

But $d\vec{r}\cdot\vec{\nabla}U \equiv dU$, for any scalar fn. $U(\vec{r}$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\frac{dU}{dt}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 + U = \mathcal{E} = \text{const.}$$

Thus, $\boxed{\text{if } \vec{F} = -\vec{\nabla}U \Rightarrow \mathcal{E} = \frac{1}{2}mv^2 + U = \text{const}}$

Note: $\vec{F}$ cannot, in general, be written as $-\vec{\nabla}U$.

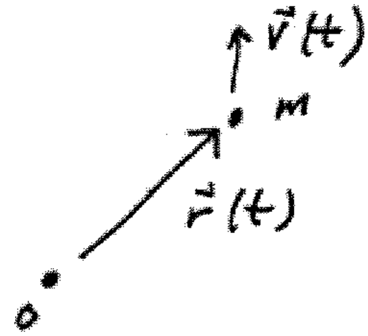
if $\vec{F}$ is conservative, then the each of the following is true + implies the other:

$\vec{F}$ is conservative $\Leftrightarrow$

- $\boxed{\vec{F} = -\vec{\nabla}U}$

- $\boxed{\vec{\nabla}\times\vec{F} = 0}$ $\Leftrightarrow$ $\boxed{\oint_C \vec{F}\cdot d\vec{r} = 0}$

Revisit Central Force Motion Using Constants of Motion



$$\boxed{\begin{aligned} m\dot{\vec{v}} &= \vec{F} \\ \dot{\vec{r}} &= \vec{v} \\ \text{where } \vec{F} &= -\hat{r} f(r) \end{aligned}} \quad -\ (1)$$

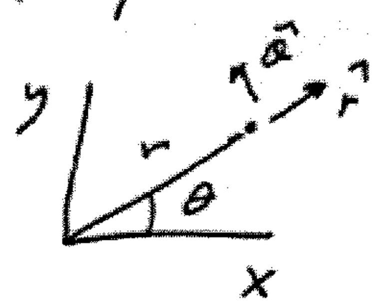
- since $\vec{F}$ is in $\vec{r}$ direction

$$\Rightarrow \quad m\vec{r}\times\vec{v} = \vec{L}_0$$

let $\vec{L_0}$ be in $\hat{z}$ direction

$$\Rightarrow \quad m\vec{r}\times\vec{v} = \hat{z}L_0 \quad -\ (2)$$

and motion confined to plane of (x, y)



- use polar coordinates

$$\Rightarrow \quad \vec{r} = r\hat{r}$$

$$\Rightarrow \quad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\Rightarrow \quad \vec{r}\times\vec{v} = \hat{z}r^2\dot{\theta} \quad -\ (3)$$

$$\therefore \quad (3) \rightarrow (2) \quad \Rightarrow \quad \boxed{r^2\dot{\theta} = L_0} \quad -\ (4)$$

- Since $\vec{F} = -\hat{r} f(r)$,

<u>by inspection</u>, $\vec{F} = -\vec{\nabla} U$

where $U = U(r)$, and $\frac{dU}{dr} = f(r)$

$$\Rightarrow \quad \mathcal{E} = \frac{1}{2} m v^2 + U(r) = \text{const}$$

But $v^2 = \dot{r}^2 + r^2 \dot{\theta}^2$, from previous

$$\Rightarrow \quad \boxed{\frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + U(r) = \mathcal{E}} \quad (5)$$

$m = 1$

- Now we can solve for $r(t)$, $\theta(t)$.

use (4) → (5) ⇒

$$\frac{1}{2} \dot{r}^2 + \frac{1}{2} \frac{L_0^2}{r^2} + U(r) = \mathcal{E}$$

$$\Rightarrow \quad \boxed{\dot{r} = \pm \sqrt{2} \left[\mathcal{E} - V(r) \right]^{1/2}}$$

$$\boxed{\text{where } V(r) \equiv U(r) + \frac{1}{2} \frac{L_0^2}{r^2}}$$

$\Rightarrow r(t)$ as before, whereupon, (4) $\Rightarrow \theta(t)$