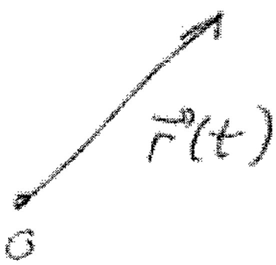
Motion in 2D and 3D

Objective: Find position of particle in time

$\Rightarrow$ Find

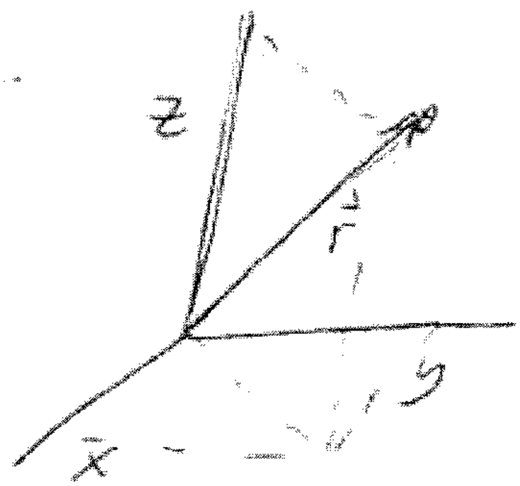


$\vec{r}(t)$ is a vector

Use $\quad m\dfrac{d\vec{v}}{dt} = \vec{F}, \quad \dfrac{d\vec{r}}{dt} = \vec{v}$

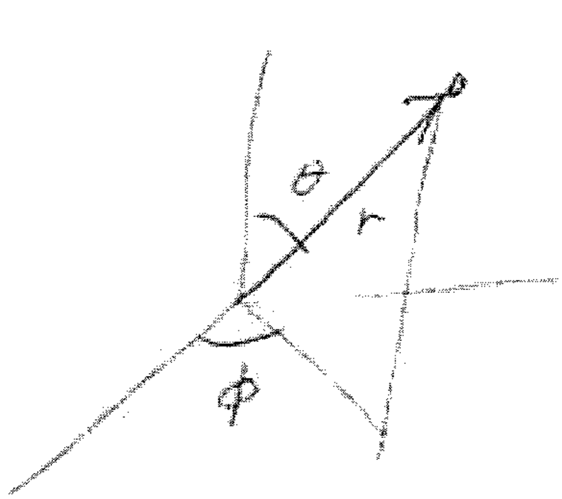
equivalent to finding

$\vec{r}(t) = [x(t), y(t), z(t)]$



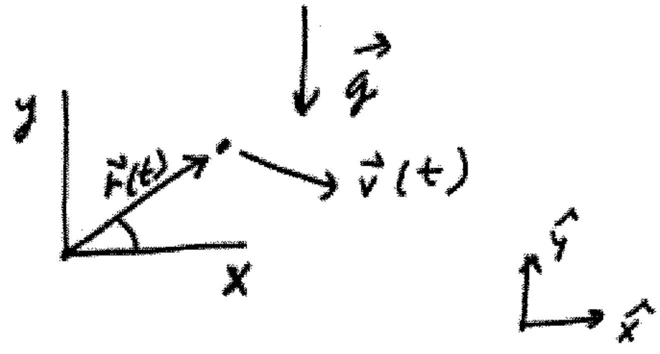
OR finding

$r(t), \theta(t), \phi(t)$



Projectile Motion

: Motion in 2D



$$m\frac{d\vec{v}}{dt} = m\vec{g}$$

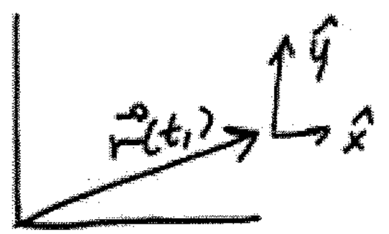
$$\frac{d\vec{r}}{dt} = \vec{v}$$

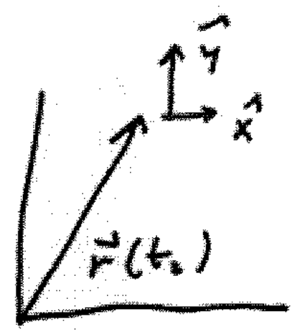
$$\Rightarrow \boxed{\frac{d\vec{v}}{dt} = -g\,\hat{y} \quad , \quad \frac{d\vec{r}}{dt} = \vec{v}}$$

Let $\boxed{\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y}}$

$$\Rightarrow \quad \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[x(t)\,\hat{x}\right] + \frac{d}{dt}\left[y(t)\,\hat{y}\right]$$

Note





$\hat{x}, \hat{y}$ are <u>constant</u> in time

Thus, $$\frac{d}{dt}\left[x(t)\,\hat{x}\right] = \left(\frac{dx}{dt}\right)\hat{x} + x\underbrace{\frac{d\hat{x}}{dt}}_{=0}$$

$$\Rightarrow \boxed{\vec{v}(t) = \dot{x}\,\hat{x} + \dot{y}\,\hat{y}}$$

$d\vec{v}/dt = -g\hat{y}$

$\Rightarrow \quad \ddot{x}\hat{x} + \ddot{y}\hat{y} = -g\hat{y}$

$\Rightarrow \quad \boxed{\ddot{x} = 0, \quad \ddot{y} = -g}$

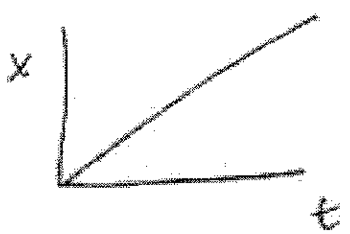
$\Rightarrow \quad x = At + B, \quad y = Ct + D - \frac{1}{2}gt^2$

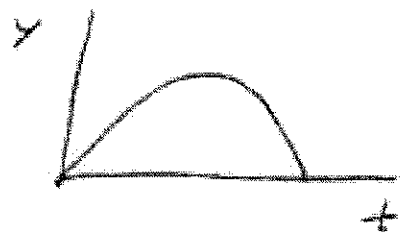
* Suppose $\vec{r}(0) = 0, \quad \vec{v}(0) = \{v_0, v_0\}$

$\Rightarrow \quad B = 0, \quad D = 0$

$A = v_0, \quad C = v_0$

$\Rightarrow \quad \boxed{x(t) = v_0 t, \quad y(t) = v_0 t - \frac{1}{2}gt^2}$



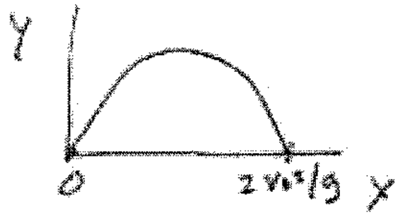


Parameterize $y(x)$

use $t = x/v_0$ in y

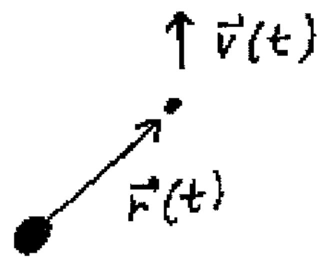
$\Rightarrow \quad y = x - \frac{1}{2} g \frac{x^2}{v_0^2}$

$$\boxed{y(x) = x - \frac{1}{2}\frac{g}{v_0^2}x^2}$$



* Note: 4 initial conditions since 2 ODE's, 2nd order, 2D

Central Force Motion



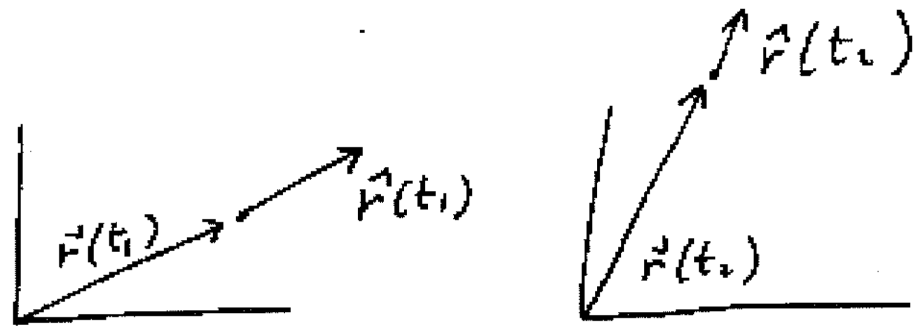
(Motion in 3D with unit vectors time-dependent)

$$m\frac{d\vec{v}}{dt} = \vec{F}, \quad \frac{d\vec{r}}{dt} = \vec{v}$$

let $\vec{F} = -\hat{r} f(r)$

(spherically symmetric, points inward)

Note



$$\hat{r}(t_1) \neq \hat{r}(t_2)$$

$$\Rightarrow \boxed{\hat{r} = \hat{r}(t)}$$

Also $\hat{\theta} = \hat{\theta}(t)$

$\hat{\phi} = \hat{\phi}(t)$ etc.

Since this is 3D, need 6 initial conditions.

Let $m = 1$

$$\Rightarrow \boxed{\frac{d\vec{v}}{dt} = -\hat{r} f(r), \quad \frac{d\vec{r}}{dt} = \vec{v}}$$

$f = \frac{GMm}{r^2}$ for gravity

<u>Note</u> $\vec{r}(t) \times \hat{r}(t) = 0 \quad \forall\, t$

$$\Rightarrow \vec{r} \times \frac{d\vec{v}}{dt} = 0$$

But $\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$

$$= \vec{v} \times \vec{v} + 0 = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$

$$\Rightarrow \vec{r}(t) \times \vec{v}(t) = \vec{L}_0$$

where $\vec{L}_0 = (L_{0x}, L_{0y}, L_{0z})$

= constant in t

= constant of the motion

Orient z axis $\ni \vec{L}_0 = (0, 0, L_0)$

$$\Rightarrow \boxed{\vec{r} \times \vec{v} = \hat{z} L_0}$$

$\vec{r}_\perp$ in x-y plane

let $\vec{r} = \vec{r}_\perp + \hat{z} z, \quad \vec{v} = \vec{v}_\perp + \hat{z} v_z$

$$\Rightarrow \vec{r}_\perp \times \vec{v}_\perp + v_z \vec{r}_\perp \times \hat{z} + z\, \hat{z} \times \vec{v}_\perp = \hat{z} L_0$$

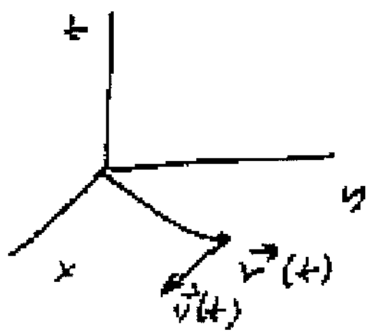
$$\Rightarrow \vec{r_\perp} \times \vec{v_\perp} = \hat{z} L_0 \qquad ①$$

$$\text{and} \quad v_z \vec{r_\perp} = z \vec{v_\perp} \qquad ②$$

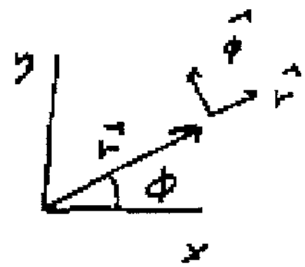
but ② → ① ⇒ contradiction ($L_0 \neq 0$) unless $v_z = 0$, $z = 0$

⇒ $\vec{r}(t)$ and $\vec{v}(t)$ stay in plane (x-y)

and $\boxed{\vec{r} \times \vec{v} = \hat{z} L_0}$ (this ⇒ only 4 degrees of freedom with L_0 specifying one)



⇒



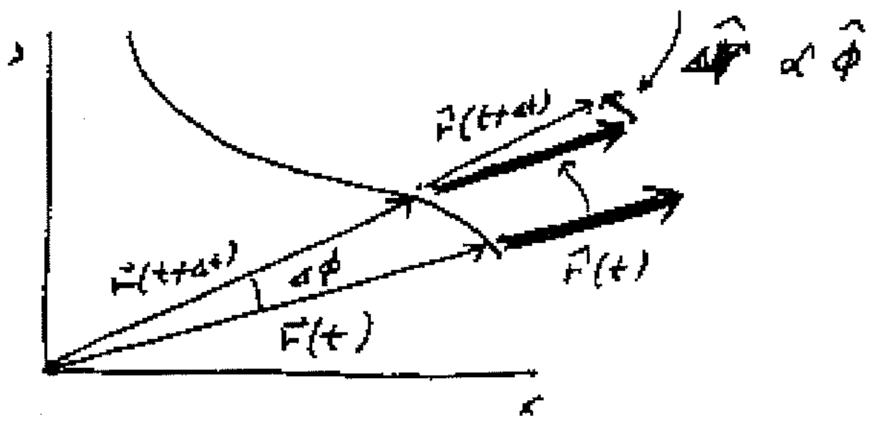
∴ Let $\boxed{\vec{r}(t) = r(t)\,\hat{r}(t)}$

Note $\phi = \phi(t)$, $\hat{\phi} = \hat{\phi}(t)$

$$\Rightarrow \vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[r(t)\,\hat{r}(t)\right]$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

Can find $\frac{d\hat{r}}{dt}$ and $\frac{d\hat{\phi}}{dt}$ graphically



length of $\Delta\hat{r}$

$$\frac{d\hat{r}}{dt} \leftarrow \frac{\Delta\hat{r}}{\Delta t} = \frac{\hat{r}(t+\Delta t) - \hat{r}(t)}{\Delta t} = \frac{\hat{\phi}\,\Delta\phi}{\Delta t}$$

$$\Rightarrow \boxed{\frac{d\hat{r}}{dt} = \hat{\phi}\frac{d\phi}{dt}}$$

← if $\Delta\phi$ does not change in Δt, $\hat{r}$ + $\hat{\phi}$ do not change in δ

likewise $$\boxed{\frac{d\hat{\phi}}{dt} = -\hat{r}\frac{d\phi}{dt}}$$

$$\Rightarrow \boxed{\vec{v}(t) = \frac{dr}{dt}\hat{r} + r\frac{d\phi}{dt}\hat{\phi}}$$

$$\Rightarrow \boxed{\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{r}}$$

Collect all $\Rightarrow$

$$\boxed{\begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -f(r) \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0 \end{aligned}}$$

③
④

$$\vec{r} \times \vec{v} = r\hat{r} \times \hat{\phi}\, r\dot{\phi} = L_0\, \hat{z}$$

$$\Rightarrow \boxed{r^2 \dot{\phi} = L_0}$$ ⑤ angular mom conserved

Note: ⑤ $\Rightarrow$ ④ since $(r^2\dot{\phi})^{\cdot} = r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = 0$

Thus, use ⑤ $\to$ ③

$$\boxed{\ddot{r} - \frac{L_0^2}{r^3} = -f(r)}$$ ⑥

Recall obtaining $r(t)$, $\phi(t)$ solves the problem $[z(t) = 0]$

Thus, ⑥ $\Rightarrow r(t)$ & , then, ⑤ $\Rightarrow \phi(t)$

Must solve ⑥

$$\boxed{\ddot{r} = \frac{L_0^2}{r^2} - f(r)}$$

$r(t)$
$L_0 = $ const
$f(r)$ given

ODE is of the "energy form"!

to use this method, must rewrite RHS as RHS $= -dU(r)/dr$

$$\Rightarrow \boxed{U(r) = \frac{L_0^2}{2r^2} - \int^r dr\, f(r)}$$

$$\Rightarrow \ddot{r} = -dU/dr$$

$$\Rightarrow \boxed{\frac{1}{2}\dot{r}^2 + U(r) = E}$$

E = constant
$\Rightarrow E \geq U(r)$

if $f(r) = \frac{1}{r^2}$, like Earth's gravitational field,

$$U(r) = \frac{L_0^2}{2r^2} - \frac{1}{r}$$

Now $r(t)$ can be found

$$\int \frac{dr}{[E - U(r)]^{1/2}} = \pm\sqrt{2}\int dt$$

<u>Summary</u>

- The 3D problem is harder – it has 3 degrees of freedom and needs 6 initial conditions.
- "Constants of the Motion" are a great help. Any combination of the variables $\{\vec{r}(t), \vec{v}(t)\}$ that remains constant is a help. e.g., angular momentum and energy.