ENERGY METHOD for 2nd order NL ODE's

- Special ODE: $\boxed{\ddot{x} = f(x)}$ $f(x)$ is arbitrary

nonlinear, homogeneous, non-const coeffs, non equidimensional.

- Importance in physics

$$m\frac{dv}{dt} = F, \quad \frac{dx}{dt} = v$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = F.$$

F could be $F(x)$, e.g. harmonic oscillator, gravitational forces etc.

- Special Method of solution

$$\ddot{x} = f(x) \quad \Rightarrow \quad \dot{x}\ddot{x} = \dot{x} f(x)$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) = \dot{x} f(x)$$

let $f(x) = -\dfrac{dU}{dx}$, $U = U(x)$

can always be found as $U(x) = -\int^x dx' f(x')$

$$\Rightarrow \quad \dot{x} f(x) = -\frac{dx}{dt}\frac{dU}{dx} = -\frac{dU}{dt}[x(t)]$$

[Note: $U = U[x(t)]$]

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) = -\frac{dU}{dt} \qquad \Rightarrow \text{integrate}$$

$$\Rightarrow \boxed{\frac{1}{2}\dot{x}^2 + U(x) = E}^{\,*} \quad -(1) \qquad E = \text{constant} \ (\text{"energy"})$$

But, we are not done yet. We want $x(t)$ and $U(t)$. We only have $\dot{x}$ related to $U(x)$.

<u>BUT!</u> (1) is a 1^{st} order ODE, and we know how to solve these.

Here's how:

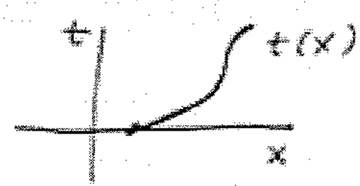
$$\dot{x}^2 = 2\left[E - U(x)\right]$$

$$\dot{x} = \pm\sqrt{2}\left[E - U(x)\right]^{1/2}$$

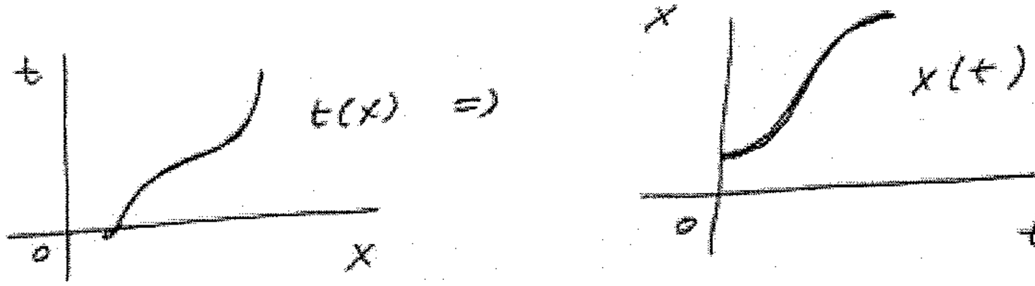
$$\Rightarrow \boxed{\frac{dx}{\left[E - U(x)\right]^{1/2}} = \pm\sqrt{2}\, dt} \qquad \text{separable!}$$

$$\Rightarrow \int_{x_0}^{x(t)} \frac{dx}{\left[E - U(x)\right]^{1/2}} = \pm\sqrt{2}\, t \qquad \Rightarrow t = t(x)$$

where $x(0) = x_0$



* Note $E = \frac{1}{2}\dot{x}^2(0) + U[x(0)]$ from initial conditions



. Significance of $(\pm)$ given by example below

Example - harmonic oscillator

$$\boxed{\ddot{x} = -x} \qquad \text{ie,} \quad f(x) = -x$$

let $x(0) = x_0$, $\dot{x}(0) = 0$

direct soln (const coeffs) $\Rightarrow$ $\boxed{x(t) = x_0 \cos t}$

Energy method

$$\ddot{x} = -x \quad \Rightarrow \quad U(x) = \tfrac{1}{2} x^2$$

$$\Rightarrow \quad \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 = E$$

$$x(0) = x_0,\ \dot{x}(0) = 0 \quad \Rightarrow \quad \tfrac{1}{2}x_0^2 = E \quad \Rightarrow E$$

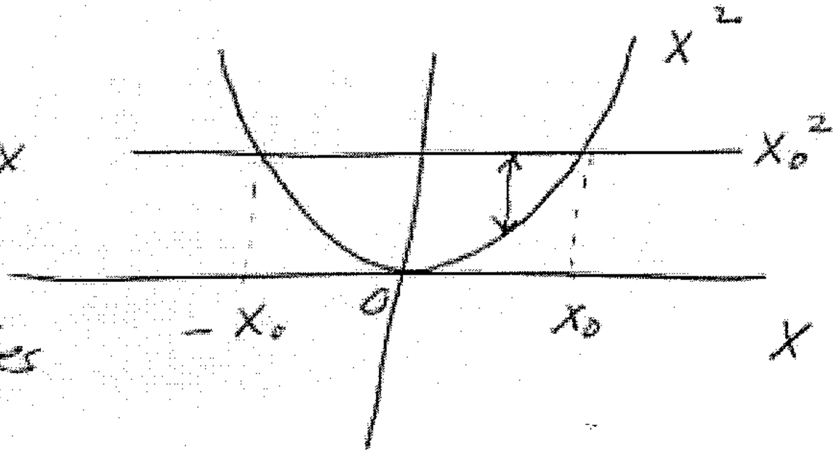
$$\Rightarrow \quad \boxed{\dot{x}^2 + x^2 = x_0^2}$$ equivalent to (1)

Note $\dot{x}^2 \geq 0$
$\Rightarrow x_0^2 \geq x^2$

$$\Rightarrow \quad \frac{dx}{(x_0^2 - x^2)^{1/2}} = \pm\, dt$$

Some information graphically:

$$\dot{x}^2 = x_0^2 - x^2$$



- Plot x^2 and x_0^2 vs x
- The arrow $\updownarrow$ indicates $x_0^2 - x^2$.
- Now $\dot{x}^2 \geq 0 \Rightarrow x_0^2 - x^2 \geq 0$

$\Rightarrow$ mass must stay between $\pm x_0$

[these are "turning points"]

- If $x(t=0) = x_0$, then $x(t>0)$, immediately after, must be $< x_0$. $\Rightarrow \dot{x} \leq 0$ immediately after.

Thus, for $t > 0$, we pick

$$\boxed{\dot{x} = -\sqrt{x_0^2 - x^2}}$$

This choice of sign is ~~good~~ until the turning point at $-x_0$ is reached. After this, we must pick the $+\sqrt{x_0^2 - x^2}$ for $\dot{x}$.

Now, we solve for $x(t)$ for $\dot{x}<0$

$$\dot{x}(t) = -\sqrt{x_0^2 - x^2}$$

$$\Rightarrow \frac{dx}{\sqrt{x_0^2-x^2}} = -dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx'}{\sqrt{x_0^2-x'^2}} = -\int_0^t dt' = -t$$

But $\dfrac{dx}{\sqrt{x_0^2-x^2}} = -d\left[\cos^{-1}\left(\dfrac{x}{x_0}\right)\right]$

$$\Rightarrow \cos^{-1}\left(\frac{x}{x_0}\right) = t$$

$$\Rightarrow \boxed{x = x_0 \cos t}$$

Summary of Energy Method

- By a "trick" ($\times$ both sides by $\dot{x}$) we were able to integrate the equation. This yielded a "constant of the motion" and reduced the system to a 1st order ODE. The latter is easy to solve.
- Thus, finding constants of the motion is a useful technique.