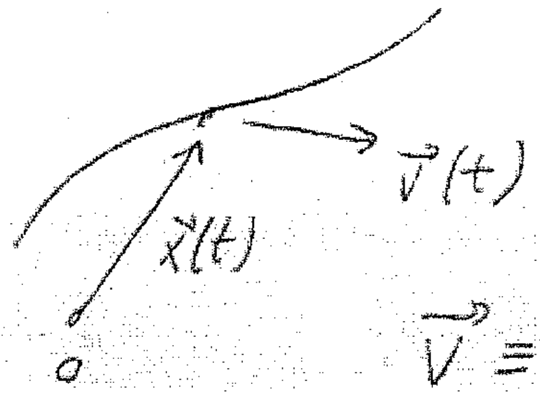
CLASSICAL MECH

Newton's Laws



$$\vec{v} \equiv \frac{d\vec{x}}{dt} \leftarrow \frac{\Delta \vec{x}}{\Delta t}$$

Then

$$\boxed{\begin{aligned} m\frac{d\vec{v}}{dt} &= \vec{F} \\ \frac{d\vec{x}}{dt} &= \vec{v} \end{aligned}}$$

$\vec{x}(t)$
$\vec{v}(t)$

$$\vec{F} = \vec{F}[\vec{x}, \vec{v}, t]$$

$$\rightarrow \vec{F}[\vec{x}(t), \vec{v}(t), t]$$

e.g. $\vec{F} = \hat{x}\, x y \sin t$

LINEAR, 1st ORDER ODE's

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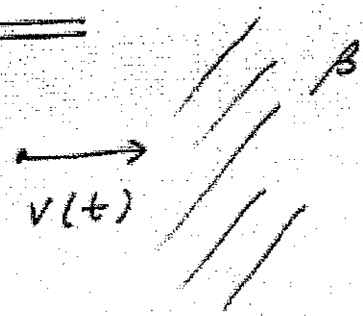
Bullet in Molasses



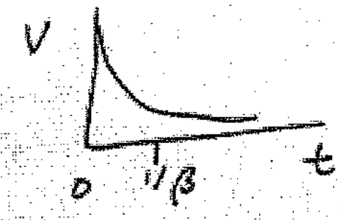
$v = v(t)$

$m\dot{v} = -\beta m v \quad \Rightarrow \quad \boxed{\dot{v} = -\beta v} \qquad v(0) = v_0$

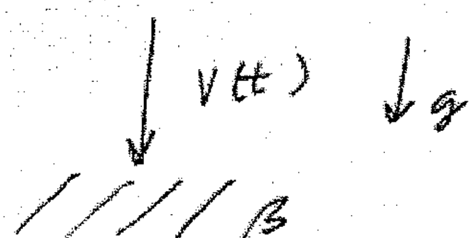
linear, homogeneous, 1st order, (const coeffs)

$\Rightarrow \quad v = Ae^{-\beta t}, \quad \Rightarrow \quad v_0 = A$

$\Rightarrow \quad \boxed{v(t) = v_0 e^{-\beta t}}$



Bullet in Molasses w/g



$m\dot{v} = -m\beta v + mg$

$\Rightarrow \boxed{\dot{v} + \beta v = g} \qquad v(0) = 0$

linear, inhomogeneous

Method ① $\quad v_h + v_p$

$\beta v_p = g$, by inspection

$v_h = Ae^{-\beta t}$

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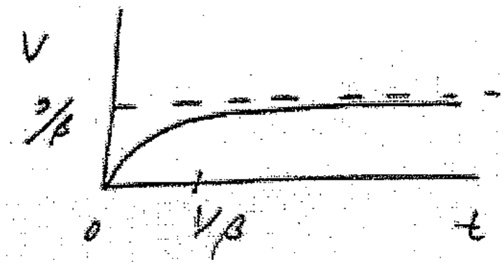
$$\Rightarrow \quad v = A e^{-\beta t} + g/\beta$$

$$\Leftrightarrow \quad 0 = A + g/\beta$$

$$\Rightarrow \quad \boxed{v(t) = \frac{g}{\beta}\left[1 - e^{-\beta t}\right]} \quad v(\infty) \to g/\beta$$



Method ② Separate variables

$$dv = (g - \beta v)\,dt \Rightarrow \frac{dv}{\frac{g}{\beta} - v} = \beta dt$$

$$\Rightarrow \int_0^v \frac{dv'}{\frac{g}{\beta} - v'} = \beta \int_0^t dt' \Rightarrow -\left[\ln\left[\frac{g}{\beta} - v\right]\right]_0^v = \beta t$$

$$\Rightarrow \ln\left(\frac{\frac{g}{\beta}}{\frac{g}{\beta} - v}\right) = \beta t \Rightarrow \left(1 - \frac{\beta v}{g}\right)^{-1} = e^{\beta t}$$

$$\Rightarrow \boxed{v(t) = \frac{g}{\beta}\left[1 - e^{-\beta t}\right]}$$

Method ③ integrating factor

$$\dot{v} + \beta v = g \Rightarrow (v e^{\beta t})^{\circ} = g e^{\beta t}$$

$$\Rightarrow v e^{\beta t} = \frac{g}{\beta} e^{\beta t} + \text{constant.}$$

$$v(0) = 0 \Rightarrow 0 = \frac{g}{\beta} + \text{const.} \Rightarrow \text{const}$$

$$\Rightarrow \boxed{v(t) = \frac{g}{\beta}\left[1 - e^{-\beta t}\right]}$$

Bullet in Molasses in changing $\beta(t)$

Let $\beta \to \alpha/t$, $t \neq 0$

$$\Rightarrow \dot{v} + \frac{\alpha v}{t} = g. \quad \text{Not separable}$$

Let $v(1) = 0$.

Use integrating factor.

$$\int \frac{dt}{t} = \ln t; \quad e^{\alpha \int \frac{dt}{t}} \Rightarrow e^{\alpha \ln t} = e^{\ln t^{\alpha}} = t^{\alpha}$$

$$\Rightarrow (v t^{\alpha})^{\cdot} = g t^{\alpha}$$

$$\Rightarrow v t^{\alpha} = g \int_1^t dt' \, t'^{\alpha}, \quad v(1) = 0.$$

$$\Rightarrow v t^{\alpha} = g \left[\frac{t'^{\alpha+1}}{\alpha+1} \right]_1^t = \frac{g}{\alpha+1} \left(t^{\alpha+1} - 1 \right)$$

$$\Rightarrow \boxed{v(t) = \frac{g t}{\alpha+1} \left(1 - \frac{1}{t^{\alpha+1}} \right)} \quad v(1) = 0$$

Note $v(t \to \infty) \to \dfrac{g t}{\alpha+1}$

SUMMARY

- 1st order ODE, in general

  ⇒ separate variables

  ⇒ integrating factor

- 1st order ODE, linear, inhomogeneous

  ⇒ homogeneous + particular solution

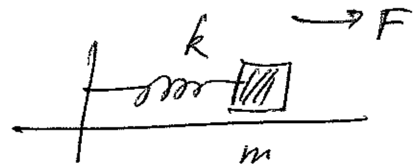
- one initial condition required

Normalizations

Consider driven H.O.

$$m\ddot{x} = -kx + F(t)$$

$$F = F_0 \cos \omega t$$



let $\dfrac{mX}{T^2} = kX \Rightarrow \boxed{T \equiv (m/k)^{1/2}}$

let $kX = F_0$; $[X] = L$, $\Rightarrow kL = F_0$

$$\Rightarrow \boxed{L \equiv F_0/k}$$

- have space and time dimensions

let $\boxed{x = \hat{x} L, \quad t = \hat{t}\, T}$

$$\Rightarrow m \frac{d^2\hat{x}}{d\hat{t}^2}\frac{L}{T^2} = -k\hat{x}L + F_0 \cos(\omega T \hat{t})$$

$$\Rightarrow \frac{d^2\hat{x}}{d\hat{t}^2} = -\frac{k}{m}\hat{x}T^2 + \frac{F_0 T^2}{mL}\cos(\omega T\hat{t})$$

$$\Rightarrow \boxed{\frac{d^2\hat{x}}{d\hat{t}^2} = -\hat{x} + \cos[(\omega T)\hat{t}]}$$

Normalized equation with single parameter (ωT).

Quicker way

$$m\frac{d^2x}{dt^2} = -kx + F_0\cos(\omega t)$$

$$\ddot{x} = -\frac{k}{m}x + \frac{F_0}{m}\cos(\omega t)$$

we have $\left\{\frac{k}{m}, \frac{F_0}{m}, \omega\right\}$ parameters

$\boxed{\text{Allowed to set at most 3 parameters } =1 \text{, provided they do not have same dimensions.}}$ Eg, $\frac{k}{m}=1$, $\frac{F_0}{m}=1$

$\left[\frac{k}{m}\right] = \frac{1}{T^2}$; $\left[\frac{F_0}{m}\right] = \frac{L}{T^2}$. Thus,

these two set T and L dimensions

But ~~can~~ cannot set $\omega = 1$, since T has been used already.

$$\Rightarrow \boxed{\ddot{x} = -x + \cos(\omega t)} \longrightarrow \text{as before}$$

Compare to previous

only parameter left.

Alternatively,

let $\frac{F_0}{m} = 1$, $\omega = 1$

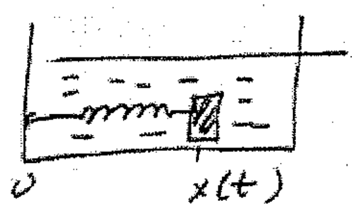
$$\Rightarrow \boxed{\ddot{x} = -\frac{k}{m} x + \cos(t)}$$

parameter left

here, the normalization of T is with respect to ω.

Linear, 2nd order, ODE's

Harmonic oscillator, damped



$x(t)$

$v(t) = x'(t)$

$$\boxed{\ddot{x} + \omega_0^2 x - 2\beta \dot{x} = 0}$$

linear, 2nd order, homogeneous

CONSTANT COEFFS ($\omega_0 = \text{const}$, $\beta = \text{const}$)

$\rightarrow$

$\Rightarrow$ Try $x(t) = e^{\alpha t}$

$$\Leftrightarrow \quad \alpha^2 + \omega_0^2 - 2\beta\alpha = 0$$

$$\alpha^2 - 2\beta\alpha + \omega_0^2 = 0$$

$$\boxed{\alpha_\pm = \beta \pm (\beta^2 - \omega_0^2)^{1/2}}$$

$$\Rightarrow \quad x(t) = A_+ e^{\alpha_+ t} + A_- e^{\alpha_- t}$$

2 free constants, 2 solns

Case 1 $\beta = 0$

$$\Rightarrow \alpha_\pm = \pm(-\omega_0^2)^{1/2} = \pm i\omega_0$$

$$\Rightarrow x(t) \sim \left\{ e^{\pm i\omega_0 t} \right\}, \text{ or } \left\{ \begin{matrix} \cos\omega_0 t \\ \sin\omega_0 t \end{matrix} \right\}$$

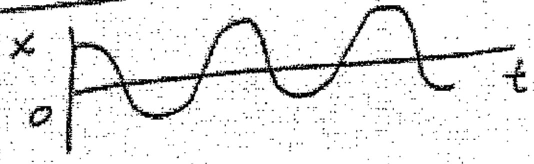
$$\text{or } \left\{ \begin{matrix} \cos(\omega_0 t + \phi) \\ \sin(\omega_0 t + \phi) \end{matrix} \right\}$$

→ Various combinations, always 2 independent solns.

Suppose $x(0) = x_0$, $v(0) = 0 = \dot{x}(0)$

$$\Rightarrow \boxed{x(t) = x_0 \cos(\omega_0 t)}$$

since $x(0) = x_0$, $\dot{x}(0) = 0$



Case 2 suppose $\beta \ll \omega_0$

$$\Rightarrow \pm(\beta^2 - \omega_0^2)^{1/2} = \pm i\omega_0 (1 - \beta^2/\omega_0^2)^{1/2}$$

$$\simeq \pm i\omega_0 \left[1 - \frac{1}{2}\frac{\beta^2}{\omega_0^2} + \cdots \right]$$

$$\Rightarrow \alpha_\pm \simeq \beta \pm i\omega_0 \mp \frac{i}{2}\frac{\beta^2}{\omega_0} \Rightarrow \boxed{\alpha_\pm \simeq \pm i\omega_0 - \beta}$$

β: small; $\frac{i}{2}\frac{\beta^2}{\omega_0}$: smaller

$$\Rightarrow \quad x(t) \sim e^{i\omega_0 t} e^{-\beta t}, \; e^{-i\omega_0 t} e^{-\beta t}$$

$$\rightarrow \quad e^{-\beta t} \begin{Bmatrix} \cos\omega_0 t \\ \sin\omega_0 t \end{Bmatrix} \; \text{etc.}$$

Suppose $x(0) = 0$

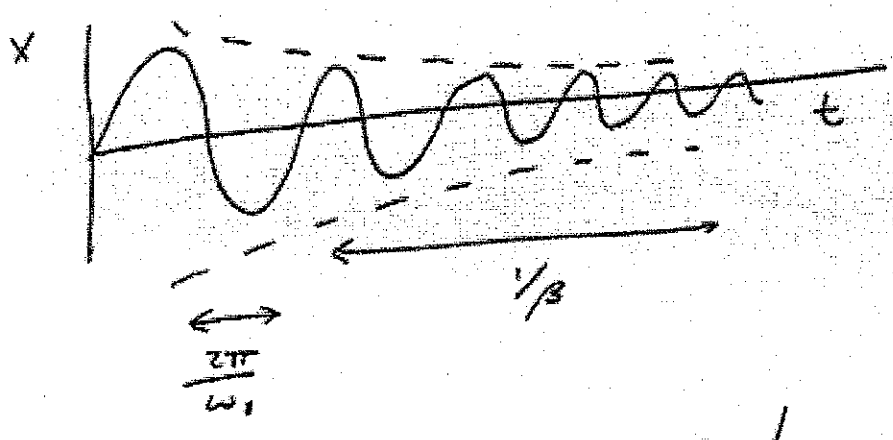
$\dot{x}(0) = V_0$

let $x(t) \simeq A e^{-\beta t} \sin(\omega_0 t)$

then $x(0) = 0$, $\dot{x}(t) = -\beta A e^{-\beta t} \sin(\omega_0 t) + \omega_0 A e^{-\beta t} \cos(\omega_0 t)$

$$\Rightarrow \quad V_0 = A\omega_0$$

$$\Rightarrow \quad \boxed{x(t) \simeq \frac{V_0}{\omega_0} e^{-\beta t} \sin(\omega_0 t)}$$



2 <u>time</u> <u>scales</u> : $\frac{2\pi}{\omega_0}$, $\frac{1}{\beta}$; $\beta \ll \omega_0$

Constant Coefficient Method

applies to all n^{th} order, linear, homogeneous ODE's. ie, $x(t) \to e^{\alpha t}$ works.

e.g.

$$\epsilon \dddot{x} + \ddot{x} + x = 0$$

Try $x \sim e^{\alpha t}$

$$\Rightarrow \epsilon \alpha^3 + \alpha^2 + 1 = 0 \text{; cubic eqn}$$

$$\Rightarrow \alpha = \{\alpha_1, \alpha_2, \alpha_3\}$$

[Aside: suppose $\epsilon \to 0$. Then, can show,

$$\alpha_{2,3}^2 \simeq -1 \Rightarrow \boxed{\alpha_{2,3} \simeq \pm i}$$

and $$\epsilon \alpha_1 + 1 \simeq 0 \Rightarrow \boxed{\alpha_1 \simeq -\frac{1}{\epsilon}}$$

$$\Rightarrow e^{\pm it}, e^{-t/\epsilon} \text{]}$$

EQUIDIMENSIONAL eqns

Suppose $x(t)$

$$\boxed{t\frac{d}{dt}\left(t\frac{dx}{dt}\right) \pm x = 0} \quad \text{—(a)}$$

$$\Rightarrow \boxed{t^2x'' + tx' \pm x = 0} \quad \text{—(b)}$$

linear, homogeneous, Non-const coeffs

BUT, equidimensional, i.e.,

all dimensions in t are same

$$t^2\frac{d^2}{dt^2} : t\frac{d}{dt} : 1 \sim 1:1:1$$

Then, try $\boxed{x(t) \sim t^\alpha}$

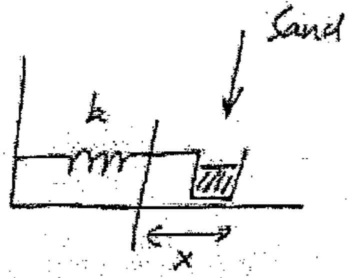
Tried on (b) $\Rightarrow \alpha(\alpha-1) + \alpha \pm 1 = 0$

Tried on (a) $\Rightarrow \alpha^2 \pm 1 = 0$

either way, $\alpha_\pm = \pm 1$

$$\Rightarrow x(t) \sim \begin{Bmatrix} t^1 \\ t^{-1} \end{Bmatrix} = \begin{Bmatrix} t \\ 1/t \end{Bmatrix}$$

Variable mass oscillator



let $m(t) = \mu t^2$

let $x(t_0) = 0$

$v(t_0) = V_0$, $t_0 > 0$

$$\frac{d}{dt}(mv) = -kx, \quad \frac{dx}{dt} = v$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(t^2 \frac{dx}{dt}\right) = -\frac{k}{\mu} x} \quad \boxed{\begin{array}{l} x(t_0) = 0 \\ v(t_0) = V_0 \end{array}}$$

Dimensions of $\frac{k}{m} = \frac{1}{T^2}$; but $\frac{k}{m} = \frac{k}{\mu T^2}$

$$\Rightarrow \frac{k}{\mu T^2} \stackrel{dim}{=} \frac{1}{T^2} \Rightarrow \boxed{\frac{k}{\mu} \text{ is dimensionless!}}$$

pure number $\Rightarrow$ cannot be used to set any dimensions for the problem $\Rightarrow \frac{k}{\mu} \neq 1$

However, t_0 and V_0 are not dimensionless and have different dimensions

$$\Rightarrow \text{set } \boxed{V_0 = 1, \; t_0 = 1}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(t^2 \frac{dx}{dt}\right) = -\frac{k}{\mu} x} \quad \boxed{\begin{array}{l} x(1) = 0 \\ v(1) = 1 \end{array}}$$

equidimensional $\Rightarrow$ try $x \sim t^{\alpha}$

$\Longleftrightarrow \quad \alpha(\alpha+1) = -k/\mu$

$\Rightarrow \quad \alpha_{\pm} = -\frac{1}{2} \pm i\left(\frac{k}{\mu} - \frac{1}{4}\right)^{1/2}$, assume $\frac{k}{\mu} > \frac{1}{4}$

let $\boxed{\frac{k}{\mu} - \frac{1}{4} \equiv \gamma^2}$

$\Rightarrow \boxed{\alpha_{\pm} = -\frac{1}{2} \pm i\gamma}$

$\Rightarrow \quad x(t) = \frac{1}{t^{1/2}}\left\{t^{i\gamma}, t^{-i\gamma}\right\}$

use $t^{i\gamma} \equiv e^{\ln t^{i\gamma}} \equiv e^{i\gamma \ln t}$

$\Rightarrow \quad x(t) = \frac{1}{t^{1/2}}\left\{\cos[\gamma \ln t], \sin[\gamma \ln t]\right\}$

Try $x(t) = \frac{1}{t^{1/2}}\left[A\cos(\gamma \ln t) + B\sin(\gamma \ln t)\right]$

$x(1) = 0 = A \quad \Rightarrow \boxed{A = 0}$

$\Rightarrow x(t) = \frac{B}{t^{1/2}}\sin(\gamma \ln t)$

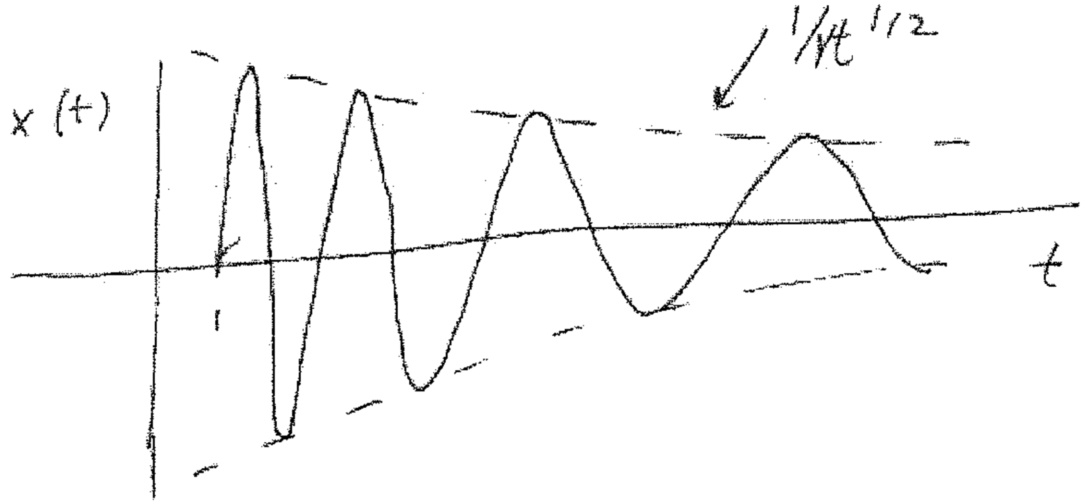
$\Rightarrow v(t) = -\frac{1}{2}\frac{B}{t^{3/2}}\sin(\gamma \ln t) + \frac{\gamma B}{t^{3/2}}\cos(\gamma \ln t)$

$v(1) = 1 = \gamma B \quad \Rightarrow \boxed{B = 1/\gamma}$

Thus $$\boxed{x(t) = \frac{1}{\gamma t^{1/2}} \cos(\gamma \ln t)}$$

$$\gamma \equiv (k/\mu - 1/4)^{1/2}$$

$\Rightarrow$



frequency $\sim \dfrac{\gamma \ln t}{t}$

amplitude $\sim \dfrac{1}{\gamma t^{1/2}}$

Inserting dimensions back, time $\sim t_0$, space $\sim v_0 t_0$

$$\Rightarrow \boxed{x(t) = \frac{v_0 t_0}{\gamma (t/t_0)^{1/2}} \cos\left[\gamma \ln (t/t_0)\right]}$$