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Homework #6 — PHYS 625 — Spring 2021
Deadline: Friday, May 7, 2021, by email to masoudma@umd.edu

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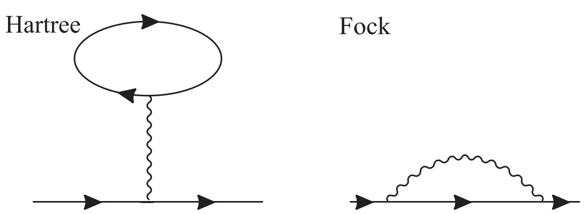
Web page: https://terpconnect.umd.edu/ galitski/PHYS625/index.html

Do not forget to write your name and the homework number!

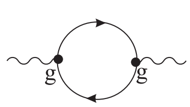
1. Consider a three-dimensional system of fermions, which interact with each other with a point-like potential $V(\mathbf{r}-\mathbf{r}') = u_0\delta(\mathbf{r}-\mathbf{r}')$. Calculate the fermionic self-energy in the Hartree-Fock approximation. I.e., calculate the following diagrams



Within the Hartree-Fock approximation, derive the general formula for the correction to the chemical potential in terms of the spin s, the fermion density n, and the interaction strength u_0. Note that in the model of "spinless fermions," the correction vanishes. Can this fact be understood without calculations?

Hint: Note that the integral of the Green's function over energy is simply the (Fermi) distribution function.

2. The leading order correction to the phonon propagator is given by the diagram,



$$\Longrightarrow \quad D^{-1}(\omega,\mathbf{k}) = D_0^{-1}(\omega,\mathbf{k}) + g^2\,\Pi(\omega,\mathbf{k})$$

where the wavy line corresponds to the phonon propagator, the expression for the bubble was given in homework 5, and each vertex corresponds to the electron-phonon coupling constant g.

Find the leading order correction to the speed of sound in three and two dimensions. Assume that the bare speed of sound is much smaller than the Fermi velocity, $c \ll v_{\mathrm{F}}$.

3. A localized magnetic impurity is introduced in a Fermi gas. The spin interacts with electrons via the following exchange interaction

$$\hat{\mathcal{H}}_{\rm int} = J \int S^i \delta(\mathbf{r}) \hat{\psi}^\dagger_\alpha(\mathbf{r}) \hat{\sigma}^i_{\alpha\beta} \psi_\beta(\mathbf{r}) d^3\mathbf{r} \equiv J S^i \hat{\sigma}^i_{\rm el}(\mathbf{r} = \mathbf{0}), \tag{1}$$

where J is a constant, which is assumed *small*, $\mathbf{S}$ is the impurity spin, $i = x, y, z$, α and $\beta = \uparrow, \downarrow$ are spin indices, $\sigma^i_{\alpha\beta}$ are the Pauli matrices, and $\hat{\sigma}^i_{\rm el}(\mathbf{r})$ is the position-dependent electron spin density operator. Find the spin polarization $\langle \hat{\sigma}^i(\mathbf{r}) \rangle$ at large distances from the localized magnetic moment (specify, what "large distances" mean in the given problem).

Hint: Using the basic formulas of the perturbative theory of many-body Green's functions, involving the time-ordered product, the Wick theorem, etc., show/argue that the electron Green's function can be represented in first-order perturbation theory as follows:

$$G^{(1)}_{\alpha\beta}(\varepsilon, \mathbf{r}; \mathbf{r}') = G_0(\varepsilon, \mathbf{r}) \delta_{\alpha\beta} + J S^i \sigma^i_{\alpha\beta} G_0(\varepsilon, \mathbf{r}) G_0(\varepsilon, -\mathbf{r}'). \tag{2}$$

Express the spin polarization density through the Green's function as

$$\left\langle \hat{\sigma}^i_{\rm el}(\mathbf{r}) \right\rangle = \lim_{t' \to t+0} \left[-i \sigma^i_{\alpha\beta} G_{\beta\alpha}(\mathbf{r}, t; \mathbf{r}', t') \right]$$

and use the Green's function $G^{(1)}_{\alpha\beta}(\varepsilon, \mathbf{r}; \mathbf{r}')$ of Eq. (2) to calculate the spin density distribution.

Note: The density oscillations that you found here are directly related to the so-called RKKY interactions, which is a leading coupling mechanism between nuclear magnetic moments in metals.

4. Tricky problem. Optional. Consider a fermionic chain (see also your lecture notes and homework 2; also, compare with the mean-field Hamiltonian in the BCS theory of superconductivity):

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int},$$

where

$$\mathcal{H}_0 = \sum_{n=-\infty}^{\infty} \left(J_1 a^\dagger_n a_{n+1} - B a^\dagger_n a_n + \text{h. c.} \right)$$

and

$$\mathcal{H}_{\rm int} = \sum_{n=-\infty}^{\infty} \left(J_2 a_n a_{n+1} + \text{h. c.} \right)$$

Find the Green's function $G_0(\varepsilon, p)$ of the "non-interacting" problem described by the Hamiltonian $\mathcal{H}_0$. Draw the Feynmann graphs, corresponding to the exact Green's function. Consider two types of interaction vertices corresponding to the hermitian conjugate terms $2iJ_2 \sin q$ and $-2iJ_2 \sin q$. Note that only even orders of perturbation theory with respect to $\mathcal{H}_{\rm int}$ give non-zero contributions. Sum up the corresponding series and find the spectrum of quasiparticles which is given by the poles of the exact Green's function.