Homework #2 — PHYS 625 — Spring 2019
Deadline: Monday, March 4, 2019, in class

Professor Victor Galitski
Office: 2330 Physics and
2270 PSC

Web page: `http://terpconnect.umd.edu/~galitski/PHYS625/`

Do not forget to write your name and the homework number!

Bogoliubov phonons. SSH model. Berry phase.

1. **Spectrum of Bogoliubov phonons in a Bose-Einstein condensate (BEC).**

   Consider a system of interacting bosons, described by the Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \frac{U}{2V} \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4} \hat{a}^\dagger_{\mathbf{p}_4} \hat{a}^\dagger_{\mathbf{p}_3} \hat{a}_{\mathbf{p}_2} \hat{a}_{\mathbf{p}_1}, \tag{1}$$

   where $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$ are creation and annihilation operators of bosons with momentum $\mathbf{p}$ correspondingly, U is the interaction constant, and V is the volume of the system. Here the same notations (and summation over momenta instead of an integration; this is unimportant) are used as in the recommended book by Abrikosov, Gor'kov, and Dzyaloshinskii. You are advised to follow the book, but present details of all calculations.

   Use the Bogoliubov approximation (where the operators corresponding to the zero-momentum of the condensate $\hat{a}_0$ are replaced by numbers of order $\sqrt{N_0}$; N_0 is the number of particles in the condensate, that is assumed to much exceed the particles outside the condensate, $N_0 \gg N - N_0$) to derive a quadratic in operators Hamiltonian.

   Use the Bogoliubov transform to diagonalize the Hamiltonian and find the spectrum of excitations in the BEC. Calculate the speed of sound.

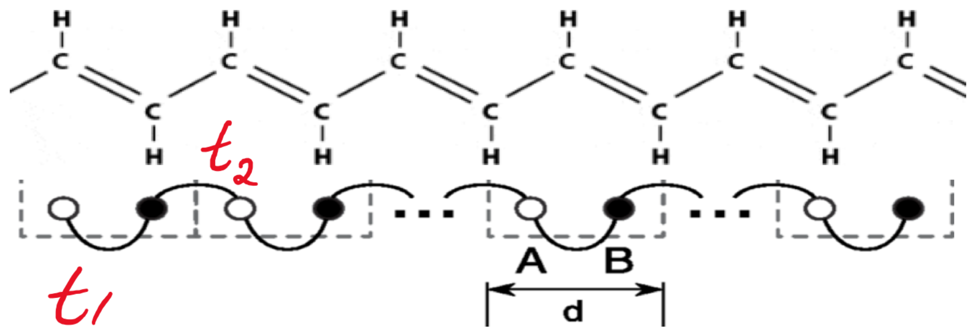
2. **Su-Schrieffer-Heeger (SSH) model**.

   Consider SSH model of fermions hopping on a one-dimensional lattice (see the figure) with alternating strong and weak hopping amplitudes. It is described by the Hamiltonian (first introduced by Shockley):

$$\hat{H} = \sum_n \left(t_1 \hat{c}^\dagger_{n,A} \hat{c}_{n,B} + t_2 \hat{c}^\dagger_{n,A} \hat{c}_{n+1,B} + \text{h.c} \right).$$

   Here $\hat{c}^\dagger_{n,A.B}$ and $\hat{c}_{n,A/B}$ represent fermion creation and anihilation operators for electrons on site n, sublattice A/B correspondingly. Diagonalize the Hamiltonian and find the spectrum of excitations in the model. Plot the spectra. You should consider an infinite fermionic chain with periodic boundary conditions and may ignore the electron spin, which is irrelevant in this problem.

H



3. Berry phase of a spin-1/2 particle in a magnetic field.
Consider the Hamiltonian of a spin-1/2 particle coupled to a time-dependent (slowly varying) Zeeman magnetic field $\vec{B}(t)$ (see lectures; below, we set the particle's magnetic moment to one),

$$\hat{\mathcal{H}}(t) = \vec{\mathbf{B}}(t) \cdot \hat{\vec{\boldsymbol{\sigma}}} \tag{2}$$

where, $\vec{\mathbf{B}}(t) = B\vec{\mathbf{n}}(t)$ (with B being a time-independent constant and $\vec{\mathbf{n}}(t)$ time-dependent *unit* vector) and $\hat{\vec{\boldsymbol{\sigma}}}$ is a vector of Pauli matrices. Assume that the system is initialized in the ground state at $t = 0$.

(a) Write down an operator that rotates spin-1/2 by an angle χ around the vector $\vec{\mathbf{n}}$ and use this operator to derive the eigenstates of the Hamiltonian given above.

(b) Calculate the Berry connection $\vec{\mathbf{A}}(\theta, \phi)$ in spherical coordinates. Assume spherical coordinates θ, ϕ to be defined by,

$$\vec{\mathbf{B}}(t) \equiv (B_x, B_y, B_z) = B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

(c) Calculate the Berry curvature $F_{\theta\phi}$ and show that the degeneracy point $B = 0$ serves as a monopole of the Berry flux. Work out the technical details of the calculation that was outlined in the lectures.

(d) Calculate the Berry phase γ for a closed path (i.e. $\vec{\mathbf{B}}(0) = \vec{\mathbf{B}}(T)$) traced out by $\vec{\mathbf{B}}(t)$ on the surface of the sphere with radius B, and show that $\gamma = \Omega/2$ where, Ω is the solid angle enclosed by this path.