CLOSED GEODESICS ON SURFACES: TOPOLOGY, GEOMETRY, ARITHMETIC

FRANCISCO ARANA–HERRERA AND POUYA HONARYAR

Abstract. We prove a law of large numbers and a central limit theorem for the norm of the homology class, i.e., the arithmetic complexity, of closed geodesics of a fixed topological type and length at most L on an arbitrary negatively curved surface as $L \to \infty$. These results contrast with analogous statements obtained when sampling among all primitive closed geodesics and lead to natural conjectures in the large genus regime. In the course of the proof we also establish laws of large numbers and central limit theorems for the action in homology of mapping class groups.

Contents

1. Introduction

Motivation. Closed geodesics on negatively curved surfaces can be studied from three different points of view: their topology - e.g., whether they are simple or not-, their geometry - e.g., their length-, and their arithmetic - i.e., their homology class. See Figure 1 for a schematic diagram. Many interesting questions arise when studying the mysterious aspects that lie at the center of this diagram, i.e., that connect the three perspectives one can use to study closed geodesics. In this regard, the main goal of this paper is to provide a concrete answer to the following illustrative question: How homologically complicated are long non-separating simple closed geodesics on negatively curved surfaces?

We answer this question in a statistical sense by proving a law of large numbers and a central limit theorem for the norm of the homology class of non-separating simple closed geodesics of length at most L on an arbitrary negatively curved surface as $L \to \infty$. More concretely, we show there exists a universal exponent $\varsigma = \varsigma(g) \in (0, 1)$, such that for any closed negatively surface X of genus $g \geq 2$, the norm of the homology class of most long non-separating simple closed geodesic on X is comparable to the ς-th power of their length. Furthermore, we describe the statistics of appropriately rescaled deviations from this exponent as a non-degenerate normal distribution.

The universal exponent $\varsigma = \varsigma(g) \in (0, 1)$ admits a dynamical interpretation as the top Lyapunov exponent of the invariant part of the Kontsevich-Zorich cocycle on the principal stratum of holomorphic quadratic differentials of genus $g \geq 2$. In particular, numerical experiments of Fougeron [Fou20] suggest that $\varsigma(g) \searrow 1/2$ as $g \nearrow \infty$; see Figure 2 for the results of these experiments. This phenomena fits into a series of more general conjectures of Zorich regarding the asymptotic behavior of the Lyapunov spectra of the Kontsevich-Zorich cocycle in large genus.

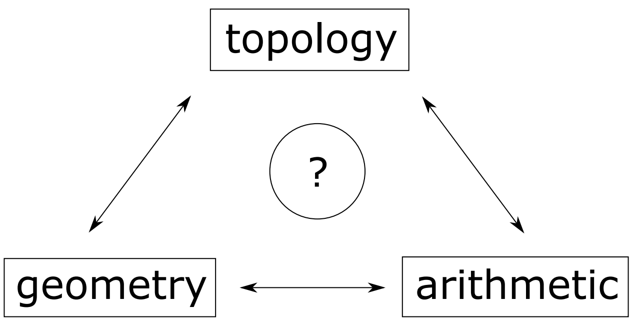

Figure 1. The study of closed geodesics on hyperbolic and negatively curved surfaces.

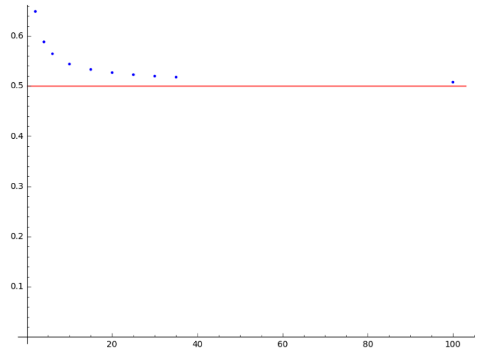

Figure 2. Numerical computation of the exponent $\varsigma(g)$ in terms of the genus $g \geq 2$.

When sampling over all primitive closed geodesics, and not only over those that are simple and non-separating, the description of the corresponding arithmetic properties is much more developed. Indeed, work of several authors [PS87, Eps87, Lal89, Pol91, BL98, Sha04] shows that on any closed negatively curved surface the norm of the homology class of most long primitive closed geodesic is comparable to the square root of their length. Furthermore, the preferred direction in homology of long closed geodesics is chosen according to the measure in projective space induced by an inner product. Moreover, a central limit theorem and a stronger local limit theorem for the joint distribution of the norm and the direction of homology classes of long primitive closed geodesics can be proved. Unfortunately, the methods in these works cannot be used to study the main question of this paper. Indeed, it is very hard to detect simplicity of closed geodesics through coding techniques.

Nevertheless, it is natural to ask whether analogous central limit theorems for the joint distribution of the norm and the direction of homology classes of long closed geodesics on closed negatively curved surfaces can hold when we sample only over simple non-separating geodesics. In this setting, our results actually show that such a central limit theorem cannot hold. Indeed, the order of the rescaling in such a central limit theorem is incompatible with the order of the rescaling in the central limit theorem we prove. These notably different behaviours are certainly compatible with the general theory given that the sampling sizes in each case are drastically different: by work of Huber and Selberg [Hub61], the number of primitive closed geodesics of length at most L grows like an exponential in L while, by work of Mirzakhani [Mir08b], the number of simple non-separating closed geodesics of length at most L grows like a polynomial in L of degree $6g-6$, where $g \geq 2$ is the genus of the corresponding closed negatively curved surface. Nevertheless, the conjecture that $\lambda(g) \searrow 1/2$ as $g \nearrow \infty$ seems to point out a connection between both regimes that fits into a more general mantra: simple closed geodesics behave like primitive ones in large genus.

The proof of the main results of this paper is based on the effective tracking principle for mapping class group actions introduced by the first author in [AH21a] and later developed by the second author in [Hon24]. Roughly speaking, this tracking principle states that the action of the mapping class group on the space of closed curves of a closed surface effectively tracks the corresponding action on Teichmüller space in the following sense: for all but quantitatively few mapping classes, the information of how a mapping class moves a given point of Teichmüller space determines, up to a power saving error term, how it changes the geometric intersection numbers of a given closed curve with respect to arbitrary geodesic currents. The relevant geodesic currents in our setting are the Liouville currents of negatively curved metrics on the surface of interest.

Using this tracking principle one can reduce the original questions about the statistics of the norm of the homology classes of long non-separating simple closed geodesics on negatively curved surfaces to questions about the statistics of the action of the mapping class group on the homology group of the surface relative to its action on Teichmüller space. These questions in turn can be thoroughly studied using the dynamics of the Teichmüller geodesic flow on the principal stratum of holomorphic quadratic differentials and, more concretely, using the Kontsevich-Zorich cocycle. In fact, a careful application of averaging and unfolding principles originally introduced by Margulis in his thesis [Mar70] allows one to reduce such questions to problems concerning the equidistribution of weighted Teichmüller balls on moduli spaces of Riemann surfaces.

Such problems in dynamics are generally studied using mixing properties of the corresponding flow, in this case the Teichmüller geodesic flow, but the additional weights in our case force upon us the use of more sophisticated mixing limit theorems. Such limit theorems were introduced by Dolgopyat and Nandori in [DN20] to study general hyperbolic flows. In our setting, mixing laws of large numbers and central limit theorems for the Kontsevich-Zorich cocycle were proved by the first author and Forni in [AF24]. This work provides a general framework for upgrading limit theorems to mixing limit theorems under mild ergodicity and hyperbolicity conditions. Its application to the Kontsevich-Zorich cocycle in turn relies on works of several authors [Fil17, BDG$^+$21, ASF22].

Statements of the main results. Let X be a closed, connected, oriented surface endowed with a negatively curved Riemannian metric. Two closed curves on X are said to have the same topological type if there exists a homeomorphism of X identifying the corresponding free homotopy classes. Notice that closed geodesics on X that have the same topological type as a simple non-separating closed curve are precisely those that are simple and non-separating. Given a closed curve γ on X, denote by $[\gamma] \in H_1(X;\mathbb{R})$ its homology class. Given a closed geodesic γ on X denote by $\ell_X(\gamma) > 0$ its corresponding length.

Given a closed curve γ_0 on X and $L > 0$ consider the finite set $\mathfrak{G}(X,\gamma_0,L)$ of all closed geodesics γ on X of the same topological type as γ_0 and length $\ell_X(\gamma) \leq L$. Endow this space with the uniform probability measure $\mathbb{P}_{X,\gamma_0,L}$.

In this paper we prove the following law of large numbers for the norm of the homology class of long closed geodesics of a fixed topological type; this is the first main result of this paper.

Theorem 1.1. *For every $g \geq 2$ there exists a constant $\varsigma = \varsigma(g) \in (0,1)$ with the following property. Let X be a closed, connected, oriented surface of genus g endowed with a negatively curved Riemannian metric, γ_0 be a closed curve on X that is non-trivial in homology, and $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|[\gamma]\|}{\log \ell_X(\gamma)} \quad \text{on} \quad (\mathfrak{G}(X,\gamma_0,L), \mathbb{P}_{X,\gamma_0,L})$$

converge in distribution to the point mass at ς as $L \to \infty$.

Remark 1.2. The convergence in distribution in Theorem 1.1 is equivalent to

$$\forall \epsilon > 0\colon \lim_{L\to\infty} \frac{\#\{\gamma \in \mathfrak{G}(X,\gamma_0,L)\colon \log\|[\gamma]\|/\log \ell_X(\gamma) \in (\varsigma-\epsilon,\varsigma+\epsilon)\}}{\#\mathfrak{G}(X,\gamma_0,L)} = 1.$$

Roughly speaking, Theorem 1.1 shows there exists a universal constant $\varsigma = \varsigma(g) \in (0,1)$, depending only on the topology, i.e., the genus $g \geq 2$, of the negatively curved surface being considered, and not on its geometry or on the topological type of the closed geodesics being sampled, such that the norm of the homology class of long closed geodesics on the surface of the given topological type is comparable to the ς-th power of their length.

In this paper we also prove the following central limit theorem for the norm of the homology class of long closed geodesics of a fixed topological type; this is the second main result of this paper.

Theorem 1.3. *For every $g \geq 2$ there exist constants $\varsigma = \varsigma(g) \in (0,1)$ and $V = V(g) > 0$ with the following property. Let X be a closed, connected, oriented surface of genus g endowed with a negatively curved Riemannian metric, γ_0 be a closed curve on X that is non-trivial in homology, and $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|[\gamma]\| - \log \ell_X(\gamma)\cdot\varsigma}{\sqrt{\log \ell_X(\gamma)}} \quad \text{on} \quad (\mathfrak{G}(X,\gamma_0,L), \mathbb{P}_{X,\gamma_0,L})$$

converge in distribution to a Gaussian of mean 0 and variance V as $L \to \infty$.

Remark 1.4. The convergence in distribution in Theorem 1.3 is equivalent to

$$\forall a < b\colon \lim_{L\to\infty} \frac{\#\{\gamma \in \mathfrak{G}(X,\gamma_0,L)\colon (\log\|[\gamma]\| - \log\ell_X(\gamma)\cdot\varsigma)/\sqrt{\log \ell_X(\gamma)} \in (a,b)\}}{\#\mathfrak{G}(X,\gamma_0,L)}$$
$$= \frac{1}{\sqrt{2\pi V}} \int_a^b e^{-x^2/2V}\, dx.$$

Remark 1.5. Notice that the statistics considered in Theorem 1.3 are not well defined if $\log \ell_X(\gamma) < 1$. This only happens for finitely many of the closed geodesics being sampled. In particular, this does not affect the limiting distribution obtained.

Remark 1.6. Theorems 1.1 and 1.3 apply in more generality. Instead of considering a negatively curved Riemannian metric, one can consider an arbitrary geodesic current on the corresponding surface; see Theorems 4.20 and 4.21 for precise statements.

Roughly speaking, Theorem 1.3 shows the deviations from the universal exponent $\varsigma = \varsigma(g) \in (0,1)$ in Theorem 1.1 converge to a normal distribution of mean zero and variance $V = V(g) > 0$ when appropriately rescaled. Furthermore, this variance depends only on the topology of the underlying surface and not on its geometry or the topological type of the closed geodesics being sampled.

Limit theorems for mapping class groups. Fix a closed, connected, oriented surface S_g of genus $g \geq 2$. Denote by $\mathcal{T}_g$ the Teichmüller space of marked complex structures on S_g. The mapping class group Mod_g of S_g acts on $\mathcal{T}_g$ by changing the markings. Denote by $d_\mathcal{T}$ the Teichmüller metric on $\mathcal{T}_g$.

Given $X, Y \in \mathcal{T}_g$ and $R \geq 0$ denote by $\mathfrak{M}(X,Y,R)$ the set of all mapping classes $\mathbf{g} \in \text{Mod}_g$ such that $0 < d_\mathcal{T}(X, \mathbf{g}.Y) \leq R$. Endow this space with the uniform probability measure $\mathbb{P}_{X,Y,R}$.

In the course of the proof of Theorem 1.1 we prove the following result of independent interest corresponding to a law of large numbers for the action in homology of mapping class groups.

Theorem 1.7. *For every $g \geq 2$ there exists a constant $\varsigma = \varsigma(g) \in (0,1)$ with the following property. Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $v_0 \in H_1(X;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}.v_0\|}{d_\mathcal{T}(X,\mathbf{g}.Y)} \quad \text{on} \quad (\mathfrak{M}(X,Y,R), \mathbb{P}_{X,Y,R})$$

converge in distribution to the point mass at ς as $L \to \infty$.

In the course of the proof of Theorem 1.3 we prove the following result of independent interest corresponding to a central limit theorem for the action in homology of mapping class groups.

Theorem 1.8. *For every $g \geq 2$ there exist constants $\varsigma = \varsigma(g) \in (0,1)$ and $V = V(g) > 0$ with the following property. Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $v_0 \in H_1(X;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}.v_0\| - d_\mathcal{T}(X,\mathbf{g}.Y)\cdot\varsigma}{\sqrt{d_\mathcal{T}(X,\mathbf{g}.Y)}} \quad \text{on} \quad (\mathfrak{M}(X,Y,R), \mathbb{P}_{X,Y,R})$$

converge in distribution to a Gaussian of mean 0 and variance V as $L \to \infty$.

Remark 1.9. Notice that the statistics considered in Theorem 1.8 are not well defined if $d_\mathcal{T}(X,\mathbf{g}.Y) < 1$. This only happens for finitely many of the mapping classes being sampled. In particular, this does not affect the limiting distribution obtained.

Remark 1.10. The constants $\varsigma = \varsigma(g) \in (0,1)$ and $V = V(g) > 0$ in Theorems 1.1, 1.3, 1.7, and 1.8 are the same. More concretely, ς is the top Lyapunov exponent of the invariant part of the Kontsevich-Zorich cocycle on the principal stratum of holomorphic quadratic differentials of genus $g \geq 2$ and V is the variance predicted by the central limit theorem of Al–Saqban and Forni [ASF22] for this cocycle. We remind the reader that numerical experiments of Fougeron [Fou20] and conjectures of Zorich predict that $\varsigma(g) \searrow 1/2$ as $g \nearrow \infty$; see Figure 2.

Remark 1.11. The proofs of Theorems 1.1 and 1.3 actually require more precise versions of Theorems 1.7 and 1.8 that apply to bisectors of Teichmüller space rather than just balls; see Theorems 3.33 and 3.34 for precise statements.

Remark 1.12. The proofs of Theorems 1.7 and 1.8 can be adapted to obtain other laws of large numbers and central limit theorems for the action in homology of mapping class groups; see Theorems 3.35 and 3.36 for examples and Remark 3.37 for an extended discussion.

Primitive closed geodesics. Let X be a closed, connected, oriented surface endowed with a negatively curved Riemannian metric. Given $L > 0$, denote by $\mathfrak{G}(X,L)$ the set of all primitive closed geodesics γ on X with $\ell_X(\gamma) \leq L$. Endow this space with the uniform probability measure $\mathbb{P}_{X,L}$.

Although it will not be used in the rest of this paper, we give a precise statement of a central limit theorem in work of Sharp [Sha04] for the homology class of long primitive closed geodesics without prescribed topological type on negatively curved surfaces. The reader might find it useful to keep this result in mind as a point of comparison while exploring the rest of this paper.

Theorem 1.13. *Let X be a closed, connected, oriented surface of genus $g \geq 2$ endowed with a negatively curved Riemannian metric. Identify $H_1(X;\mathbb{R}) = \mathbb{R}^{2g}$. Then, there exists a positive definite $2g \times 2g$ matrix $\Sigma = \Sigma(X)$ such that, as $L \to \infty$, the $H_1(X;\mathbb{R})$–valued random variables*

$$[\gamma]/\sqrt{\ell_X(\gamma)} \quad \text{on} \quad (\mathfrak{G}(X,L), \mathbb{P}_{X,L})$$

converge in distribution to a Gaussian multivariate of mean $\vec{0} \in \mathbb{R}^{2g}$ and covariance matrix Σ.

Remark 1.14. The convergence in distribution in Theorem 1.13 is equivalent to

$$\forall A \subseteq \mathbb{R}^{2g} \text{ with } \text{Leb}(\partial A) = 0\colon \lim_{L\to\infty} \frac{\#\{\gamma \in \mathfrak{G}(X,L)\colon [\gamma]/\sqrt{\ell_X(\gamma)} \in A\}}{\#\mathfrak{G}(X,L)}$$
$$= \frac{1}{\sqrt{(2\pi)^{2g}\det(\Sigma)}} \int_A \exp\left(\frac{-\vec{x}^T\Sigma^{-1}\vec{x}}{2}\right) d\vec{x}.$$

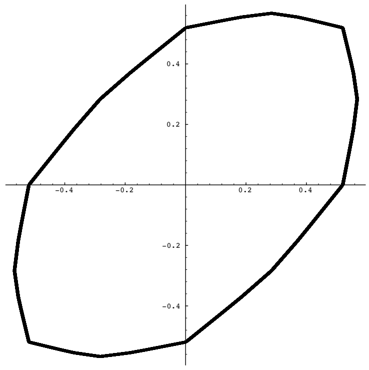

FIGURE 3. The unit ball of the stable norm of a hyperbolic once punctured torus.

Roughly speaking, Theorem 1.13 shows that the norm of the homology class of most long primitive closed geodesics on an arbitrary negatively curved surface X is comparable to the square root of their length. Furthermore the direction in homology of these geodesics is chosen according to an inner product, the one defined by the positive definite matrix $\Sigma = \Sigma(X)$.

As a direct consequence of Theorem 1.13 one can deduce the following corollary.

Corollary 1.15. *Let X be a closed, connected, oriented surface of genus $g \geq 2$ endowed with a negatively curved Riemannian metric and let $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Given $0 < r < R$, denote by $A(r,R) \subseteq \mathbb{R}^{2g}$ the annulus of inner radius r and outer radius R. Then, there exists a positive definite $2g \times 2g$ matrix $\Sigma = \Sigma(X)$ such that*

$$\forall a < b\colon \lim_{L\to\infty} \frac{\#\{\gamma \in \mathfrak{G}(X,L)\colon \log\|[\gamma]\| - \log \ell_X(\gamma)/2 \in (a,b)\}}{\#\mathfrak{G}(X,L)} = \frac{1}{\sqrt{(2\pi)^{2g}\det(\Sigma)}} \int_{A(e^a,e^b)} \exp\left(\frac{-\vec{x}^T\Sigma^{-1}\vec{x}}{2}\right) d\vec{x}.$$

In particular, Theorem 1.3 and Corollary 1.15 show that a central limit theorem like Theorem 1.13 cannot hold when sampling over closed geodesics of a fixed topological type. Indeed, the scalings in Theorem 1.3 and Corollary 1.15 needed to get nontrivial limits are incompatible.

Directions in homology. Let X be a connected, oriented surface endowed with a complete negatively curved Riemannian metric; we allow this surface to have punctures. The stable norm of X is the norm on the homology group $H_1(X;\mathbb{R})$ defined as follows:

$$\|c\| := \inf\left\{\sum_{i=1}^n r_i \ell_X(\gamma_i)\colon \begin{array}{l}\gamma_1,\ldots,\gamma_n \text{ piecewise smooth closed curves on } X,\\ \sum_{i=1}^n r_i[\gamma_i] = c,\\ r_1,\ldots,r_n > 0.\end{array}\right\}.$$

See Figure 3 for a numerical computation of the unit ball of this norm for a hyperbolic once punctured torus [MR95b]. Given a non-zero homology class $c \in H_1(X;\mathbb{R})$, denote by $[c] \in \mathbb{P}H_1(X;\mathbb{R})$ its projective class. The stable norm $\|\cdot\|$ induces a Borel measure μ on $\mathbb{P}H_1(X;\mathbb{R})$ given by

$$\mu(A) = \mathrm{Leb}(\{c \in H_1(X;\mathbb{R})\colon 0 < \|c\| \leq 1 \text{ and } [c] \in A\}).$$

Regardless of the incompatibility between Theorems 1.3 and 1.13, one can still consider the question of preferred directions in homology for long closed geodesics of a fixed topological type. Although the general case seems out of reach for the moment, we highlight the following result of McShane and Rivin [MR95a] for hyperbolic once punctured tori.

Theorem 1.16. *Let X be a hyperbolic once punctured torus and γ_0 be a simple non-separating closed curve on X. Denote by μ the measure induced by the stable norm of X on $\mathbb{P}H_1(X;\mathbb{R})$. Then, the $\mathbb{P}H_1(X;\mathbb{R})$–valued random variables*

$$[[\gamma]] \quad on \quad (\mathfrak{G}(X,\gamma_0,L), \mathbb{P}_{X,\gamma_0,L})$$

converge in distribution to μ as $L \to \infty$.

Remark 1.17. In [MR95a], McShane and Rivin show that the stable norm on the homology group $H_1(X;\mathbb{R})$ of any hyperbolic once punctured torus X is highly non-smooth in the sense that its unit ball has a corner at every line intersecting the integer lattice $H_1(X;\mathbb{Z})$; compare to the case of Theorem 1.13 where the direction in homology is chosen according to an inner product.

Open problems. The previous discussion naturally leads to the following open problems of interest.

Problem 1.18. *Show that the universal exponent $\varsigma = \varsigma(g) \in (0,1)$ in Theorems 1.1, 1.3, 1.7, and 1.8 satisfies the asymptotics $\varsigma(g) \searrow 1/2$ as $g \nearrow \infty$.*

Problem 1.19. *Given a negatively curved surface, characterize the measure on the projectivized homology group of the surface that records the statistics of the direction in homology of long closed geodesics of a prescribed topological type. In particular, does this measure coincide with the measure induced by the stable norm?*

Problem 1.20. *For an arbitrary negatively curved surface, give an asymptotic formula, as $L \to \infty$, for the number of closed geodesics of a given topological type and length at most L that belong to a prescribed homology class.*

Organization of the paper. In §2 we cover the preliminaries on Teichmüller theory that will be used throughout the rest of the paper. In §3 we prove Theorems 1.7 and 1.8 above. Furthermore, we prove stronger versions of these theorems that apply to bisectors of Teichmüller space; see Theorems 3.33 and 3.34. The corresponding mixing limit theorems, key ingredients in the proofs, are introduced as Theorems 3.1 and 3.2. In §4 we prove Theorems 1.1 and 1.3 above. Furthermore, we prove stronger versions that apply to arbitrary filling geodesic currents; see Theorems 4.20 and 4.21. The tracking principle, see 4.7, and the technical tools needed to handle the case of non-filling closed curves are also discussed in detail in this section.

Acknowledgements. The authors would like to thank Alex Eskin, Giovanni Forni, and Anton Zorich for very enlightening conversations on the subject of this paper.

2. Preliminaries

Outline of this section. In this section we give a brief overview of the objects and terminology that will be used throughout the rest of this paper. With the exception of the discussion on Hubbard-Masur functions and on central limit theorems for the Kontsevich-Zorich cocycle, see Theorem 2.5, the material presented in this section is standard.

Quadratic differentials. A holomorphic quadratic differential q on a Riemann surface X is a differential which in local coordinates z has the form $f(z)\,dz^2$ for some holomorphic function $f(z)$. Such a differential has a well defined notion of area,

$$\|q\| := \text{Area}(q) = \int_X |q|.$$

More precisely, the differential q induces a singular flat metric on X. If in local coordinates $z = x + iy$ then the metric is given by $dx^2 + dy^2$; the zeroes of the differential correspond to singularities of this metric. The area of q is the total area of this metric. Denote by $Q(X)$ the complex vector space of all holomorphic quadratic differentials on a Riemann surface X and by $S(X) \subseteq Q(X)$ the sphere of all such differentials of unit area. We sometimes denote quadratic differentials by (X,q) to record the Riemann surface X they are defined on.

Teichmüller and moduli spaces. Fix a closed, connected, oriented surfaces S_g of genus $g \geq 2$. Denote by $\mathcal{T}_g$ the Teichmüller space of marked complex structures on S_g. The spaces $Q(X)$ and $S(X)$ for X ranging over $\mathcal{T}_g$ can be arranged into bundles $\mathcal{QT}_g$ and $\mathcal{Q}^1\mathcal{T}_g$ of marked quadratic differentials on S_g. Denote by Mod_g the mapping class group of S_g. This group acts properly discontinuously on $\mathcal{T}_g$, $\mathcal{QT}_g$, and $\mathcal{Q}^1\mathcal{T}_g$ by changing the markings. The corresponding quotients $\mathcal{M}_g$, $\mathcal{QM}_g$, and $\mathcal{Q}^1\mathcal{M}_g$ are moduli spaces of complex structures and quadratic differentials on S_g. Denote by $\widehat{\pi}\colon \mathcal{QT}_g \to \mathcal{T}_g$ and by $\pi\colon \mathcal{QM}_g \to \mathcal{M}_g$ the natural forgetful maps.

The Masur-Veech measure. The bundle $\mathcal{QT}_g$ can be identified with the cotangent bundle of $\mathcal{T}_g$. In particular, it supports a canonical volume form. The restriction of this volume form to $\mathcal{Q}^1\mathcal{T}_g$ induces a smooth measure $\widehat{\mu}$ called the Masur-Veech measure. This measure is invariant under the marking changing action of Mod_g on $\mathcal{Q}^1\mathcal{T}_g$. Denote by μ its local pushforward to the quotient space $\mathcal{Q}^1\mathcal{M}_g$; we also refer to this measure as the Masur-Veech measure. Independent works of Hubbard and Masur [Mas82, Vee82] show that the measure μ on $\mathcal{Q}^1\mathcal{M}_g$ is finite. Later we introduce a particular normalization of this measure that will be useful for our purposes; see (2).

Singular measured foliations. Denote by $\mathcal{MF}_g$ the space of singular measured foliations on S_g up to isotopy and Whitehead moves. The set of isotopy classes of weighted simple closed curves on S_g embeds densely into $\mathcal{MF}_g$. Furthermore, geometric intersection numbers extend continuously to a pairing $i(\cdot,\cdot)$ on $\mathcal{MF}_g$. Train track coordinates induce a natural integral piecewise linear structure on $\mathcal{MF}_g$. A particular example of such coordinate system is provided by Dehn-Thurston coordinates, which identify $\mathcal{MF}_g$ with Σ^{3g-3}, where $\Sigma := \mathbb{R}^2/\langle \pm 1 \rangle$, via intersection and twisting numbers with respect to a fixed pair of pants decomposition; see [PH92, §1.2]. In particular, $\mathcal{MF}_g$ carries a natural Lebesgue class measure ν called the Thurston measure. This measure is invariant under the natural Mod_g-action on $\mathcal{MF}_g$. We consider the normalization of the Thurston measure induced by the symplectic structure described in [PH92, §3.2]. Denote by $\mathcal{PMF}_g$ the projectivization of $\mathcal{MF}_g$ under the $\mathbb{R}_+$ action that scales transverse measures and by $[\lambda] \in \mathcal{PMF}_g$ the projective class of $\lambda \in \mathcal{MF}_g$.

The Hubbard-Masur theorems. Every quadratic differential q on a Riemann surface X gives rise to a pair of singular measured foliations $\Re(q)$ and $\Im(q)$ on X. If in local coordinates $z = x + iy$ the differential q corresponds to dz^2 then $\Re(q)$ corresponds to the measured foliation induced by $|dx|$ and $\Im(q)$ corresponds to the measured foliation induced by $|dy|$; the zeroes of q correspond to the singularities of the foliations. We refer to $\Re(q)$ and $\Im(q)$ as the real/vertical and imaginary/horizontal foliations of q. These constructions give rise to Mod_g-equivariant maps $\Re, \Im \colon \mathcal{QT}_g \to \mathcal{MF}_g$.

The following theorem of Hubbard and Masur [HM79] allows us to parametrize $Q(X)$ in terms of vertical foliations across all $X \in \mathcal{T}_g$.

Theorem 2.1. *Given $X \in \mathcal{T}_g$ and a singular measured foliation $\eta \in \mathcal{MF}_g$, there exists a unique quadratic differential $q = q(X,\eta) \in Q(X)$ such that $\Re(q) = \eta$. Furthermore, the map $q \in Q(X) \mapsto \Re(q) \in \mathcal{MF}_g$ is a homeomorphism. The analogous result holds for imaginary foliations.*

We say that a pair of singular measured foliations $(\eta,\zeta) \in \mathcal{MF}_g \times \mathcal{MF}_g$ fills the surface if the sum of their geometric intersection numbers with any singular measured foliation is positive. Denote by $\Delta_{\mathcal{MF}_g} \subseteq \mathcal{MF}_g \times \mathcal{MF}_g$ the set of non-filling pairs of singular measured foliations. The following theorem of Gardiner and Masur [GM91] allows us to globally parametrize $\mathcal{QT}_g$ in terms of real and imaginary foliations.

Theorem 2.2. *Given a filling pair of singular measured foliations $(\eta,\zeta) \in \mathcal{MF}_g \times \mathcal{MF}_g \setminus \Delta_{\mathcal{MF}_g}$, there exists a unique quadratic differential $q \in \mathcal{QT}_g$ such that $\Re(q) = \eta$ and $\Im(q) = \zeta$. Furthermore, the map $(\Re,\Im) \colon \mathcal{QT}_g \to \mathcal{MF}_g \times \mathcal{MF}_g \setminus \Delta_{\mathcal{MF}_g}$ is a homeomorphism.*

The Teichmüller geodesic flow. A half-translation structure on a surface S is an atlas of charts to $\mathbb{C}$ on the complement of a finite set of points $\Sigma \subseteq S$ whose transition functions are of the form $z \mapsto \pm z + c$ with $c \in \mathbb{C}$. Every quadratic differential q induces a half-translation structure on the Riemann surface it is defined on by considering local coordinates on the complement of the zeroes of q for which $q = dz^2$. Viceversa, every half-translation structure induces a quadratic differential on its underlying surface by pulling back the differential dz^2 on the corresponding charts.

The group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on half-translation structures by postcomposing the corresponding charts with the linear action of this group on $\mathbb{C} = \mathbb{R}^2$. In particular, the group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on $\mathcal{QT}_g$ preserving $\mathcal{Q}^1\mathcal{T}_g$. For every $t \in \mathbb{R}$ and every $\theta \in [0,2\pi]$ denote

$$a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1}$$

The flow induced by the action of the one-parameter subgroup $\{a_t\}_{t\in\mathbb{R}} \subseteq \mathrm{SL}(2,\mathbb{R})$ is the Teichmüller geodesic flow. For every $\theta \in [0,2\pi]$ and every $q \in \mathcal{QT}_g$ one can show $r_\theta = e^{2\theta i} q$. In particular, the action of $\mathrm{SO}(2) \subseteq \mathrm{SL}(2,\mathbb{R})$ preserves the fibers $Q(X)$ of $\mathcal{QT}_g$ and the fibers $S(X)$ of $\mathcal{Q}^1\mathcal{T}_g$. The $\mathrm{SL}(2,\mathbb{R})$ and Mod_g actions on $\mathcal{QT}_g$ commute. In particular, there is a well defined $\mathrm{SL}(2,\mathbb{R})$ action and a well defined Teichmüller geodesic flow on $\mathcal{QM}_g$.

Stable and unstable foliations. Every $q_0 \in \mathcal{Q}^1\mathcal{T}_g$ supports strongly stable, central, and strongly unstable leaves given respectively by

$$\alpha^{ss}(q_0) = \{q \in \mathcal{Q}^1\mathcal{T}_g \mid \Re(q) = \Re(q_0)\},$$
$$\alpha^{c}(q_0) = \{a_t q_0 \mid t \in \mathbb{R}\},$$
$$\alpha^{uu}(q_0) = \{q \in \mathcal{Q}^1\mathcal{T}_g \mid \Im(q) = \Im(q_0)\}.$$

These leaves give rise to topological foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{c}$, and $\mathcal{F}^{uu}$ called the strongly stable, central, and strongly unstable foliations of $\mathcal{Q}^1\mathcal{T}_g$. The stable and unstable leaves of $q_0 \in \mathcal{Q}^1\mathcal{T}_g$ are given by

$$\alpha^s(q_0) := \{q \in \mathcal{Q}^1\mathcal{T}_g \mid [\Re(q)] = [\Re(q_0)]\} = \bigcup_{t\in\mathbb{R}} a_t\alpha^{uu}(q_0),$$
$$\alpha^u(q_0) := \{q \in \mathcal{Q}^1\mathcal{T}_g \mid [\Im(q)] = [\Im(q_0)]\} = \bigcup_{t\in\mathbb{R}} a_t\alpha^{ss}(q_0).$$

These leaves give rise to topological foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ called the stable and unstable foliations.

Leafwise measures. Given $\eta \in \mathcal{MF}_g$, denote by $\mathcal{MF}_g(\eta) \subseteq \mathcal{MF}_g$ the open, dense, full-measure subset of singular measured foliations on S_g that together with η fill the surface. Fix $q_0 \in \mathcal{Q}^1\mathcal{T}_g$. By Theorem [GM91], the restriction $\Im|_{\alpha^s(q_0)} \colon \alpha^s(q_0) \to \mathcal{MF}_g(\Re(q_0))$ is a homeomorphism. Denote by $\mu_{\alpha^s(q_0)}$ the pullback to $\alpha^s(q_0)$ of the Thurston measure ν on $\mathcal{MF}_g(\Re(q_0))$. Analogously, one can define a measure $\mu_{\alpha^u(q_0)}$ on the unstable leaf $\alpha^u(q_0)$.

Given $\eta \in \mathcal{MF}_g$, denote by $\mathcal{MF}_g^1(\eta) \subseteq \mathcal{MF}_g$ the subset of singular measured foliations $\zeta \in \mathcal{MF}_g(\eta)$ such that $i(\eta,\zeta) = 1$. Fix $q_0 \in \mathcal{Q}^1\mathcal{T}_g$. By Theorem [GM91], the restriction $\Im|_{\alpha^{ss}(q_0)} \colon \alpha^{ss}(q_0) \to \mathcal{MF}_g^1(\Re(q_0))$ is a homeomorphism. Denote by $\mu_{\alpha^{ss}(q_0)}$ the pullback to $\alpha^s(q_0)$ of the conned-off Thurston measure on $\mathcal{MF}_g^1(\Re(q_0))$. Analogously, one can define a measure $\mu_{\alpha^{uu}(q_0)}$ on the strongly unstable leaf $\alpha^{uu}(q_0)$.

We normalize the Masur-Veech measure μ on $\mathcal{Q}^1\mathcal{T}_g$ so that locally

$$d\mu = d\mu_{\alpha^u}\, d\mu_{\alpha^{ss}} = d\mu_{\alpha^s}\, d\mu_{\alpha^{uu}}. \tag{2}$$

Strata and compactness. Denote by $\mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \subseteq \mathcal{Q}^1\mathcal{T}_g$ and $\mathcal{Q}^1\mathcal{M}_g(\mathbf{1}) \subseteq \mathcal{Q}^1\mathcal{M}_g$ the principal strata of marked/unmarked, unit area, holomorphic quadratic differentials on S_g, that is, the corresponding subsets of differentials with $4g-4$ distinct zeroes of multiplicity one. The complements of these strata, the so-called multiple zero loci, are zero measure subsets of the respective Masur-Veech measure classes.

A saddle connection of a quadratic differential is a geodesic in the corresponding singular flat metric connecting two singularities and having no other singularities in its interior. Given $q \in \mathcal{Q}^1\mathcal{M}_g$, denote by $\ell_{\min}(q) > 0$ the length of the shortest saddle connections of q. For every $\delta > 0$ consider the subset $K_\delta \subseteq \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$ of quadratic differentials $q \in \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$ such that $\ell_{\min}(q) \geq \delta$. By work of Masur, see for instance [MT02, Proposition 3.6], these subsets are a compact exhaustion of $\mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$.

Fiberwise measures. Denote by $\widehat{\mathbf{m}}$ the pushforward to $\mathcal{T}_g$ of the Masur-Veech measure $\widehat{\mu}$ on $\mathcal{Q}^1\mathcal{T}_g$ under the natural forgetful map; we refer to this measure as the Masur-Veech measure on $\mathcal{T}_g$. This measure is Mod_g invariant and smooth. Denote by $\{s_X\}_{X\in\mathcal{T}_g}$ the disintegration of the Masur-Veech measure $\widehat{\mu}$ along the fibers of the forgetful map $\widehat{\pi} \colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{T}_g$. More precisely, for every $X \in \mathcal{T}_g$ there exists a unique probability measure s_X on $S(X) = \widehat{\pi}^{-1}(X)$ such that the following disintegration formula holds,

$$d\widehat{\mu}(X,q) = ds_X(q)\, d\widehat{\mathbf{m}}(X). \tag{3}$$

The fiberwise measures s_X are actually smooth and one can make sense of the disintegration above at the level of volume forms. By [ABEM12a, Theorem 2.2], the measures s_X give zero mass to the multiple zero locus.

Denote by $\mathbf{m}$ the local pushforward to $\mathcal{M}_g$ of the measure $\widehat{\mathbf{m}}$ on $\mathcal{T}_g$ under the natural quotient map. Equivalently, $\mathbf{m}$ is the pushforward to $\mathcal{M}_g$ of the Masur-Veech measure μ on $\mathcal{Q}^1\mathcal{M}_g$ under the natural projection. We refer to this measure as the Masur-Veech measure on $\mathcal{M}_g$. The following disintegration formula at the level of moduli spaces also holds,

$$d\mu(X,q) = ds_X(q)\, d\mathbf{m}(X).$$

Extremal length. Given a Riemann surface X and a simple closed curve γ on it, one can define the extremal length of γ with respect to X in two equivalent ways. Analytically, it can be defined as

$$\text{Ext}_X(\gamma) := \sup_\rho \frac{\ell_\rho(\gamma)^2}{\text{Area}(\rho)},$$

where the supremum runs over all conformal metrics ρ on X of non-zero, finite area, and $\ell_\rho(\gamma)$ denotes the infimum of the ρ-lengths of simple closed curves homotopic to γ. Equivalently, it can be defined geometrically as

$$\text{Ext}_X(\gamma) := \inf_C \frac{1}{\text{mod}(C)},$$

where the infimum ranges over all embedded cylinders C on X with core curve homotopic to γ and $\text{mod}(C)$ denotes the modulus of the cylinder.

In independent works, Jenkins and Strebel [Jen57, Str66, Str75, Str76] showed these two a priori different notions of extremal length are actually equivalent. As a direct consequence of Theorem 2.1, the notion of extremal length with respect to any $X \in \mathcal{T}_g$ can be extended to a unique continuous 2-homogeneous function on $\mathcal{MF}_g$.

The Hubbard-Masur functions. Theorem 2.1 implies that, for every $\eta \in \mathcal{MF}_g$, the projection $\pi\colon \mathcal{QT}_g \to \mathcal{T}_g$ restricts to a bijection from $\Re^{-1}(\eta)$ onto $\mathcal{T}_g$. In [HM79], Hubbard and Masur actually proved that this restriction is a homeomorphism and, furthermore, a diffeomorphism onto its image when restricted to $\Re^{-1}(\eta) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$. In particular, for every $q \in \mathcal{Q}^1\mathcal{T}_g$, the projection $\pi\colon \mathcal{QT}_g \to \mathcal{T}_g$ is a homeomorphism onto $\mathcal{T}_g$ when restricted to $\alpha^s(q)$ and a diffeomorphism onto its image when restricted to $\alpha^s(q) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$. The analogous fact also holds for unstable leaves. The Hubbard-Masur functions, introduced in [ABEM12b], are the unique smooth, positive functions $\lambda^-, \lambda^+\colon \mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \to \mathbb{R}_{>0}$ such that for every $(X,q) \in \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$,

$$d\widehat{\mathbf{m}}(X) = \lambda^-(q)\, d\widehat{\pi}_*(\mu_{\alpha^s(q)})(X),$$
$$d\widehat{\mathbf{m}}(X) = \lambda^+(q)\, d\widehat{\pi}_*(\mu_{\alpha^u(q)})(X).$$

The Hubbard-Masur functions can be defined in the following alternative way. By Theorem 2.1, for every $X \in \mathcal{T}_g$, the maps $\Re, \Im\colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{MF}_g$ are homeomorphism onto $\mathcal{MF}_g$ when restricted to $Q(X)$. Moreover, these maps are piecewise smooth isomorphisms onto their image when restricted to $Q(X) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$; see for instance [Mir08a, Lemma 4.3] or [Dum15, Theorem 5.8]. Consider the subset $E_X \subseteq \mathcal{MF}_g$ of singular measured foliations of unit extremal length with respect to X and denote by ν_X the conned-off Thurston measure on E_X. By [ABEM12b, Proposition 2.3], the Hubbard-Masur functions $\lambda^-, \lambda^+\colon \mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \to \mathbb{R}_{>0}$ are the unique smooth, positive functions such that, for every $(X,q) \in \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$,

$$d((\Re|_{S(X)})^*\nu_X)(q) = \lambda^-(q)\, ds_X(q),$$
$$d((\Im|_{S(X)})^*\nu_X)(q) = \lambda^+(q)\, ds_X(q).$$

Directly from this definition one can check that the Hubbard-Masur functions are Mod_g-invariant. Furthermore, in [AH23] it is proved that the Hubbard-Masur functions are SO(2)-invariant and coincide everywhere on $\mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$. We denote by $\lambda\colon \mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \to \mathbb{R}_{>0}$ the corresponding function and refer to it as the Hubbard-Masur function.

The Hubbard-Masur constant. For every $X \in \mathcal{T}_g$ consider

$$\Lambda(X) := \nu(\{\eta \in \mathcal{MF}_g \mid \mathrm{Ext}_X(\eta) \leq 1\}).$$

A direct computation using the definitions above shows that

$$\Lambda(X) = \int_{S(X)} \lambda(q)\, ds_X(q). \tag{4}$$

Combining results of Dumas [Dum15] and Gardiner [Gar84], Mirzakhani showed that the value of $\Lambda(X)$ is independent of $X \in \mathcal{T}_g$. We denote this constant by $\Lambda = \Lambda(g) > 0$ and refer to it as the Hubbard-Masur constant.

The Teichmüller metric. The Teichmüller metric $d_\mathcal{T}$ on $\mathcal{T}_g$ quantifies the minimal dilation among quasiconformal maps between complex structures on S_g. More precisely, for $X, Y \in \mathcal{T}_g$ one defines

$$d_\mathcal{T}(X,Y) := \log\left(\inf_{f\colon X \to Y} K(f)\right),$$

where the infimum runs over all quasiconformal maps $f\colon X \to Y$ in the homotopy class given by the markings of X and Y, and where $K(f)$ denotes the dilations of such maps. See [FM12, Chapter 11] for more details. The action of Mod_g on $\mathcal{T}_g$ preserves this metric. This metric is complete and its geodesics correspond to projections to $\mathcal{T}_g$ of Teichmüller geodesic flow orbits on $\mathcal{Q}^1\mathcal{T}_g$. In [Ker80], Kerckhoff proved the following formula.

Theorem 2.3. *For any pair of marked complex structures $X, Y \in \mathcal{T}_g$,*

$$d_\mathcal{T}(X,Y) = \max_{\eta \in \mathcal{MF}_g} \log\left(\frac{\sqrt{\mathrm{Ext}_Y(\eta)}}{\sqrt{\mathrm{Ext}_X(\eta)}}\right).$$

The Hodge inner product. Let X be a closed Riemann surface. Consider the Hodge decomposition of its complex cohomology group

$$H^1(X;\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

into holomorphic and anti-holomorphic 1-forms. On this cohomology group consider the Hodge intersection pairing

$$(\alpha,\beta)_X = \frac{i}{2}\int_X \alpha \wedge \overline{\beta}.$$

This pairing is Hermitian, positive definite on $H^{1,0}(X)$, and negative definite on $H^{0,1}(X)$. The Hodge inner product $\langle\cdot,\cdot\rangle_X$ on $H^1(X;\mathbb{C})$ is the unique Hermitian inner product given by the Hodge intersection pairing on $H^{1,0}(X)$, the negative of the Hodge intersection pairing on $H^{0,1}(X)$, and which makes $H^{1,0}(X)$ and $H^{0,1}(X)$ orthogonal. The Hodge inner product restricts to a real inner product on $H^1(X,\mathbb{R})$. By Poincaré duality, this induces a real inner product on the homology group $H_1(X;\mathbb{R})$. The corresponding norm, the Hodge norm, will be denoted by $\|\cdot\|_X$.

The homology bundle. Denote by $\mathbb{H}_g$ the bundle over $\mathcal{Q}^1\mathcal{M}_g$ whose fiber above every point (X,q) is given by the homology group $H_1(X;\mathbb{R})$; such a fiber can be endowed with the Hodge norm $\|\cdot\|_X$ induced by X via Poincaré duality. The natural $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{Q}^1\mathcal{M}_g$ lifts to an action on $\mathbb{H}_g$ via parallel transport with respect to the Gauss-Manin connection.

More concretely, consider the trivial bundle $\widehat{\mathbb{H}}_g := \mathcal{Q}^1\mathcal{T}_g \times H_1(S_g;\mathbb{R})$; the fiber of this bundle above $(X,q) \in \mathcal{Q}^1\mathcal{T}_g$ can be canonically identified with the homology group $H_1(X;\mathbb{R})$ by using the underlying marking. The mapping class group Mod_g acts on $\widehat{\mathbb{H}}_g$ diagonally. The $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{Q}^1\mathcal{T}_g$ extends to an action on $\mathbb{H}_g$ by declaring the action on $H_1(S_g;\mathbb{R})$ to be trivial. These actions commute and thus one obtains an $\mathrm{SL}(2,\mathbb{R})$ action on the quotient bundle $\mathbb{H}_g := \widehat{\mathbb{H}}_g/\mathrm{Mod}_g$.

The following important bound is due to Forni [For02, Corollary 2.2].

Theorem 2.4. *For every quadratic differential $(X,q) \in \mathcal{Q}^1\mathcal{T}_g$, every homology class $v_0 \in H_1(X;\mathbb{R})$, and every $t \in \mathbb{R}$,*

$$e^{-t}\|v_0\|_{\pi(q)} \leq \|a_t v_0\|_{\pi(a_t q)} \leq e^t \|v_0\|_{\pi(q)}.$$

As a consequence of Theorem 2.4, it is possible to compute Lyapunov exponents for the cocycle defined by the action of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}_g$ with respect to the Masur-Veech measure μ on $\mathcal{Q}^1\mathcal{M}_g$. Work of Forni [For02, Corollary 2.2] shows that the top Lyapunov exponent of this cocycle belongs to $(0,1)$; denote this exponent by $\varsigma = \varsigma(g) \in (0,1)$.

Central limit theorems. Let μ^* be the unique probability measure on the projectivized bundle $\mathbb{PH}_g$ which projects to the Masur-Veech measure μ on $\mathcal{Q}^1\mathcal{M}_g$ and whose conditional measures on fibers are equal to Lebesgue probability measures. For every $(q,v) \in \mathbb{PH}_g$ and every $t \in \mathbb{R}$ denote

$$\sigma(q,v,t) := \log \frac{\|a_t v\|_{\pi(a_t q)}}{\|v\|_{\pi(q)}}.$$

By the Oseledets theorem, for μ^*-almost-every $(q,v) \in \mathbb{PH}_g$,

$$\lim_{t\to\infty} \frac{\sigma(\omega,v,t)}{t} = \varsigma.$$

The following central limit theorem follows from work of Al-Saqban and Forni [ASF22, Theorem 2.1] but also uses work of Bell, Delecroix, Gadre, Gutiérrez-Romo, and Schleimer [BDG+21, Theorem 10.1] as a crucial input to guarantee positivity of the variance.

Theorem 2.5. *There exists $V = V(g) > 0$ such that the random variables*

$$\frac{\sigma(\omega,v,t) - t\cdot\varsigma}{\sqrt{t}} \quad on \quad (\mathbb{PH}_g, \mu^*)$$

converge in distribution to a Gaussian of mean 0 and variance V as $t \to \infty$.

The variance $V = V(g) > 0$ in Theorem 2.5 will be featured throughout the rest of this paper.

Geodesic currents. In [Bon88], Bonahon gave a unified treatment of several seemingly unrelated notions of length for closed curves on closed, orientable surfaces using the concept of geodesic currents. To define geodesic currents let us endow the surface S_g with an auxiliary hyperbolic metric. The projective tangent bundle PTS_g admits a 1-dimensional foliation by lifts of geodesics on S_g. A geodesic current on S_g is a Radon transverse measure of the geodesic foliation of PTS_g. Equivalently, a geodesic current on S_g is a $\pi_1(S_g)$-invariant Radon measure on the space of unoriented geodesics of the universal cover of S_g. Endow the space of geodesic currents on S_g with the weak-$\star$ topology. Different choices of auxiliary hyperbolic metrics on S_g yield canonically identified spaces of geodesic currents [Bon88, Fact 1]. Denote the space of geodesic currents on S_g by $\mathcal{C}_g$. This space supports a natural $\mathbb{R}_+$ action that scales transverse measures and a natural Mod_g action [RS19, §2]. Denote by $\mathcal{PC}_g$ the space of projective geodesic currents on S_g. This space is compact [Bon88, Corollary 5].

Free homotopy classes of weighted, unoriented closed curves on S_g embed into $\mathcal{C}_g$ by considering their geodesic representatives with respect to any auxiliary hyperbolic metric. By work of Bonahon [Bon88, Proposition 2], this embedding is dense. Moreover, the geometric intersection number pairing for closed curves on S_g extends in a unique way to a continuous, symmetric, bilinear pairing $i(\cdot,\cdot)$ on $\mathcal{C}_g$ [Bon88, Proposition 3]. This pairing is invariant with respect to the diagonal action of Mod_g. Singular measured foliations on S_g also embed into $\mathcal{C}_g$ by considering their geodesic representatives with respect to any auxiliary hyperbolic metric [Lev83].

Many different spaces of metrics on S_g embed into $\mathcal{C}_g$ in such a way that the geometric intersection number of any metric with any closed curve is equal to the length of the geodesic representatives of the closed curve with respect to the metric. We refer to the geodesic current corresponding to any such metric as its Liouville current. Examples of metrics admitting Liouville currents include:

- Hyperbolic metrics [Bon88],
- Negatively curved Riemannian metrics [Ota90],
- Negatively curved Riemannian metrics with cone singularities of angle at least 2π [HP97],
- Singular flat metrics induced by quadratic differentials [DLR10],
- Singular flat metrics with cone singularities of angle at least 2π [BL18].

A closed curve on S_g is said to be filling if it intersects every homotopically non-trivial closed curve on S_g. A geodesic current $\alpha \in \mathcal{C}_g$ is said to be filling if $i(\alpha,\beta) > 0$ for every non-zero $\beta \in \mathcal{C}_g$. Relevant examples of filling geodesic currents include free homotopy classes of unoriented filling closed curves and the Liouville currents listed above. Denote by $\mathcal{C}_g^* \subseteq \mathcal{C}_g$ the open subset of filling geodesic currents on S_g.

Constants. Let $A, B \in \mathbb{R}$ be real quantities and $*$ be a set of parameters. We write $A \preceq_* B$ if there exists a constant $C = C(*) > 0$ depending only on the parameters $*$ such that $A \leq C \cdot B$. We write $A \asymp_* B$ if $A \preceq_* B$ and $B \preceq_* A$. We write $A = O_*(B)$ if there exists a constant $C = C(*) > 0$ depending only on the parameters $*$ such that $|A| \leq C \cdot B$. When the dependencies on $*$ are made clear by the context, we sometimes drop the subscript $*$.

3. Limit theorems for mapping class groups

Outline of this section. Following [ABEM12b, AH23, ASF22, AF24], we begin with a brief overview of some of the technical tools that will be needed in the proofs of this section. After proving Theorems 1.7 and 1.8, restated in this section as Theorems 3.16 and 3.17, we prove more refined versions of them that apply to sectors and bisectors of Teichmüller space; see Theorems 3.25, 3.26, 3.33, and 3.34. These results will be crucial in the proofs of Theorems 1.1 and 1.3, the main results of this paper. We finish this section with a brief discussion of other limit theorems for mapping class groups that can be proved using the same techniques; see Theorems 3.35 and 3.36.

Mixing limit theorems for the Kontsevich–Zorich cocycle. Fix a closed, connected, oriented surface S_g of genus $g \geq 2$. Recall that $\mathcal{Q}^1\mathcal{M}_g$ denotes the moduli space of unit area, holomorphic quadratic differentials on S_g and that $\mathbb{H}_g$ denotes the bundle over $\mathcal{Q}^1\mathcal{M}_g$ whose fiber above every point $(X,q) \in \mathcal{Q}^1\mathcal{M}_g$ is given by the homology group $H_1(X;\mathbb{R})$; such a fiber can be endowed with the Hodge norm $\|\cdot\|_X$ induced by X via Poincaré duality. Recall that the one-parameter diagonal subgroup $\{a_t\}_{t\in\mathbb{R}} \subseteq \mathrm{SL}(2,\mathbb{R})$ introduced in (1) acts on $\mathcal{Q}^1\mathcal{M}_g$ by the Teichmüller geodesic flow and on $\mathbb{H}_g$ by parallel transport with respect to the Gauss-Manin connection; as explained in §2, these actions extend naturally to all $\mathrm{SL}(2,\mathbb{R})$. Recall that $\pi\colon \mathcal{Q}^1\mathcal{M}_g \to \mathcal{M}_g$ denotes the natural forgetful map to the moduli space of complex structures on S_g. Recall that μ denotes the Masur-Veech measure on $\mathcal{Q}^1\mathcal{M}_g$. Let $\varsigma = \varsigma(g) \in (0,1)$ be the top Lyapunov exponent of $\mathbb{H}_g$ as introduced in §2. Denote by $\mathcal{C}_c^+(\mathbb{R})$ the space of non-negative, continuous, compactly supported functions $\xi\colon \mathbb{R} \to \mathbb{R}$. The following mixing law of large numbers corresponds to [AF24, Theorem 4.25].

Theorem 3.1. *Let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an $\mathrm{SO}(2)$-equivariant, nowhere vanishing, measurable section. Then, for the random variables*

$$I_t(s,q) := \frac{1}{t}\log\frac{\|a_t s(q)\|_{\pi(a_t q)}}{\|s(q)\|_{\pi(q)}} \quad \text{on} \quad (\mathcal{Q}^1\mathcal{M}_g,\mu), \tag{5}$$

for every pair of essentially bounded functions $\phi_1,\phi_2 \in L^\infty(\mathcal{Q}^1\mathcal{M}_g,\mu)$, and for every function $\xi \in \mathcal{C}_c^+(\mathbb{R})$, the following holds,

$$\lim_{t\to\infty}\int_{\mathcal{Q}^1\mathcal{M}_g}\phi_1(q)\,\xi(I_t(s,q))\,\phi_2(a_t q)\,d\mu(q) = \frac{\mu(\phi_1)\cdot\xi(\varsigma)\cdot\mu(\phi_2)}{\mu(\mathcal{Q}^1\mathcal{M}_g)}. \tag{6}$$

Let $V = V(g) > 0$ be the variance of $\mathbb{H}_g$ as introduced in §2 and $\mathcal{N}_V$ be a Gaussian distribution on $\mathbb{R}$ of mean 0 and variance V. The following mixing central limit theorem corresponds to [AF24, Theorem 4.28].

Theorem 3.2. *Let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an $\mathrm{SO}(2)$-equivariant, nowhere vanishing, measurable section. Then, for the random variables*

$$J_t(s,q) := \frac{1}{\sqrt{t}}\left(\log\frac{\|a_t s(q)\|_{\pi(a_t q)}}{\|s(q)\|_{\pi(q)}} - t\cdot\varsigma\right) \quad \text{on} \quad (\mathcal{Q}^1\mathcal{M}_g,\mu), \tag{7}$$

for every pair of essentially bounded functions $\phi_1, \phi_2 \in L^\infty(\mathcal{Q}^1\mathcal{M}_g, \mu)$, and for every function $\xi \in \mathcal{C}_c^+(\mathbb{R})$, the following holds,

$$\lim_{t\to\infty} \int_{\mathcal{Q}^1\mathcal{M}_g} \phi_1(q)\, \xi(J_t(s,q))\, \phi_2(a_t q)\, d\mu(q) = \frac{\mu(\phi_1)\cdot \mathcal{N}_V(\xi)\cdot \mu(\phi_2)}{\mu(\mathcal{Q}^1\mathcal{M}_g)}. \tag{8}$$

The Hubbard-Masur function. Recall $\mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \subseteq \mathcal{Q}\mathcal{T}_g$ and $\mathcal{Q}^1\mathcal{M}_g(\mathbf{1}) \subseteq \mathcal{Q}^1\mathcal{M}_g$ denote the principal strata of the Teichmüller/moduli spaces of marked/unmarked unit area, holomorphic quadratic differentials on S_g. Recall that $\ell_{\min}(q) > 0$ denotes the length of the shortest saddle connections of a quadratic differential q. Recall that $\lambda\colon \mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \to \mathbb{R}_{>0}$ denotes the Hubbard-Masur function introduced in §2. As this function is Mod_g-invariant, one can also consider it as a function defined on $\mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$. Denote $h = h(g) := 6g - 6$. The following estimate is proved in [AH23, Proposition 3.5].

Proposition 3.3. *Let $\mathcal{K} \subseteq \mathcal{M}_g$ be a compact subset. Then, for every quadratic differential $q \in \mathcal{Q}^1\mathcal{M}_g(\mathbf{1}) \cap \pi^{-1}(\mathcal{K})$, the following estimate holds,*

$$\lambda(q) \preceq_{\mathcal{K}} \ell_{\min}(q)^{-(h-1)}.$$

Recall that $S(X) \subseteq \mathcal{Q}^1\mathcal{M}_g$ denotes the sphere of unit area, holomorphic quadratic differentials on $X \in \mathcal{M}_g$ and that s_X denotes the fiberwise measure on $S(X)$ induced by the Masur-Veech measure μ on $\mathcal{Q}^1\mathcal{M}_g$ via disintegration. Recall that for every $\delta > 0$ we consider the compact set $K_\delta \subseteq \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$ of quadratic differentials $q \in \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$ with $\ell_{\min}(q) \geq \delta$. The following estimate is proved in [AH23, Proposition 3.6].

Proposition 3.4. *Let $\mathcal{K} \subseteq \mathcal{M}_g$ be a compact subset. Then, for every Riemann surface $X \in \mathcal{K}$ and every $\delta > 0$, the following estimate holds,*

$$\int_{S(X)\setminus K_\delta} \lambda(q)\, ds_X(q) \preceq_{\mathcal{K}} \delta.$$

The Masur-Veech measure in polar coordinates. Recall that $\mathcal{T}_g$ denotes the Teichmüller space of marked complex structures on S_g. Fix $X \in \mathcal{T}_g$. Consider the polar coordinates map $\Phi_X \colon S(X) \times \mathbb{R}_{>0} \to \mathcal{T}_g$ which to every $q \in S(X)$ and every $t > 0$ assigns the marked Riemann surface $\Phi_X(q,t) := \pi(a_t q) \in \mathcal{T}_g$. This map is a homeomorphism onto $\mathcal{T}_g \setminus \{X\}$ and a diffeomorphism onto its image when restricted to $(S(X) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})) \times \mathbb{R}_{>0}$. Denote by $\Delta_X \colon (S(X) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})) \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ the unique smooth, positive function such that for every $q \in S(X) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$ and every $t > 0$,

$$d((\Phi_X)^*(\mathbf{m}))(q,t) = \Delta_X(q,t)\, ds_X(q)\, dt. \tag{9}$$

For every $\epsilon > 0$ and every $t > 0$ consider the subset $K_\epsilon(t) \subseteq \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$ of quadratic differentials in the principal stratum of moduli space whose Teichmüller geodesic flow orbit between times 0 and t spends at least half of the time in the compact subset K_ϵ, i.e.,

$$K_\epsilon(t) := \{q \in \mathcal{Q}^1\mathcal{M}_g(\mathbf{1}) \colon |\{s \in [0,t] \colon a_s q \in K_\epsilon\}| \geq t/2\}.$$

Recall that $\widehat{\pi}\colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{T}_g$ denotes the natural forgetful map. The following estimate for the function $\Delta_X \colon (S(X) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})) \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is proved in [AH23, Theorem 3.7]; see also [ABEM12b, Proposition 2.5].

Theorem 3.5. *For every $\epsilon > 0$ there exists a constant $\kappa_1 = \kappa_1(g,\epsilon) > 0$ with the following property. Fix a compact subset $\mathcal{K} \subseteq \mathcal{T}_g$. Let $X \in \mathrm{Mod}_g \cdot \mathcal{K}$, let $q \in S(X) \cap \mathcal{Q}^1\mathcal{T}_g(\mathbf{1})$, and let $t > 0$ be such that $q \in K_\epsilon(t)$ and $\widehat{\pi}(a_t q) \in \mathrm{Mod}_g \cdot \mathcal{K}$. Then, the following estimate holds,*

$$\Delta_X(q,t) = \lambda(a_t q)\cdot \lambda(q) \cdot e^{ht} + O_{\mathcal{K}}\left(\ell_{\min}(a_t q)^{-(h-1)} \cdot \ell_{\min}(q)^{-(h-1)} \cdot e^{(h-\kappa_1)t}\right).$$

Estimates near the multiple zero locus. Denote by $B_R(X) \subseteq \mathcal{T}_g$ the ball of radius $R > 0$ centered at $X \in \mathcal{T}_g$ with respect to the Teichmüller metric. Given $X \in \mathcal{T}_g$, $R > 0$, $\mathcal{K} \subseteq \mathcal{T}_g$ compact, and $\epsilon > 0$, denote by $B_R(X, \mathcal{K}, K_\epsilon) \subseteq \mathcal{T}_g$ the subset of points $Y \in B_R(X) \cap \mathrm{Mod}_g \cdot \mathcal{K}$ such that the projection to the moduli space of quadratic differentials of the Teichmüller geodesic segment from X to Y spends less than half of the time in the compact subset $K_\epsilon \subseteq \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$. The following estimate corresponds to [AH23, Theorem 3.8]; see also [EMR19, Theorem 1.7] and [ABEM12b, Theorem 2.7].

Theorem 3.6. *There exist constants $\epsilon_1 = \epsilon_1(g) > 0$ and $\kappa_2 = \kappa_2(g) > 0$ such that for every compact subset $\mathcal{K} \subseteq \mathcal{T}_g$, every $X \in \mathcal{K}$, and every $0 < \epsilon < \epsilon_1$,*

$$\widehat{\mathbf{m}}(B_R(X, \mathcal{K}, K_\epsilon))) \preceq_{\mathcal{K}} e^{(h-\kappa_2)R}.$$

Denote by $\Delta_{\mathcal{T}_g} \subseteq \mathcal{T}_g \times \mathcal{T}_g$ the corresponding diagonal. Consider the map $q_s \colon \mathcal{T}_g \times \mathcal{T}_g \setminus \Delta_{\mathcal{T}_g} \to \mathcal{Q}^1\mathcal{T}_g$ which to every pair $X, Y \in \mathcal{T}_g$ with $X \neq Y$ assigns the quadratic differential $q_s(X,Y) \in S(X)$ corresponding to the cotangent direction at X of the unique Teichmüller geodesic segment from X to Y. For every $X \in \mathcal{T}_g$ and every $V \subseteq S(X)$ consider the sector

$$\mathrm{Sect}_V(X) := \{Y \in \mathcal{T}_g \setminus \{X\} \mid q_s(X,Y) \in V\}.$$

Denote by $p\colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{Q}^1\mathcal{M}_g$ the quotient map. The following estimate is proved in [AH23, Theorem 3.9]; see also [ABEM12b, Theorem 2.6].

Theorem 3.7. *There exists a constant $\kappa_3 = \kappa_3(g) > 0$ with the following property. Let $\mathcal{K} \subseteq \mathcal{T}_g$ compact, let $X \in \mathcal{K}$, let $\delta > 0$, and let $V := p^{-1}(K_\delta) \cap S(X)$. Then, for every $R > 0$,*

$$\widehat{\mathbf{m}}(B_R(X) \cap \mathrm{Sect}_V(X) \cap \mathrm{Mod}_g \cdot \mathcal{K}) \preceq_{\mathcal{K}} \delta \cdot e^{hR} + e^{(h-\kappa_3)R}.$$

The following large deviations estimate is proved in [AH23, Theorem 3.10]; see also [Ath06, Theorem 1.1] for a series of important related results.

Theorem 3.8. *There exist constants $\epsilon_2 = \epsilon_2(g) > 0$ and $\kappa_4 = \kappa_4(g) > 0$ such that for every $0 < \epsilon < \epsilon_2$ and every $t > 0$, the following estimate holds,*

$$\mu(\mathcal{Q}^1\mathcal{M}_g \setminus K_\epsilon(t)) \preceq_g e^{-\kappa_4 t}.$$

Counting mapping class group orbits. Recall that $\Lambda = \Lambda(g) > 0$ denotes the Hubbard-Masur constant introduced in §2. Recall that, given $X, Y \in \mathcal{T}_g$ and $R > 0$, we denote by $\mathfrak{M}(X,Y,R)$ the set of all mapping classes $\mathbf{g} \in \mathrm{Mod}_g$ such that $0 < d_{\mathcal{T}}(X, \mathbf{g}.Y) \leq R$. The following effective counting result corresponds to [AH23, Theorem 1.1].

Theorem 3.9. *There exists a constant $\kappa = \kappa(g) > 0$ such that for every compact set $\mathcal{K} \subseteq \mathcal{T}_g$, every $X, Y \in \mathcal{K}$, and every $R > 0$,*

$$\#\mathfrak{M}(X,Y,R) = \frac{\Lambda^2}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot e^{hR} + O_{\mathcal{K}}\left(e^{(h-\kappa)R}\right).$$

Mean equidistribution of statistical balls. Recall that $\widehat{\mu}$ denotes the Masur-Veech measure on $\mathcal{Q}^1\mathcal{T}_g$ normalized as in (2) and that $\widehat{\mathbf{m}}$ denotes its pushforward to $\mathcal{T}_g$ under the natural forgetful map. Recall that $d_{\mathcal{T}}$ denotes the Teichmüller metric on $\mathcal{T}_g$. The natural marking changing actions of the mapping class group Mod_g of S_g on $\mathcal{Q}^1\mathcal{T}_g$ and $\mathcal{T}_g$ preserve the measures $\widehat{\mu}, \widehat{\mathbf{m}}$, and the metric $d_{\mathcal{T}}$.

We now define a particular class of measures on $\mathcal{T}_g$ that keep track of the statistics of the random variables I and J introduced in (5) and (7) along Teichmüller metric balls. Fix $X \in \mathcal{T}_g$, a nowhere vanishing, measurable section $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$, a function $\xi \in \mathcal{C}_c^+(\mathbb{R})$, and $R > 0$. Consider the measure $\widehat{\mathbf{m}}^R_{X,I,s,\xi}$ on $\mathcal{T}_g$ given for every $Y \in \mathcal{T}_g \setminus \{X\}$ by

$$d\widehat{\mathbf{m}}^R_{X,I,s,\xi}(Y) = \xi(I_{d_{\mathcal{T}}(X,Y)}(s, p(q_s(X,Y))))\, \mathbb{1}_{B_R(X)}(Y)\, d\widehat{\mathbf{m}}(Y). \tag{10}$$

Analogously, consider the measure on $\mathcal{T}_g$ given for every $Y \in \mathcal{T}_g \setminus \{X\}$ by

$$d\widehat{\mathbf{m}}^R_{X,J,s,\xi}(Y) = \xi(J_{d_{\mathcal{T}}(X,Y)}(s, p(q_s(X,Y))))\, \mathbb{1}_{B_R(X)}(Y)\, d\widehat{\mathbf{m}}(Y).$$

Denote by $\mathbf{m}^R_{X,I,s,\xi}$ and $\mathbf{m}^R_{X,J,s,\xi}$ the pushforward of the corresponding measures on $\mathcal{T}_g$ under the forgetful map to $\mathcal{M}_g$. These measures do not depend on the marking of $X \in \mathcal{T}_g$ but only on its underlying complex structure in $\mathcal{M}_g$.

The following mean equidistribution result, which we deduce as a consequence of Theorem 3.1 and the technical results discussed above, is the main tool used in the proof of Theorem 1.7.

Theorem 3.10. *Let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an $\mathrm{SO}(2)$-equivariant, nowhere vanishing, measurable section, let $\phi_1, \phi_2 \in L^\infty(\mathcal{M}_g, \mathbf{m})$ be essentially bounded functions with compact essential support, and let $\xi \in \mathcal{C}_c^+(\mathbb{R})$ be a non-negative, continuous, compactly supported function. Then,*

$$\lim_{R\to\infty} e^{-hR} \int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,I,s,\xi}(Y) \right) d\mathbf{m}(X) = \frac{\Lambda^2 \cdot \mathbf{m}(\phi_1) \cdot \mathbf{m}(\phi_2) \cdot \xi(\varsigma)}{h \cdot \mathbf{m}(\mathcal{M}_g)}.$$

Proof. For simplicity we assume $g \geq 3$; the proof for $g = 2$ requires keeping track of constant orbifold factors that muddle up the computations. Fix a measurable fundamental domain $\mathcal{F}_g \subseteq \mathcal{T}_g$ of the Mod_g action on Teichmüller space, a compact subset $\mathcal{K} \subseteq \mathcal{M}_g$, and a pair of essentially bounded functions $\phi_1, \phi_2 \in L^\infty(\mathcal{M}_g, \mathbf{m})$ with $\operatorname{ess\,supp}(\phi_1), \operatorname{ess\,supp}(\phi_2) \subseteq \mathcal{K}$. Denote by $\widehat{\phi}_1, \widehat{\phi}_2 \in L^\infty(\mathcal{T}_g, \widehat{\mathbf{m}})$ the lifts of these functions to $\mathcal{T}_g$. Let $R > 0$ and $0 < \delta < 1$ be arbitrary. Recall that $\mathbf{m}_{X,I,\xi}$ is the pushforward to $\mathcal{M}_g$ of the measure $\widehat{\mathbf{m}}_{X,I,\xi}$ on $\mathcal{T}_g$. Using (10) and (9) we can write

$$\int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y) \, d\mathbf{m}^R_{X,I,s,\xi}(Y) \right) d\mathbf{m}(X) \tag{11}$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_{\mathcal{T}_g} \widehat{\phi}_2(Y) \, \xi(I_{d_{\mathcal{T}}(X,Y)}(s, p(q_s(X,Y)))) \, \mathbb{1}_{B_R(X)}(Y) \, d\widehat{\mathbf{m}}(Y) \right) d\widehat{\mathbf{m}}(X)$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \xi(I_t(s, p(q))) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X).$$

To apply Theorem 3.5 we first bound the contributions near the multiple zero locus. By Theorem 3.7,

$$\int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \xi(I_t(s, p(q))) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X) \tag{12}$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X)$$
$$+ O_{\mathcal{K}} \left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot \left(\delta \cdot e^{hR} + e^{(h-\kappa_3)R} \right) \right).$$

A symmetric argument using Fubini's theorem and Theorem 3.7 shows that

$$\int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X) \tag{13}$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X)$$
$$+ O_{\mathcal{K}} \left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot \left(\delta \cdot e^{hR} + e^{(h-\kappa_3)R} \right) \right).$$

Let $\epsilon_1 = \epsilon_1(g) > 0$ be as in Theorem 3.6 and $\epsilon_2 = \epsilon_2(g) > 0$ be as in Theorem 3.8. Fix an arbitrary $0 < \epsilon = \epsilon(g) < \min\{\epsilon_1, \epsilon_2\}$. By Theorem 3.6,

$$\int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X) \tag{14}$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \mathbb{1}_{K_{\epsilon(t)}}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X)$$
$$+ O_{\mathcal{K}} \left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot e^{(h-\kappa_2)R} \right).$$

Let $\kappa = \kappa(g) := \min\{\kappa_2, \kappa_3\} > 0$. Putting (11), (12), (13), and (14) together we deduce

$$\int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y) \, d\mathbf{m}^R_{X,I,s,\xi}(Y) \right) d\mathbf{m}(X) \tag{15}$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \mathbb{1}_{K_{\epsilon(t)}}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X)$$
$$+ O_{\mathcal{K}} \left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot \left(\delta \cdot e^{hR} + e^{(h-\kappa)R} \right) \right).$$

We are now in a good position to apply Theorem 3.5:

$$\int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \mathbb{1}_{K_{\epsilon(t)}}(p(q)) \, \Delta_X(q,t) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X) \tag{16}$$
$$= \int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R e^{ht} \int_{S(X)} \widehat{\phi}_2(\pi(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \lambda(a_t q) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \mathbb{1}_{K_{\epsilon(t)}}(p(q)) \, \lambda(q) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X)$$
$$+ O_{\mathcal{K}} \left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot \delta^{-2(h-1)} \cdot e^{(h-\kappa_1)R} \right).$$

Using Fubini's theorem and (3) we can write

$$\int_{\mathcal{F}_g} \widehat{\phi}_1(X) \left(\int_0^R e^{ht} \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \lambda(a_t q) \, \xi(I_t(s, p(q))) \, \mathbb{1}_{K_\delta}(p(q)) \, \mathbb{1}_{K_{\epsilon(t)}}(p(q)) \, \lambda(q) \, ds_X(q) \, dt \right) d\widehat{\mathbf{m}}(X) \tag{17}$$
$$= \int_0^R e^{ht} \left(\int_{\mathcal{F}_g} \int_{S(X)} \widehat{\phi}_2(\widehat{\pi}(a_t q)) \, \mathbb{1}_{K_\delta}(p(a_t q)) \, \lambda(a_t q) \, \xi(I_t(s, p(q))) \, \widehat{\phi}_1(X) \, \mathbb{1}_{K_\delta}(p(q)) \, \mathbb{1}_{K_{\epsilon(t)}}(p(q)) \, \lambda(q) \, ds_X(q) \, d\widehat{\mathbf{m}}(X) \right) dt$$
$$= \int_0^R e^{ht} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(a_t q)) \, \mathbb{1}_{K_\delta}(a_t q) \, \lambda(a_t q) \, \xi(I_t(s, q)) \, \phi_1(\pi(q)) \, \mathbb{1}_{K_\delta}(q) \, \mathbb{1}_{K_{\epsilon(t)}}(q) \, \lambda(q) \, d\mu(q) \right) dt.$$

To apply Theorem 3.1 we first use Theorem 3.8 to write

$$\begin{aligned}\tag{18} &\int_0^R e^{ht} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(a_t q))\, \mathbb{1}_{K_\delta}(a_t q)\, \lambda(a_t q)\, \xi(I_t(s,q))\, \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \mathbb{1}_{K_{\epsilon(t)}}(q)\, \lambda(q)\, d\mu(q) \right) dt \\ &= \int_0^R e^{ht} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(a_t q))\, \mathbb{1}_{K_\delta}(a_t q)\, \lambda(a_t q)\, \xi(I_t(s,q))\, \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right) dt \\ &\quad + O_g\left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot e^{(h-\kappa_4)R} \right).\end{aligned}$$

Recall that the Hubbard-Masur function $\lambda\colon \mathcal{Q}^1\mathcal{M}_g(\mathbf{1}) \to \mathbb{R}_{>0}$ is continuous and thus it is bounded on the compact set $K_\delta \subseteq \mathcal{Q}^1\mathcal{M}_g(\mathbf{1})$; a more explicit bound is provided by Proposition 3.3. Thus, applying Theorem 3.1, we deduce

$$\begin{aligned} &\lim_{t\to\infty} \int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(a_t q))\, \mathbb{1}_{K_\delta}(a_t q)\, \lambda(a_t q)\, \xi(I_t(s,q))\, \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \\ &= \frac{\xi(\varsigma)}{\mu(\mathcal{Q}^1\mathcal{M}_g)} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right) \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right).\end{aligned}$$

In particular, it follows that,

$$\begin{aligned}\tag{19} &\lim_{R\to\infty} e^{-hR} \int_0^R e^{ht} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(a_t q))\, \mathbb{1}_{K_\delta}(a_t q)\, \lambda(a_t q)\, \xi(I_t(s,q))\, \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right) dt \\ &= \frac{\xi(\varsigma)}{h\cdot\mu(\mathcal{Q}^1\mathcal{M}_g)} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right) \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right).\end{aligned}$$

Using (15), (16), (17), (18), (19), we deduce

$$\begin{aligned}\tag{20} &\lim_{R\to\infty} e^{-hR} \int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,I,s,\xi}(Y) \right) d\mathbf{m}(X) \\ &= \frac{\delta_c(\xi)}{h\cdot\mu(\mathcal{Q}^1\mathcal{M}_g)} \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_1(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right) \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \phi_2(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) \right) \\ &\quad + O_{\mathcal{K}}\left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot \delta \right).\end{aligned}$$

We now incorporate the contributions near the multiple zero locus to the leading term in the previous estimate. By (3) and Proposition 3.4, for $i \in \{1,2\}$,

$$\tag{21} \int_{\mathcal{Q}^1\mathcal{M}_g} \phi_i(\pi(q))\, \mathbb{1}_{K_\delta}(q)\, \lambda(q)\, d\mu(q) = \int_{\mathcal{Q}^1\mathcal{M}_g} \phi_i(\pi(q))\, \lambda(q)\, d\widehat{\mu}(q) + O_{\mathcal{K}}\left(\|\phi_i\|_\infty \cdot \delta \right).$$

Using (3) and the definition of the Hubbard-Masur constant $\Lambda > 0$ in (4) we can write, for $i \in \{1,2\}$,

$$\tag{22} \int_{\mathcal{Q}^1\mathcal{M}_g} \phi_i(\pi(q))\, \lambda(q)\, d\mu(q) = \Lambda \cdot \left(\int_{\mathcal{M}_g} \phi_i(X)\, d\mathbf{m}(X) \right) = \Lambda \cdot \mathbf{m}(\phi_i).$$

Putting (20), (21), and (22) together we deduce

$$\begin{aligned}\tag{23} &\lim_{R\to\infty} e^{-hR} \int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,I,s,\xi}(Y) \right) d\mathbf{m}(X) \\ &= \frac{\Lambda^2 \cdot \mathbf{m}(\phi_1) \cdot \mathbf{m}(\phi_2) \cdot \xi(\varsigma)}{h\cdot\mu(\mathcal{Q}^1\mathcal{M}_g)} + O_{\mathcal{K}}\left(\|\phi_1\|_\infty \cdot \|\phi_2\|_\infty \cdot \|\xi\|_\infty \cdot \delta \right).\end{aligned}$$

Taking $\delta \to 0$ and using the fact that $\mu(\mathcal{Q}^1\mathcal{M}_g) = \mathbf{m}(\mathcal{M}_g)$ we conclude

$$\begin{aligned} &\lim_{R\to\infty} e^{-hR} \int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,I,s,\xi}(Y) \right) d\mathbf{m}(X) \\ &= \frac{\Lambda^2 \cdot \mathbf{m}(\phi_1) \cdot \mathbf{m}(\phi_2) \cdot \xi(\varsigma)}{h\cdot\mathbf{m}(\mathcal{M}_g)}.\end{aligned}$$ □

The same arguments, with Theorem 3.2 used in place of Theorem 3.1, yield the following mean equidistribution result; this is the main tool used in the proof of Theorem 1.8.

Theorem 3.11. *Let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an* SO(2)*-equivariant, nowhere vanishing, measurable section, let $\phi_1, \phi_2 \in L^\infty(\mathcal{M}_g, \mathbf{m})$ be essentially bounded functions with compact essential support, and let $\xi \in \mathcal{C}_c^+(\mathbb{R})$ be a non-negative, continuous, compactly supported function. Then,*

$$\lim_{R\to\infty} e^{-hR} \int_{\mathcal{M}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,J,s,\xi}(Y) \right) d\mathbf{m}(X) = \frac{\Lambda^2 \cdot \mathbf{m}(\phi_1)\cdot \mathbf{m}(\phi_2) \cdot \mathcal{N}_V(\xi)}{h \cdot \mathbf{m}(\mathcal{M}_g)}.$$

Limit theorems for mapping class groups. Recall that, given $X \in \mathcal{M}_g$, $\|\cdot\|_X$ denotes the Hodge norm induced by X on $H_1(X;\mathbb{R})$. We also denote by $\|\cdot\|_X$ the Hodge norm induced by $X \in \mathcal{T}_g$ on $H_1(S_g;\mathbb{R})$, where, we recall, S_g is the underlying topological surface describing the markings of $\mathcal{T}_g$. Given $v_0 \in H_1(S_g;\mathbb{R})$ a non-zero vector and $X \neq Y \in \mathcal{T}_g$, denote

$$\sigma_{v_0}(X,Y) := \frac{1}{d_{\mathcal{T}}(X,Y)} \log \frac{\|v_0\|_Y}{\|v_0\|_X}.$$

With the aim of proving Theorem 1.7, we establish the following bound.

Proposition 3.12. *Let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero vector in homology. Suppose that $\mathbf{g} \in \mathrm{Mod}_g$ and $X, X', Y, Y' \in \mathcal{K}$ are such that $\mathbf{g}.Y \neq X$ and $\mathbf{g}.Y' \neq X'$. Then,*

$$|\sigma_{v_0}(X', \mathbf{g}.Y') - \sigma_{v_0}(X, \mathbf{g}.Y)| \leq \frac{2d_{\mathcal{T}}(X,X') + 2d_{\mathcal{T}}(Y,Y')}{d_{\mathcal{T}}(X,\mathbf{g}.Y)}.$$

Proof. Under the assumptions of Proposition 3.12, not only $d_{\mathcal{T}}(X,\mathbf{g}.Y) \geq 1$ but also $d_{\mathcal{T}}(X',\mathbf{g}.Y') \geq 1$. In particular, the quantities considered are well defined. Directly from Theorem 2.4 we deduce

$$|\log\|v_0\|_{X'} - \log\|v_0\|_X| \leq d_{\mathcal{T}}(X,X') \leq \delta,$$
$$|\log\|v_0\|_{\mathbf{g}.Y'} - \log\|v_0\|_{\mathbf{g}.Y}| \leq d_{\mathcal{T}}(\mathbf{g}.Y,\mathbf{g}.Y') \leq \delta,$$
$$|\log\|v_0\|_{\mathbf{g}.Y} - \log\|v_0\|_X| \leq d_{\mathcal{T}}(X,\mathbf{g}.Y).$$

Using these bounds, together with the triangle inequality, we deduce

$$|\sigma_{v_0}(X', \mathbf{g}.Y') - \sigma_{v_0}(X, \mathbf{g}.Y)| \leq \frac{2d_{\mathcal{T}}(X,X') + 2d_{\mathcal{T}}(Y,Y')}{d_{\mathcal{T}}(X,\mathbf{g}.Y)}. \qquad \square$$

Given $v_0 \in H_1(S_g;\mathbb{R})$ a non-zero vector and $X \neq Y \in \mathcal{T}_g$, denote

$$\tau_{v_0}(X,Y) := \frac{1}{\sqrt{d_{\mathcal{T}}(X,Y)}} \left(\log \frac{\|v_0\|_Y}{\|v_0\|_X} - d_{\mathcal{T}}(X,Y) \cdot \varsigma \right).$$

The following estimate will be used in the proof of Theorem 1.3; the proof uses the same types of arguments as Proposition 3.13.

Proposition 3.13. *Let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero vector in homology. Suppose that $\mathbf{g} \in \mathrm{Mod}_g$ and $X, X', Y, Y' \in \mathcal{K}$ are such that $\mathbf{g}.Y \neq X$ and $\mathbf{g}.Y' \neq X'$. Then,*

$$|\tau_{v_0}(X', \mathbf{g}.Y') - \tau_{v_0}(X, \mathbf{g}.Y)| \leq \frac{2d_{\mathcal{T}}(X,X') + 2d_{\mathcal{T}}(Y,Y')}{\sqrt{d_{\mathcal{T}}(X,\mathbf{g}.Y)}} + 2\sqrt{d_{\mathcal{T}}(X,X') + d_{\mathcal{T}}(Y,Y')}.$$

Recall that, given $X, Y \in \mathcal{T}_g$ and $R > 0$, we denote by $\mathfrak{M}(X,Y,R)$ the set of all mapping classes $\mathbf{g} \in \mathrm{Mod}_g$ such that $0 < d_{\mathcal{T}}(X,\mathbf{g}.Y) \leq R$. Given $\xi \in \mathcal{C}_c^+(\mathbb{R})$, $v_0 \in H_1(S_g;\mathbb{R})$ a non-zero vector, $X, Y \in \mathcal{T}_g$, and $R > 0$, consider the counting function

$$F_{\xi,v_0}(X,Y,R) := \sum_{\mathbf{g} \in \mathfrak{M}(X,Y,R)} \xi(\sigma_{v_0}(X,\mathbf{g}.Y)).$$

Notice that this counting function does not depend on the marking of $Y \in \mathcal{T}_g$ but only on its underlying complex structure, i.e., on its projection to $\mathcal{M}_g$.

Recall $h := 6g-6$. With the aim of proving Theorem 1.7, we establish the following bound.

Proposition 3.14. *Let $\mathcal{K} \subseteq \mathcal{T}_g$ compact, $\xi \in \mathcal{C}_c^+(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, $v_0 \in H_1(S_g;\mathbb{R})$ non-zero, and $\delta > 0$. Suppose that $X, X', Y, Y' \in \mathcal{K}$ are such that $d_{\mathcal{T}}(X,X') \leq \delta$ and $d_{\mathcal{T}}(Y,Y') \leq \delta$. Then, for every $R > 0$,*

$$F_{\xi,v_0}(X,Y,R) \leq F_{\xi,v_0}(X',Y',R+2\delta) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1} \cdot e^{hR/2}) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1} \cdot \delta \cdot e^{hR}/R).$$

Proof. For $R > 0$ denote

$$F^{\dagger}_{\xi,v_0}(X,Y,R) := \sum_{\mathbf{g}\in\mathfrak{M}(X,Y,R)\setminus\mathfrak{M}(X,Y,R/2)} \xi(\sigma_{v_0}(X,\mathbf{g}.Y)).$$

By Theorem 3.9,

$$F_{\xi,v_0}(X,Y,R) = F^{\dagger}_{\xi,v_0}(X,Y,R) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot e^{hR/2}).$$

By Proposition 3.12 and Theorem 3.9,

$$F^{\dagger}_{\xi,v_0}(X,Y,R) = \sum_{\mathbf{g}\in\mathfrak{M}(X,Y,R)\setminus\mathfrak{M}(X,Y,R/2)} \xi(\sigma_{v_0}(X',\mathbf{g}.Y')) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot\delta\cdot e^{hR}/R).$$

By the triangle inequality,

$$\sum_{\mathbf{g}\in\mathfrak{M}(X,Y,R)\setminus\mathfrak{M}(X,Y,R/2)} \xi(\sigma_{v_0}(X',\mathbf{g}.Y')) \leq \sum_{\mathbf{g}\in\mathfrak{M}(X',Y',R+2\delta)} \xi(\sigma_{v_0}(X',\mathbf{g}.Y')) + O_g(\|\xi\|_{\mathcal{C}^1})$$

Putting everything together we conclude

$$F_{\xi,v_0}(X,Y,R) \leq F_{\xi,v_0}(X',Y',R+2\delta) + O_{\mathcal{K}}(\|\xi\|_{\infty}\cdot e^{hR/2}) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot\delta\cdot e^{hR}/R). \qquad \square$$

Given $\xi\in\mathcal{C}_c^+(\mathbb{R})$, $v_0\in H_1(S_g;\mathbb{R})$ non-zero, $X,Y\in\mathcal{T}_g$, and $R>0$, consider the counting function

$$H_{\xi,v_0}(X,Y,R) := \sum_{\mathbf{g}\in\mathfrak{M}(X,Y,R)} \xi(\tau_{v_0}(X,\mathbf{g}.Y)).$$

Notice that, as in the case of $F_{\xi,v_0}(X,Y,R)$, this counting function does not depend on the marking of $Y\in\mathcal{T}_g$ but only on its underlying complex structure, i.e., on its projection to $\mathcal{M}_g$.

With the aim of proving Theorem 1.8, we establish the following bound; the proof is analogous to that of Proposition 3.14 but uses Proposition 3.13 in place of Propostion 3.12.

Proposition 3.15. *Let $\mathcal{K}\subseteq\mathcal{T}_g$ compact, $\xi\in\mathcal{C}_c^+(\mathbb{R})\cap\mathcal{C}^1(\mathbb{R})$, $v_0\in H_1(S_g;\mathbb{R})$ non-zero, and $\delta>0$. Suppose that $X,X',Y,Y'\in\mathcal{K}$ are such that $d_{\mathcal{T}}(X,X')\leq\delta$ and $d_{\mathcal{T}}(Y,Y')\leq\delta$. Then, for every $R>0$,*

$$H_{\xi,v_0}(X,Y,R) \leq H_{\xi,v_0}(X',Y',R+2\delta) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot e^{hR/2}) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot(\delta\cdot e^{hR}/\sqrt{R}+\sqrt{\delta}\cdot e^{hR})).$$

Recall that, given $X,Y\in\mathcal{T}_g$ and $R>0$ sufficiently large, we endow $\mathfrak{M}(X,Y,R)$ with the uniform probability measure $\mathbb{P}_{X,Y,R}$. We are now ready to prove Theorem 1.7, which we restate here for the reader's convenience.

Theorem 3.16. *Let $X,Y\in\mathcal{T}_g$ be marked complex structures on S_g, let $v_0\in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|\mathbf{g}^{-1}.v_0\|}{d_{\mathcal{T}}(X,\mathbf{g}.Y)} \quad \text{on} \quad (\mathfrak{M}(X,Y,R),\mathbb{P}_{X,Y,R})$$

converge in distribution to the Dirac mass at ς as $R\to\infty$.

Proof. Because norms on finite dimensional vector spaces are comparable, it is enough to prove the desired statement for the Hodge norm $\|\cdot\|_Y$ induced by $Y\in\mathcal{T}_g$ on the homology group $H_1(X;\mathbb{R})$. Furthermore, by Theorem 3.9, it is equivalent for our purposes to consider the random variables

$$\frac{1}{d_{\mathcal{T}}(X,\mathbf{g}.Y)}\log\frac{\|v_0\|_{\mathbf{g}.Y}}{\|v_0\|_X} \quad \text{on} \quad (\mathfrak{M}(X,Y,R),\mathbb{P}_{X,Y,R}).$$

Fix a test function $\xi\in\mathcal{C}_c^+(\mathbb{R})$. Our goal is to show that

$$\xi(\varsigma) \leq \liminf_{R\to\infty}\frac{F_{\xi,v_0}(X,Y,R)}{\#\mathfrak{M}(X,Y,R)} \leq \limsup_{R\to\infty}\frac{F_{\xi,v_0}(X,Y,R)}{\#\mathfrak{M}(X,Y,R)} \leq \xi(\varsigma). \tag{24}$$

Standard approximation arguments show that, without loss of generality, we can assume $\xi\in\mathcal{C}_c^+(\mathbb{R})\cap\mathcal{C}^1(\mathbb{R})$. We now focus on proving the upper bound in (24); the lower bound follows by analogous arguments. For the rest of this discussion we consider $\mathcal{K}:=B_1(X)\cup B_1(Y)$, $0<\delta<1$, and $R>0$.

Supppose now $X',Y'\in\mathcal{T}_g$ are such that $d_{\mathcal{T}}(X,X')<\delta$, $d_{\mathcal{T}}(Y,Y')<\delta$. Then, by Proposition 3.14,

$$F_{\xi,v_0}(X,Y,R) \leq F_{\xi,v_0}(X',Y',R+2\delta) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot e^{hR/2}) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot\delta\cdot e^{hR}/R).$$

Multiplying by $\mathbb{1}_{B_\delta(X)}(X')\cdot\mathbb{1}_{B_\delta(Y)}(Y')$ we obtain the following inequality, valid for every $X',Y'\in\mathcal{T}_g$,

$$\begin{aligned}&\mathbb{1}_{B_\delta(X)}(X')\cdot\mathbb{1}_{B_\delta(Y)}(Y')\cdot F_{\xi,v_0}(X,Y,R)\\ &\leq \mathbb{1}_{B_\delta(X)}(X')\cdot\mathbb{1}_{B_\delta(Y)}(Y')\cdot F_{\xi,v_0}(X',Y',R+2\delta) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot e^{hR/2}) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1}\cdot\delta\cdot e^{hR}/R).\end{aligned}$$

Integrating with respect to $d\widehat{\mathbf{m}}(Y')\, d\widehat{\mathbf{m}}(X')$ we deduce

$$\begin{aligned}\tag{25} &\widehat{\mathbf{m}}(B_\delta(X)) \cdot \widehat{\mathbf{m}}(B_\delta(Y)) \cdot F_{\xi,v_0}(X,Y,R) \\ &\leq \int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(X)}(X') \left(\int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(Y')\, F_{\xi,v_0}(X',Y',R+2\delta)\, d\widehat{\mathbf{m}}(Y') \right) d\widehat{\mathbf{m}}(X') \\ &+O_{\mathcal{K}}(\widehat{\mathbf{m}}(B_\delta(X)) \cdot \widehat{\mathbf{m}}(B_\delta(Y)) \cdot \|\xi\|_{\mathcal{C}^1} \cdot e^{hR/2}) + O_{\mathcal{K}}(\widehat{\mathbf{m}}(B_\delta(X)) \cdot \widehat{\mathbf{m}}(B_\delta(Y)) \cdot \|\xi\|_{\mathcal{C}^1} \cdot \delta \cdot e^{hR}/R).\end{aligned}$$

Fix a locally finite, measurable fundamental domain $\mathcal{D}_g \subseteq \mathcal{T}_g$ for the action of Mod_g on $\mathcal{T}_g$; such a fundamental domain exists because the corresponding action is properly discontinuous. Then,

$$\begin{aligned}\tag{26} &\int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(X)}(X') \left(\int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(Y')\, F_{\xi,v_0}(X',Y',R+2\delta)\, d\widehat{\mathbf{m}}(Y') \right) d\widehat{\mathbf{m}}(X') \\ &= \sum_{\mathbf{g}\in \mathrm{Mod}_g} \sum_{\mathbf{h}\in \mathrm{Mod}_g} \int_{\mathcal{T}_g} \mathbb{1}_{\mathbf{g}.\mathcal{D}_g}(X')\, \mathbb{1}_{B_\delta(X)}(X') \left(\int_{\mathcal{T}_g} \mathbb{1}_{\mathbf{h}.\mathcal{D}_g}(Y')\, \mathbb{1}_{B_\delta(Y)}(Y')\, F_{\xi,v_0}(X',Y',R+2\delta)\, d\widehat{\mathbf{m}}(Y') \right) d\widehat{\mathbf{m}}(X').\end{aligned}$$

Fix $\mathbf{g},\mathbf{h} \in \mathrm{Mod}_g$. Recall that $\underline{p}\colon \mathcal{T}_g \to \mathcal{M}_g$ denotes the corresponding quotient map to moduli space. An unfolding argument shows that, for every $X' \in \mathcal{M}_g$,

$$\begin{aligned}\tag{27} &\int_{\mathcal{T}_g} \mathbb{1}_{\mathbf{h}.\mathcal{D}_g}(Y')\, \mathbb{1}_{B_\delta(Y)}(Y')\, F_{\xi,v_0}(X',Y',R+2\delta)\, d\widehat{\mathbf{m}}(Y') \\ &= \int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(Y'))\, F_{\xi,v_0}(X',Y',R+2\delta)\, d\mathbf{m}(Y') \\ &= \int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(\underline{p}(Y')))\, \xi(\sigma(X',Y'))\, \mathbb{1}_{B_{R+2\delta}(X')}(Y')\, d\widehat{\mathbf{m}}(Y').\end{aligned}$$

Consider the natural projection $P\colon \mathcal{Q}^1\mathcal{T}_g \times H_1(S_g,\mathbb{R}) \to \mathbb{H}_g$. Notice that the map $p\colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{Q}^1\mathcal{M}_g$ restricts to a measurable bijection on $\underline{\pi}^{-1}(\mathbf{g}.\mathcal{D}_g)$. Consider the section $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ given by

$$s_{\mathbf{g}}(q) := P(p|^{-1}_{\underline{\pi}^{-1}(\mathbf{g}.\mathcal{D}_g)}(q), v_0).$$

Informally, this section chooses the parallel transport of $v_0 \in H_1(S_g;\mathbb{R})$ above every point $q \in \mathcal{Q}^1\mathcal{M}_g$ after identification with a measurable fundamental domain of $\mathcal{Q}^1\mathcal{T}_g$. This section is SO(2)-equivariant by construction. Furthermore, for every $X' \in \mathbf{g}.\mathcal{D}_g$ and every $Y' \in \mathcal{T}_g$,

$$I_{d_{\mathcal{T}}(X',Y')}(s_{\mathbf{g}}, p(q_s(X',Y'))) = \xi(\sigma_{v_0}(X',Y')).$$

In particular, for every $X' \in \mathbf{g}.\mathcal{D}_g$,

$$\xi(\sigma(X',Y'))\, \mathbb{1}_{B_{R+2\delta}(X')}(Y')\, d\widehat{\mathbf{m}}(Y') = d\widehat{\mathbf{m}}^{R+2\delta}_{X,I,s,\xi}(Y')$$

It follows that, for every $X' \in \mathbf{g}.\mathcal{D}_g$,

$$\begin{aligned}\tag{28} &\int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(\underline{p}(Y')))\, \xi(\sigma(X',Y'))\, \mathbb{1}_{B_{R+2\delta}(X')}(Y')\, d\widehat{\mathbf{m}}(Y') \\ &= \int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(\underline{p}(Y')))\, d\widehat{\mathbf{m}}^{R+2\delta}_{X',I,s,\xi}(Y').\end{aligned}$$

As $\mathbf{m}^{R+2\delta}_{X,I,s,\xi}$ is the pushforward to $\mathcal{M}_g$ of the measure $\widehat{\mathbf{m}}^{R+2\delta}_{X,I,s,\xi}$ on $\mathcal{T}_g$,

$$\tag{29} \int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(\underline{p}(Y')))\, d\widehat{\mathbf{m}}^{R+2\delta}_{X',I,s,\xi}(Y') = \int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(Y'))\, d\mathbf{m}^{R+2\delta}_{X',I,s,\xi}(Y').$$

As the measures $\mathbf{m}^{R+2\delta}_{X',I,s,\xi}$ do not depend on the marking of $X' \in \mathcal{T}_g$,

$$\begin{aligned}\tag{30} &\int_{\mathcal{T}_g} \mathbb{1}_{\mathbf{g}.\mathcal{D}_g}(X')\, \mathbb{1}_{B_\delta(X)}(X') \left(\int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(Y'))\, d\mathbf{m}^{R+2\delta}_{X',I,s,\xi} \right) d\widehat{\mathbf{m}}(X') \\ &= \int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(X)}(\underline{p}|^{-1}_{\mathbf{g}.\mathcal{D}_g}(X')) \left(\int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(Y)}(\underline{p}|^{-1}_{\mathbf{h}.\mathcal{D}_g}(Y'))\, d\mathbf{m}^{R+2\delta}_{X',I,s,\xi} \right) d\mathbf{m}(X').\end{aligned}$$

Putting together (26), (27), (28), (29), and (30), we deduce

$$\int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(X)}(X') \left(\int_{\mathcal{T}_g} \mathbb{1}_{B_\delta(Y)}(Y')\, F_{\xi,v_0}(X',Y',R+2\delta)\, d\widehat{\mathbf{m}}(Y') \right) d\widehat{\mathbf{m}}(X') \tag{31}$$

$$= \sum_{\mathbf{g}\in \mathrm{Mod}_g} \sum_{\mathbf{h}\in \mathrm{Mod}_g} \int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(X)}(p|^{-1}_{\mathbf{g}.\mathcal{D}_g}(X')) \left(\int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(Y)}(p|^{-1}_{\mathbf{h}.\mathcal{D}_g}(Y'))\, d\mathbf{m}^{R+2\delta}_{X',I,s,\xi} \right) d\mathbf{m}(X').$$

As $\mathcal{D}_g \subseteq \mathcal{T}_g$ is locally finite, Theorem 3.10 guarantees that

$$\lim_{R\to\infty} e^{-h(R+2\delta)} \sum_{\mathbf{g}\in \mathrm{Mod}_g} \sum_{\mathbf{h}\in \mathrm{Mod}_g} \int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(X)}(p|^{-1}_{\mathbf{g}.\mathcal{D}_g}(X')) \left(\int_{\mathcal{M}_g} \mathbb{1}_{B_\delta(Y)}(p|^{-1}_{\mathbf{h}.\mathcal{D}_g}(Y'))\, d\mathbf{m}^{R+2\delta}_{X',I,s,\xi} \right) d\mathbf{m}(X') \tag{32}$$

$$= \sum_{\mathbf{g}\in \mathrm{Mod}_g} \sum_{\mathbf{h}\in \mathrm{Mod}_g} \frac{\Lambda^2 \cdot \widehat{\mathbf{m}}(B_\delta(X)\cap \mathbf{g}.\mathcal{D}_g)\cdot \widehat{\mathbf{m}}(B_\delta(Y)\cap \mathbf{h}.\mathcal{D}_g)\cdot \xi(\varsigma)}{h\cdot \mathbf{m}(\mathcal{M}_g)}$$

$$= \frac{\Lambda^2 \cdot \widehat{\mathbf{m}}(B_\delta(X))\cdot \widehat{\mathbf{m}}(B_\delta(Y))\cdot \xi(\varsigma)}{h\cdot \mathbf{m}(\mathcal{M}_g)}.$$

Recall that, by Theorem 3.9,

$$\lim_{R\to\infty} e^{-hR}\cdot \#\mathfrak{M}(X,Y,R) = \frac{\Lambda^2}{h\cdot \mathbf{m}(\mathcal{M}_g)}. \tag{33}$$

Putting together (25), (32), and (33), we deduce

$$\limsup_{R\to\infty} \frac{F_{\xi,v_0}(X,Y,R)}{\#\mathfrak{M}(X,Y,R)} \leq e^{2h\delta}\cdot \xi(\varsigma).$$

Letting $\delta\to 0$ finishes the proof of (24) and thus the proof of Theorem 3.16. □

Recall that $V = V(g) > 0$ denotes the variance of $\mathbb{H}_g$ as introduced in §2. The same arguments discussed in the proof of Theorem 3.16, but using Theorem 3.11 in place of Theorem 3.10 and Proposition 3.15 in place of Proposition 3.14, yield a proof of Theorem 1.8; we restate this result here for the reader's convenience.

Theorem 3.17. *Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|\mathbf{g}^{-1}.v_0\| - d_{\mathcal{T}}(X,\mathbf{g}.Y)\cdot\varsigma}{\sqrt{d_{\mathcal{T}}(X,\mathbf{g}.Y)}} \quad \text{on} \quad (\mathfrak{M}(X,Y,R), \mathbb{P}_{X,Y,R})$$

converge in distribution to a Gaussian of mean 0 *and variance V as $R\to\infty$.*

Mean equidistribution of statistical sectors. To prove Theorems 1.1 and 1.3 we will need more precise versions of Theorems 3.16 and 3.17 that keep track of so-called bisector information. To ease the reader into these more technical statements we first present the corresponding results for sectors of Teichmüller space; the proofs follow arguments similar to those discussed in the proofs of Theorems 3.16 and 3.17, so we focus mainly on setting up the right notation to state the results precisely.

Recall that $\mathcal{MF}_g$ denotes the space of singular measured foliations on S_g up to isotopy and Whitehead moves, that $\mathcal{PMF}_g$ denotes its projectivization under the natural $\mathbb{R}_{>0}$ action that scales transverse measures, and that $[\lambda]\in\mathcal{PMF}_g$ denotes the projective class of $\lambda\in\mathcal{MF}_g$. Consider the maps $\Re,\Im\colon \mathcal{Q}^1\mathcal{T}_g\to\mathcal{MF}_g$ which to every marked, unit area, holomorphic quadratic differential $q\in\mathcal{Q}^1\mathcal{T}_g$ assign its real/vertical and imaginary/horizontal foliations $\Re(q),\Im(q)\in\mathcal{MF}_g$. Consider also the induced maps $[\Re],[\Im]\colon\mathcal{Q}^1\mathcal{T}_g\to\mathcal{PMF}_g$. Given $X\in\mathcal{T}_g$ and $\mathcal{U}\subseteq\mathcal{PMF}_g$ denote

$$\mathrm{Sect}_{\mathcal{U}}(X) := \{Y\in\mathcal{T}_g\setminus\{X\} \mid [\Re(q_s(X,Y)]\in\mathcal{U}\} = \mathrm{Sect}_{[\Re]^{-1}(\mathcal{U})\cap S(X)}(X).$$

Now fix $X\in\mathcal{T}_g$, a measurable subset $\mathcal{U}\subseteq\mathcal{PMF}_g$, an SO(2)-equivariant, nowhere vanishing, measurable section $s\colon\mathcal{Q}^1\mathcal{M}_g\to\mathbb{H}_g$, a function $\xi\in\mathcal{C}_c^+(\mathbb{R})$, and $R>0$. Consider the measure $\widehat{\mathbf{m}}^R_{X,\mathcal{U},I,s,\xi}$ on $\mathcal{T}_g$ given for every $Y\in\mathcal{T}_g\setminus\{X\}$ by

$$d\widehat{\mathbf{m}}^R_{X,\mathcal{U},I,s,\xi}(Y) = \xi(I_{d_{\mathcal{T}}(X,Y)}(s,p(q_s(X,Y))))\,\mathbb{1}_{B_R(X)\cap\mathrm{Sect}_{\mathcal{U}}(X)}(Y)\,d\widehat{\mathbf{m}}(Y). \tag{34}$$

Analogously, consider the measure on $\mathcal{T}_g$ given for every $Y\in\mathcal{T}_g\setminus\{X\}$ by

$$d\widehat{\mathbf{m}}^R_{X,\mathcal{U},J,s,\xi}(Y) = \xi(J_{d_{\mathcal{T}}(X,Y)}(s,p(q_s(X,Y))))\,\mathbb{1}_{B_R(X)\cap\mathrm{Sect}_{\mathcal{U}}(X)}(Y)\,d\widehat{\mathbf{m}}(Y). \tag{35}$$

Denote by $\mathbf{m}^R_{X,\mathcal{U},I,s,\xi}$ and $\mathbf{m}^R_{X,\mathcal{U},J,s,\xi}$ the pushforwards to $\mathcal{M}_g$ of the corresponding measures on $\mathcal{T}_g$ under the natural forgetful map.

Recall that $\widehat{\mu}$ denotes the Masur-Veech measure on $\mathcal{Q}^1\mathcal{T}_g$. The following mean equidistribution result for sectors can be deduced using similar arguments as in the proof of Theorem 3.10.

Theorem 3.18. *Let $\mathcal{U} \subseteq \mathcal{PMF}_g$ be a measurable subset, let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an* SO(2)*-equivariant, nowhere vanishing, measurable section, let $\phi_1 \in L^\infty(\mathcal{T}_g, \widehat{\mathbf{m}})$ and $\phi_2 \in L^\infty(\mathcal{M}_g, \mathbf{m})$ be essentially bounded functions with compact essential support, and let $\xi \in \mathcal{C}_c^+(\mathbb{R})$. Then,*

$$\lim_{R\to\infty} e^{-hR} \int_{\mathcal{T}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,\mathcal{U},I,s,\xi}(Y) \right) d\widehat{\mathbf{m}}(X)$$
$$= \frac{\Lambda \cdot \mathbf{m}(\phi_2) \cdot \xi(\varsigma)}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot \int_{\mathcal{Q}^1\mathcal{T}_g} \mathbb{1}_{\mathcal{U}}([\Re(q)])\, \phi_1(\pi(q))\, \lambda(q)\, d\widehat{\mu}(q).$$

Analogously, the following mean equidistribution result can be deduced using the same arguments as in the proof of Theorem 3.11.

Theorem 3.19. *Let $\mathcal{U} \subseteq \mathcal{PMF}_g$ be a measurable subset, let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an* SO(2)*-equivariant, nowhere vanishing, measurable section, let $\phi_1 \in L^\infty(\mathcal{T}_g, \widehat{\mathbf{m}})$ and $\phi_2 \in L^\infty(\mathcal{M}_g, \mathbf{m})$ be essentially bounded functions with compact essential support, and let $\xi \in \mathcal{C}_c^+(\mathbb{R})$. Then,*

$$\lim_{R\to\infty} e^{-hR} \int_{\mathcal{T}_g} \phi_1(X) \left(\int_{\mathcal{M}_g} \phi_2(Y)\, d\mathbf{m}^R_{X,\mathcal{U},J,s,\xi}(Y) \right) d\widehat{\mathbf{m}}(X)$$
$$= \frac{\Lambda \cdot \mathbf{m}(\phi_2) \cdot \mathcal{N}_V(\xi)}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot \int_{\mathcal{Q}^1\mathcal{T}_g} \mathbb{1}_{\mathcal{U}}([\Re(q)])\, \phi_1(\pi(q))\, \lambda(q)\, d\widehat{\mu}(q).$$

Limit theorems for sectors. To promote Theorems 3.18 and 3.19 to results about statistics of mapping classes in sectors, we need to control how these statistics vary as we change the data prescribing the sectors of interest. This requires some technical results from [AH23] which we now summarize.

Given $X, Y \in \mathcal{T}_g$, $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, and $R > 0$, denote by $\mathfrak{M}(X, Y, \mathcal{U}, R)$ the set of all mapping classes $\mathbf{g} \in \mathrm{Mod}_g$ such that $\mathbf{g}.Y \in B_R(X) \cap \mathrm{Sect}_{\mathcal{U}}(X)$. Recall that ν denotes the Thurston measure on $\mathcal{MF}_g$ with the normalization described in §2. Recall that $\mathrm{Ext}_X(\eta) > 0$ denotes the extremal length of $\eta \in \mathcal{MF}_g$ with respect to $X \in \mathcal{T}_g$. Given $X \in \mathcal{T}_g$, denote by $\overline{\nu}_X$ the conned-off measure on $\mathcal{PMF}_g$ defined for every measurable subset $\mathcal{A} \subseteq \mathcal{PMF}_g$ by

$$\overline{\nu}_X(\mathcal{A}) := \nu(\{\eta \in \mathcal{MF}_g \colon [\eta] \in \mathcal{A},\ \mathrm{Ext}_X(\eta) \leq 1\}). \tag{36}$$

Fix a set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$. These coordinates identify $\mathcal{MF}_g$ with Σ^{3g-3}, where $\Sigma := \mathbb{R}^2/\langle \pm 1\rangle$. We can then identify $\mathcal{PMF}_g$ with the L^1 unit sphere in Σ^{3g-3} and endow this space with the corresponding L^1 metric, which we denote by d_Ξ. Given $\mathcal{V} \subseteq \mathcal{PMF}_g$ and $\delta > 0$, denote by $\mathcal{V}(\Xi, \delta) \subseteq \mathcal{PMF}_g$ the subset of all projective measured foliations at distance at most δ from $\mathcal{V}$ with respect to the metric induced by Ξ.

The following technical estimate can be deduced directly from the arguments introduced in the proof of [AH23, Proposition 9.3].

Proposition 3.20. *There exist $C = C(g) > 0$ and $\kappa = \kappa(g) > 0$ such that for every set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and every $\mathcal{K} \subseteq \mathcal{T}_g$ compact , there exists $\delta_0 = \delta_0(\Xi, \mathcal{K}) > 0$ with the following property. Let $\mathcal{U} \subseteq \mathcal{PMF}_g$ be a measurable set, let $0 < \delta < \delta_0$, and let $X, X', Y, Y' \in \mathcal{K}$ be marked Riemann surfaces such that $d_{\mathcal{T}}(X, X') < \delta$ and $d_{\mathcal{T}}(Y, Y') < \delta$. Then, for every $R > 0$,*

$$\#(\mathfrak{M}(X, Y, \mathcal{U}, R) \setminus \mathfrak{M}(X', Y', \mathcal{U}, R + 2\delta)) \preceq_{\mathcal{K}} \overline{\nu}_X(\partial\mathcal{U}(\Xi, C \cdot e^{-\kappa R})) \cdot e^{hR} + e^{(h-\kappa)R}.$$

Given $\xi \in \mathcal{C}_c^+(\mathbb{R})$ a non-negative, continuous, compactly supported function, $v_0 \in H_1(S_g; \mathbb{R})$ a non-zero vector, $X, Y \in \mathcal{T}_g$, $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, and $R > 0$, consider the counting function

$$F_{\xi, v_0}(X, Y, \mathcal{U}, R) := \sum_{\mathbf{g} \in \mathfrak{M}(X, Y, \mathcal{U}, R)} \xi(\sigma_{v_0}(X, \mathbf{g}.Y)).$$

Notice that this counting function does not depend on the marking of $Y \in \mathcal{T}_g$ but only on its underlying complex structure, i.e., on its projection to $\mathcal{M}_g$.

The following estimate provides a comparison for statistics of mapping classes in sectors of Teichmüller space under small changes of the defining data; the proof is similar to that of Proposition 3.14 and only requires the additional incorporation of Proposition 3.20.

Proposition 3.21. *There exist constants $C = C(g) > 0$ and $\kappa = \kappa(g) > 0$ such that for every set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and every compact set $\mathcal{K} \subseteq \mathcal{T}_g$, there exists a constant $\delta_0 = \delta_0(\Xi, \mathcal{K}) > 0$ with the following property. Let $\xi \in \mathcal{C}_c^+(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, let $v_0 \in H_1(S_g; \mathbb{R})$ non-zero, let $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, let $0 < \delta < \delta_0$, and let $X, X', Y, Y' \in \mathcal{K}$ such that $d_{\mathcal{T}}(X, X') \leq \delta$ and $d_{\mathcal{T}}(Y, Y') \leq \delta$. Then, for every $R > 0$,*

$$F_{\xi, v_0}(X, Y, \mathcal{U}, R)$$
$$\leq F_{\xi, v_0}(X', Y', \mathcal{U}, R + 2\delta) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1} \cdot (\overline{\nu}_X(\partial\mathcal{U}(\Xi, C \cdot e^{-\kappa R})) \cdot e^{hR} + e^{(h-\kappa)R})) + O_{\mathcal{K}}(\|\xi\|_{\mathcal{C}^1} \cdot \delta \cdot e^{hR}/R).$$

Given $\xi \in \mathcal{C}_c^+(\mathbb{R})$ a non-negative, continuous, compactly supported function, $v_0 \in H_1(S_g;\mathbb{R})$ a non-zero vector, $X, Y \in \mathcal{T}_g$, $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, and $R > 0$, consider the counting function

$$H_{\xi,v_0}(X,Y,\mathcal{U},R) := \sum_{\mathbf{g} \in \mathfrak{M}(X,Y,\mathcal{U},R)} \xi(\tau_{v_0}(X,\mathbf{g}.Y)).$$

Notice that, as in the case of $F_{\xi,v_0}(X,Y,\mathcal{U},R)$, this counting function does not depend on the marking of $Y \in \mathcal{T}_g$ but only on its underlying complex structure, i.e., on its projection to $\mathcal{M}_g$.

The following estimate provides a comparison for statistics of mapping classes in sectors of Teichmüller space under small changes of the defining data; the proof is similar to that of Proposition 3.15 and only requires the additional incorporation of Proposition 3.20.

Proposition 3.22. *There exist constants $C = C(g) > 0$ and $\kappa = \kappa(g) > 0$ such that for every set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and every compact set $\mathcal{K} \subseteq \mathcal{T}_g$, there exists a constant $\delta_0 = \delta_0(\Xi,\mathcal{K}) > 0$ with the following property. Let $\xi \in \mathcal{C}_c^+(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, let $v_0 \in H_1(S_g;\mathbb{R})$ non-zero, let $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, let $0 < \delta < \delta_0$, and let $X, X', Y, Y' \in \mathcal{K}$ such that $d_\mathcal{T}(X,X') \leq \delta$ and $d_\mathcal{T}(Y,Y') \leq \delta$. Then, for every $R > 0$,*

$$\begin{aligned} &H_{\xi,v_0}(X,Y,\mathcal{U},R) \\ &\leq H_{\xi,v_0}(X',Y',\mathcal{U},R+2\delta) + O_\mathcal{K}(\|\xi\|_{\mathcal{C}^1} \cdot (\overline{\nu}_X(\partial\mathcal{U}(\Xi, C \cdot e^{-\kappa R})) \cdot e^{hR} + e^{(h-\kappa)R})) \\ &\quad + O_\mathcal{K}(\|\xi\|_{\mathcal{C}^1} \cdot (\delta \cdot e^{hR}/\sqrt{R} + \sqrt{\delta} \cdot e^{hR})). \end{aligned}$$

Recall that s_X denotes the component above $X \in \mathcal{T}_g$ of the disintegration of the Masur-Veech measure $\mathbf{m}$ on $\mathcal{T}_g$ along the fiber $S(X) \subseteq \mathcal{Q}^1\mathcal{T}_g$ of the forgetful map $\pi\colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{T}_g$. Recall also the definition of the Hubbard-Masur function $\lambda\colon \mathcal{Q}^1\mathcal{T}_g(\mathbf{1}) \to \mathbb{R}_{>0}$ introduced in §2. The following result corresponding to [ABEM12b, Theorem 2.9] is a non-effective analogue of Theorem 3.9 for sectors of Teichmüller space. An effective version also holds, see [AH23, Theorems 9.1 and 9.2], but it requires introducing technical definitions that will not be needed for our purposes.

Theorem 3.23. *For every $X, Y \in \mathcal{T}_g$ and every $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable with $\overline{\nu}_X(\partial\mathcal{U}) = 0$,*

$$\lim_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}(X,Y,\mathcal{U},R) = \frac{\Lambda}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot \left(\int_{S(X)} \mathbb{1}_\mathcal{U}([\Re(q)])\, \lambda(q)\, ds_X(q) \right).$$

In view of Theorem 3.23, given $X \in \mathcal{T}_g$ and $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, we consider

$$\mathcal{I}(X,\mathcal{U}) := \int_{S(X)} \mathbb{1}_\mathcal{U}([\Re(q)])\, \lambda(q)\, ds_X(q).$$

The following result, corresponding to [AH23, Proposition 9.13], provides a comparison of these integral for varying base points $X \in \mathcal{T}_g$ in terms of the Teichmüller metric.

Proposition 3.24. *Let $0 < \delta < 1$ and $X, X' \in \mathcal{T}_g$ with $d_\mathcal{T}(X,X') < \delta$. Then, for every $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, the following estimate holds,*

$$\mathcal{I}(X,\mathcal{U}) = \mathcal{I}(X',\mathcal{U}) + O_g(\delta).$$

Given $X, Y \in \mathcal{T}_g$, a measurable set $\mathcal{U} \subseteq \mathcal{PMF}_g$ with $\overline{\nu}_X(\mathcal{U}) > 0$, and $R > 0$ sufficiently large, endow $\mathfrak{M}(X,Y,\mathcal{U},R)$ with the uniform probability measure $\mathbb{P}_{X,Y,\mathcal{U},R}$. The following law of large numbers follows from similar arguments to those introduced in the proof of Theorem 3.16; one should use Theorem 3.18 in place of Theorem 3.10, Proposition 3.21 in place of Proposition 3.14, and Theorem 3.23 in place of Theorem 3.9, while also appealing to Proposition 3.24.

Theorem 3.25. *Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $\mathcal{U} \subseteq \mathcal{PMF}_g$ be a measurable set with $\overline{\nu}_X(\mathcal{U}) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = 0$, let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|\mathbf{g}^{-1}.v_0\|}{d_\mathcal{T}(X,\mathbf{g}.Y)} \quad \text{on} \quad (\mathfrak{M}(X,Y,\mathcal{U},R), \mathbb{P}_{X,Y,\mathcal{U},R})$$

converge in distribution to the Dirac mass at ς as $R \to \infty$.

Analogously, the following central limit theorem follows from similar arguments to those introduced in the proof of Theorem 3.17; one should use Theorem 3.19 in place of Theorem 3.11, Proposition 3.22 in place of Proposition 3.15, Theorem 3.23 in place of 3.9, while also appealing to Proposition 3.24.

Theorem 3.26. *Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $\mathcal{U} \subseteq \mathcal{PMF}_g$ be a measurable set with $\overline{\nu}_X(\mathcal{U}) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = 0$, let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}^{-1}.v_0\| - d_{\mathcal{T}}(X,\mathbf{g}.Y)\cdot\varsigma}{\sqrt{d_{\mathcal{T}}(X,\mathbf{g}.Y)}} \quad \text{on} \quad (\mathfrak{M}(X,Y,\mathcal{U},R), \mathbb{P}_{X,Y,\mathcal{U},R})$$

converge in distribution to a Gaussian of mean 0 and variance V as $R \to \infty$.

Mean equidistribution of statistical bisectors. We now discuss mean equidistribution results for statistical bisectors of Teichmüller space. These results lead to limit theorems that will be crucial in the proofs of Theorems 1.1 and 1.3.

Recall that $\Delta_{\mathcal{T}_g} \subseteq \mathcal{T}_g \times \mathcal{T}_g$ denotes the corresponding diagonal. Consider the map $q_e\colon \mathcal{T}_g \times \mathcal{T}_g \setminus \Delta_{\mathcal{T}_g} \to \mathcal{Q}^1\mathcal{T}_g$ which to every pair $X, Y \in \mathcal{T}_g$ with $X \neq Y$ assigns the quadratic differential $q_e(X,Y) \in S(Y)$ corresponding to the cotangent direction at Y of the unique Teichmüller geodesic segment from X to Y. Given $X \in \mathcal{T}_g$, denote by $q_X\colon \mathcal{T}_g \setminus \{X\} \to \mathcal{Q}^1\mathcal{T}_g$ the map which to every $Y \neq X \in \mathcal{T}_g$ assigns the quadratic differential $q_X(Y) := q_E(X,Y) \in S(Y)$. Fix $X \in \mathcal{T}_g$, a measurable subset $\mathcal{U} \subseteq \mathcal{PMF}_g$, an SO(2)-equivariant, nowhere vanishing, measurable section $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$, a function $\xi \in \mathcal{C}_c^+(\mathbb{R})$, and $R > 0$. Consider the measures $\widehat{\mathbf{m}}^R_{X,\mathcal{U},I,s,\xi}$ and $\widehat{\mathbf{m}}^R_{X,\mathcal{U},I,s,\xi}$ on $\mathcal{T}_g$ introduced in (34) and (35). Lift these measures to $\mathcal{Q}^1\mathcal{T}_g$ by considering the pushforwards

$$\widehat{\mu}^R_{X,\mathcal{U},I,\xi} = (q_X)_*(\widehat{\mathbf{m}}^R_{X,\mathcal{U},I,s,\xi}), \quad \widehat{\mu}^R_{X,\mathcal{U},J,\xi} = (q_X)_*(\widehat{\mathbf{m}}^R_{X,\mathcal{U},J,s,\xi}).$$

Denote by $\mu_{X,\mathcal{U},I,s,\xi}$ and $\mu_{X,\mathcal{U},J,s,\xi}$ the pushforwards to $\mathcal{Q}^1\mathcal{M}_g$ of the corresponding measures on $\mathcal{Q}^1\mathcal{T}_g$ under the natural forgetful map.

The following mean equidistribution result can be deduced using the same arguments as in the proofs of Theorems 3.10 and 3.18.

Theorem 3.27. *Let $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an SO(2)-equivariant, nowhere vanishing, measurable section, let $\phi_1 \in L^\infty(\mathcal{T}_g, \widehat{\mathbf{m}})$, $\varphi_2 \in L^\infty(\mathcal{Q}^1\mathcal{M}_g, \mu)$ be essentially bounded functions with compact essential support, and let $\xi \in \mathcal{C}_c^+(\mathbb{R})$. Then,*

$$\lim_{R\to\infty} e^{-hR} \int_{\mathcal{T}_g} \phi_1(X) \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \varphi_2(q)\, d\mu^R_{X,\mathcal{U},I,s,\xi}(q) \right) d\widehat{\mathbf{m}}(X)$$
$$= \frac{\xi(\varsigma)}{h\cdot \mathbf{m}(\mathcal{M}_g)} \cdot \left(\int_{\mathcal{Q}^1\mathcal{T}_g} \mathbb{1}_{\mathcal{U}}([\Re(q)])\, \phi_1(\pi(q))\, \lambda(q)\, d\widehat{\mu}(q) \right) \cdot \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \varphi_2(q)\, \lambda(q) d\mu(q) \right).$$

Analogously, the following mean equidistribution result can be deduced using the same arguments as in the proofs of Theorems 3.11 and 3.19.

Theorem 3.28. *Let $\mathcal{U} \subseteq \mathcal{PMF}_g$ measurable, let $s\colon \mathcal{Q}^1\mathcal{M}_g \to \mathbb{H}_g$ be an SO(2)-equivariant, nowhere vanishing, measurable section, let $\phi_1 \in L^\infty(\mathcal{T}_g, \widehat{\mathbf{m}})$, $\varphi_2 \in L^\infty(\mathcal{Q}^1\mathcal{M}_g, \mu)$ be essentially bounded functions with compact essential support, and let $\xi \in \mathcal{C}_x^+(\mathbb{R})$. Then,*

$$\lim_{R\to\infty} e^{-hR} \int_{\mathcal{T}_g} \phi_1(X) \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \varphi_2(q)\, d\mu^R_{X,\mathcal{U},J,s,\xi}(q) \right) d\widehat{\mathbf{m}}(X)$$
$$= \frac{\mathcal{N}_V(\xi)}{h\cdot \mathbf{m}(\mathcal{M}_g)} \cdot \left(\int_{\mathcal{Q}^1\mathcal{T}_g} \mathbb{1}_{\mathcal{U}}([\Re(q)])\, \phi_1(\pi(q))\, \lambda(q)\, d\widehat{\mu}(q) \right) \cdot \left(\int_{\mathcal{Q}^1\mathcal{M}_g} \varphi_2(q)\, \lambda(q) d\mu(q) \right).$$

Limit theorems for bisectors. Given $X, Y \in \mathcal{T}_g$, measurable subsets $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$, and $R > 0$, denote by $\mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R)$ the set of all mapping classes $\mathbf{g} \in \mathrm{Mod}_g$ such that $0 < d_{\mathcal{T}}(X,\mathbf{g}.Y) \leq R$, $\mathbf{g}.Y \in \mathrm{Sect}_{\mathcal{U}}(X)$, and $\mathbf{g}^{-1}.X \in \mathrm{Sect}_{\mathcal{V}}(Y)$. The following technical estimate is an analogue of Proposition 3.20 for bisectors of Teichmüller space.

Proposition 3.29. *There exist constants $C = C(g) > 0$ and $\kappa = \kappa(g) > 0$ such that for every set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and every compact set $\mathcal{K} \subseteq \mathcal{T}_g$, there exists a constant $\delta_0 = \delta_0(\Xi,\mathcal{K}) > 0$ with the following property. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable sets, let $0 < \delta < \delta_0$, and let $X, X', Y, Y' \in \mathcal{K}$ marked Riemann surfaces such that $d_{\mathcal{T}}(X,X') < \delta$ and $d_{\mathcal{T}}(Y,Y') < \delta$. Then, for every $R > 0$,*

$$\#(\mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R) \setminus \mathfrak{M}(X',Y',\mathcal{U},\mathcal{V},R+2\delta)) \preceq_{\mathcal{K}} (\overline{\nu}_X(\partial\mathcal{U}(\Xi, C\cdot e^{-\kappa R})) + \overline{\nu}_Y(\partial\mathcal{V}(\Xi, C\cdot e^{-\kappa R})))\cdot e^{hR} + e^{(h-\kappa)R}.$$

Proof. Let $\mathcal{K} \subseteq \mathcal{T}_g$ compact and $\delta_0 = \delta_0(\mathcal{K}) > 0$ as in Proposition 3.20. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable, $0 < \delta < \delta_0$, and $X, X', Y, Y' \in \mathcal{K}$ with $d_{\mathcal{T}}(X,X') < \delta$ and $d_{\mathcal{T}}(Y,Y') < \delta$. Then, for every $R > 0$,

$$\mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R) \setminus \mathfrak{M}(X',Y',\mathcal{U},\mathcal{V},R+2\delta)$$
$$\subseteq (\mathfrak{M}(X,Y,\mathcal{U},R) \setminus \mathfrak{M}(X',Y',\mathcal{U},R+2\delta)) \cup (\mathfrak{M}(Y,X,\mathcal{V},R) \setminus \mathfrak{M}(Y',X',\mathcal{V},R+2\delta))^{-1}$$

The desired estimate then follows directly from Proposition 3.20. □

Given $\xi \in \mathcal{C}_c^+(\mathbb{R})$ a non-negative, continuous, compactly supported function, $v_0 \in H_1(S_g;\mathbb{R})$ a non-zero vector, $X, Y \in \mathcal{T}_g$, $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable, and $R > 0$, consider the counting function

$$F_{\xi,v_0}(X, Y, \mathcal{U}, \mathcal{V}, R) := \sum_{\mathbf{g} \in \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R)} \xi(\sigma_{v_0}(X, \mathbf{g}.Y)).$$

The following estimate provides a comparison for statistics of mapping classes in bisectors of Teichmüller space under small changes of the defining data; the proof is similar to those of Propositions 3.14 and 3.21 but requires using Proposition 3.29 in place of Proposition 3.20.

Proposition 3.30. *There exist constants $C = C(g) > 0$ and $\kappa = \kappa(g) > 0$ such that for every set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and every compact set $\mathcal{K} \subseteq \mathcal{T}_g$, there exists a constant $\delta_0 = \delta_0(\Xi, \mathcal{K}) > 0$ with the following property. Let $\xi \in \mathcal{C}_c^+(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, let $v_0 \in H_1(S_g;\mathbb{R})$ non-zero, let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable, let $0 < \delta < \delta_0$, and let $X, X', Y, Y' \in \mathcal{K}$ such that $d_\mathcal{T}(X, X') \leq \delta$ and $d_\mathcal{T}(Y, Y') \leq \delta$. Then, for every $R > 0$,*

$$\begin{aligned} &F_{\xi,v_0}(X, Y, \mathcal{U}, \mathcal{V}, R) \\ &\leq F_{\xi,v_0}(X', Y', \mathcal{U}, \mathcal{V}, R + 2\delta) + O_\mathcal{K}(\|\xi\|_{\mathcal{C}^1} \cdot \delta \cdot e^{hR}/R). \\ &+O_\mathcal{K}(\|\xi\|_{\mathcal{C}^1} \cdot ((\overline{\nu}(\partial\mathcal{U}(\Xi, C \cdot e^{-\kappa R})) + \overline{\nu}(\partial\mathcal{V}(\Xi, C \cdot e^{-\kappa R}))) \cdot e^{hR} + e^{(h-\kappa)R})). \end{aligned}$$

Given $\xi \in \mathcal{C}_c^+(\mathbb{R})$ a non-negative, continuous, compactly supported function, $v_0 \in H_1(S_g;\mathbb{R})$ a non-zero vector, $X, Y \in \mathcal{T}_g$, $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable, and $R > 0$, consider the counting function

$$H_{\xi,v_0}(X, Y, \mathcal{U}, \mathcal{V}, R) := \sum_{\mathbf{g} \in \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R)} \xi(\tau_{v_0}(X, \mathbf{g}.Y)).$$

The following estimate provides a comparison for statistics of mapping classes in bisectors of Teichmüller space under small changes of the defining data; the proof is similar to those of Propositions 3.15 and 3.22 but requires using Proposition 3.29 in place of Proposition 3.20.

Proposition 3.31. *There exist constants $C = C(g) > 0$ and $\kappa = \kappa(g) > 0$ such that for every set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and every compact set $\mathcal{K} \subseteq \mathcal{T}_g$, there exists a constant $\delta_0 = \delta_0(\Xi, \mathcal{K}) > 0$ with the following property. Let $\xi \in \mathcal{C}_c^+(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, let $v_0 \in H_1(S_g;\mathbb{R})$ non-zero, let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable, let $0 < \delta < \delta_0$, and let $X, X', Y, Y' \in \mathcal{K}$ such that $d_\mathcal{T}(X, X') \leq \delta$ and $d_\mathcal{T}(Y, Y') \leq \delta$. Then, for every $R > 0$,*

$$\begin{aligned} &H_{\xi,v_0}(X, Y, \mathcal{U}, \mathcal{V}, R) \\ &\leq H_{\xi,v_0}(X', Y', \mathcal{U}, \mathcal{V}, R + 2\delta) + O_\mathcal{K}(\|\xi\|_{\mathcal{C}^1} \cdot (\delta \cdot e^{hR}/\sqrt{R} + \sqrt{\delta} \cdot e^{hR}). \\ &+O_\mathcal{K}(\|\xi\|_{\mathcal{C}^1} \cdot ((\overline{\nu}(\partial\mathcal{U}(\Xi, C \cdot e^{-\kappa R})) + \overline{\nu}(\partial\mathcal{V}(\Xi, C \cdot e^{-\kappa R}))) \cdot e^{hR} + e^{(h-\kappa)R})). \end{aligned}$$

The following result corresponding to [ABEM12b, Theorem 2.10] is a non-effective analogue of Theorems 3.9 and Theorems 3.23 for bisectors of Teichmüller space. An effective version also holds, see [AH23, Theorems 10.6 and 10.7], but will not be needed for our purposes.

Theorem 3.32. *For $X, Y \in \mathcal{T}_g$ and $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ measurable with $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$,*

$$\begin{aligned} &\lim_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}(X, Y, \mathcal{U}, \mathcal{V}, R) \\ &= \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot \left(\int_{S(X)} \mathbb{1}_\mathcal{U}([\Re(q)])\, \lambda(q)\, ds_X(q) \right) \cdot \left(\int_{S(Y)} \mathbb{1}_\mathcal{V}([\Re(q)])\, \lambda(q)\, ds_Y(q) \right). \end{aligned}$$

Given $X, Y \in \mathcal{T}_g$, measurable sets $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$, and $R > 0$ sufficiently large, endow $\mathfrak{M}(X, Y, \mathcal{U}, \mathcal{V}, R)$ with the uniform probability measure $\mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R}$. The following law of large numbers follows from similar arguments to those introduced in the proof of Theorems 3.16 and 3.25; one should use Theorem 3.27 in place of Theorems 3.10 and 3.18, Proposition 3.30 in place of Propositions 3.14 and 3.21, and Theorem 3.32 in place of Theorems 3.9 and 3.23.

Theorem 3.33. *Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$, let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}^{-1}.v_0\|}{d_\mathcal{T}(X, \mathbf{g}.Y)} \quad \text{on} \quad (\mathfrak{M}(X, Y, \mathcal{U}, \mathcal{V}, R), \mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R})$$

converge in distribution to the Dirac mass at ς as $R \to \infty$.

Analogously, the following CLT follows from similar arguments to those introduced in the proofs of Theorems 3.17 and 3.26; one should use Theorem 3.28 in place of Theorems 3.11 and 3.19, Proposition 3.31 in place of Propositions 3.15 and 3.22, and Theorem 3.32 in place of Theorems 3.9 and 3.23.

Theorem 3.34. *Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$ and $\overline{\nu}_X(\partial \mathcal{U}) = \overline{\nu}_Y(\partial \mathcal{V}) = 0$, let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}^{-1}.v_0\| - d_{\mathcal{T}}(X, \mathbf{g}.Y) \cdot \varsigma}{\sqrt{d_{\mathcal{T}}(X, \mathbf{g}.Y)}} \quad on \quad (\mathfrak{M}(X, Y, \mathcal{U}, \mathcal{V}, R), \mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R})$$

converge in distribution to a Gaussian of mean 0 and variance V as $R \to \infty$.

Other limit theorems for mapping class groups. Although Theorems 1.7 and 1.8 make reference to a given vector in homology, it is possible to prove, via similar arguments, analogous limit theorems for the operator norm, i.e. the largest singular value, of the symplectic matrix representing the linear action of a mapping class on the homology group of a surface.

More precisely, let $\rho\colon \mathrm{Mod}_g \to \mathrm{Sp}(H_1(S_g;\mathbb{R}))$ be the natural symplectic linear representation of the mapping class group of S_g onto the group of symplectic linear automorphisms of its homology group. Given a norm $\|\cdot\|$ on $H_1(S_g;\mathbb{R})$, denote also by $\|\cdot\|$ the operator norm on $\mathrm{Sp}(H_1(S_g;\mathbb{R}))$, i.e., the function that records the largest singular value of the corresponding symplectic matrix. The following limit theorems are analogues of Theorems 1.7 and 1.8 and can be proved using similar arguments; see [AF24, Theorems 4.24 and 4.27] for the mixing limit theorems that play the role of Theorems 3.1 and 3.2 in the corresponding proofs.

Theorem 3.35. *Let $X, Y \in \mathcal{T}_g$ and $\|\cdot\|$ be a norm on $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\rho(\mathbf{g})^{-1}\|}{d_{\mathcal{T}}(X, \mathbf{g}.Y)} \quad on \quad (\mathfrak{M}(X, Y, R), \mathbb{P}_{X,Y,R})$$

converge in distribution to the Dirac mass at ς as $R \to \infty$.

Theorem 3.36. *Let $X, Y \in \mathcal{T}_g$ and $\|\cdot\|$ be a norm on $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\rho(\mathbf{g})^{-1}\| - d_{\mathcal{T}}(X, \mathbf{g}.Y) \cdot \varsigma}{\sqrt{d_{\mathcal{T}}(X, \mathbf{g}.Y)}} \quad on \quad (\mathfrak{M}(X, Y, R), \mathbb{P}_{X,Y,R})$$

converge in distribution to a Gaussian of mean 0 and variance V as $R \to \infty$.

Remark 3.37. Sector and bisector limit theorems analogous to Theorems 3.25, 3.26, 3.33, and 3.34 but for the operator norm of the symplectic matrix representing the linear action of a mapping class on the homology group of a surface also hold and can be proved using similar arguments; we avoid stating these results in detail as they will not be needed in the discussions that follow.

Remark 3.38. Analogous limit theorems for exterior powers also hold; see [AF24, Theorems 4.24 and 4.27] for the mixing limit theorems needed in the proofs. This allows one to describe the statistics of all the singular values of the symplectic matrix representing the linear action of a mapping class on the homology group of a surface in terms of the full Lyapunov spectrum of the Kontsevich-Zorich cocycle. We highlight that, as explained in [ASF22], the variances of the central limit theorems for higher exterior powers of the Kontsevich-Zorich cocycle are not known to be positive.

4. Limit theorems for closed curves

Outline of this section. In this section we give complete proofs of Theorems 1.1 and 1.3. We begin by proving slight variations of Theorems 3.33 and 3.34 that allow one to consider more general notions of distance; see Theorems 4.2, 4.3, 4.5, and 4.6 for precise statements. We then proceed to recall the tracking principle for mapping class group actions introduced in [AH21a] and exploited in [AH21b]. We also recall the tools introduced in [Hon24] to deal with the case of non-filling closed curves. We end this section with the proofs of more general versions of Theorems 1.1 and 1.3; see Theorems 4.20 and 4.21 below.

Generalized distances. For the rest of this section fix a closed, connected, oriented surface S_g of genus $g \geq 2$. Recall that $\mathcal{T}_g$ denotes the Teichmüller space of marked complex structures on S_g, that $d_{\mathcal{T}}$ denotes the Teichmüller metric on $\mathcal{T}_g$, and that $\mathcal{Q}^1\mathcal{T}_g$ denotes the Teichmüller space of marked, unit area, holomorphic quadratic differentials on S_g. Recall that $\mathcal{MF}_g$ denotes the space of singular measured foliations on S_g and that $\Re, \Im\colon \mathcal{Q}^1\mathcal{T}_g \to \mathcal{MF}_g$ denote the maps that to every quadratic differential $q \in \mathcal{Q}^1\mathcal{T}_g$ assign its vertical and horizontal foliations $\Re(q), \Im(q) \in \mathcal{MF}_g$, respectively. Recall that $\mathcal{PMF}_g$ denotes the space of projective singular measured foliations on S_g and that $[\eta] \in \mathcal{PMF}_g$ denotes the projective class of $\eta \in \mathcal{MF}_g$. Recall that $\Delta_{\mathcal{T}_g} \subseteq \mathcal{T}_g \times \mathcal{T}_g$ denotes the corresponding diagonal and that $q_s\colon \mathcal{T}_g \times \mathcal{T}_g \setminus \Delta_g \to \mathcal{Q}^1\mathcal{T}_g$ denotes the map which to every pair $X, Y \in \mathcal{T}_g$ with $X \neq Y$ assigns the quadratic differential $q_s(X,Y) \in \mathcal{Q}^1\mathcal{T}_g$ corresponding to the cotangent direction at X of the unique Teichmüller geodesic segment from X to Y. Recall that, given $X \in \mathcal{T}_g$ and $\mathcal{U} \subseteq \mathcal{PMF}_g$, we denote

$$\mathrm{Sect}_{\mathcal{U}}(X) := \{Y \in \mathcal{T}_g \setminus \{X\} \colon [\Re(q_s(X,Y))] \in \mathcal{U}\}.$$

We now introduce a notion of distance function that generalizes the Teichmüller metric on $\mathcal{T}_g$. This definition will later be used to state Theorems 4.2 and 4.3. Consider $X, Y \in \mathcal{T}_g$, $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$, and a continuous function $A \colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$. Denote by $\mathfrak{M}(X,Y,\mathcal{U},\mathcal{V})$ the set of all mapping classes $\mathbf{g} \in \mathrm{Mod}_g$ such that $\mathbf{g}.Y \neq X$, $\mathbf{g}.Y \in \mathrm{Sect}_{\mathcal{U}}(X)$, and $\mathbf{g}^{-1}.X \in \mathrm{Sect}_{\mathcal{V}}(Y)$. We say a function $D \colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a generalized distance function with adjustment A if there exists a function $o \colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ such that $o(\mathbf{g}) \to 0$ uniformly as $d(X,\mathbf{g}.Y) \to \infty$ and

$$D(\mathbf{g}) = d_{\mathcal{T}}(X,\mathbf{g}.Y) - A([\Re(q_s(X,\mathbf{g}.Y))], [\Re(q_s(Y,\mathbf{g}^{-1}.X))]) + o(\mathbf{g}).$$

For such a generalized distance function and every $R > 0$ denote

$$\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) := \{\mathbf{g} \in \mathcal{M}(X,Y,\mathcal{U},\mathcal{V}) \colon D(\mathbf{g}) \leq R\}. \tag{37}$$

Recall that $h := 6g-6$, that $\mathcal{Q}^1\mathcal{M}_g$ denotes the moduli space of unit area, holomorphic quadratic differentials on S_g, that μ denotes the Masur-Veech measure on $\mathcal{Q}^1\mathcal{M}_g$ normalized as in (3), and that $\mathbf{m}$ denotes the pushforward of μ to $\mathcal{M}_g$ with respect to the natural forgetful map. Recall that ν denotes the Thurston measure on $\mathcal{MF}_g$ with the normalization described in §2 and that, given $X \in \mathcal{T}_g$, $\overline{\nu}_X$ denotes the conned-off Thurston measure on $\mathcal{PMF}_g$ defined in (36).

Before discussing statistics we focus our attention on bisector counts with respect to generalized distance functions. In this context we have the following result generalizing Theorem 3.32.

Theorem 4.1. *Let $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g and $\mathcal{U}, \mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$. Suppose $A \colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is a continuous function and $D \colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a generalized distance function with adjustment A. Then,*

$$\lim_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) = \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \int_{\mathcal{U}\times\mathcal{V}} e^{hA([\eta],[\zeta])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\zeta]).$$

Proof. Fix partitions $P(\mathcal{U})$ of $\mathcal{U}$ and $P(\mathcal{V})$ of $\mathcal{V}$ such that $\overline{\nu}_X(\partial\mathcal{C}_1) = 0$ for every $\mathcal{C}_1 \in P(\mathcal{U})$ and $\overline{\nu}_Y(\partial\mathcal{C}_2) = 0$ for every $\mathcal{C}_2 \in P(\mathcal{V})$. For every pair $(\mathcal{C}_1,\mathcal{C}_2) \in P(\mathcal{U}) \times P(\mathcal{V})$ denote

$$A_+(\mathcal{C}_1,\mathcal{C}_1) := \sup\{A([\eta],[\zeta]) \colon ([\eta],[\zeta]) \in \mathcal{C}_1 \times \mathcal{C}_2\},$$
$$A_-(\mathcal{C}_1,\mathcal{C}_2) := \inf\{A([\eta],[\zeta]) : ([\eta],[\zeta]) \in \mathcal{C}_1 \times \mathcal{C}_2\}.$$

Let $\epsilon > 0$ be arbitrary. Since D is a generalized distance function with adjustment A, there exists $R_0 > 0$ such that for every $\mathbf{g} \in \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V})$ with $d(X,\mathbf{g}.Y) \geq R_0$, we have

$$d(X,\mathbf{g}.Y) \leq D(\mathbf{g}) + A([\Re(q_s(X,\mathbf{g}.Y))], [\Re(q_s(Y,\mathbf{g}^{-1}.X)]) + \epsilon.$$

Thus, by Theorem 3.9, there exists a constant $C = C(X,Y,R_0) > 0$ such that for every $\mathcal{C}_1 \in P(\mathcal{U})$, every $\mathcal{C}_2 \in P(\mathcal{V})$, and every $R > 0$,

$$\#(\mathfrak{M}(X,Y,\mathcal{C}_1,\mathcal{C}_2,R + A_+(\mathcal{C}_1,\mathcal{C}_2) + \epsilon) \setminus \mathfrak{M}_D(X,Y,\mathcal{C}_1,\mathcal{C}_2,R)) \leq C. \tag{38}$$

Notice that, by Theorem 3.32, for every $\mathcal{C}_1 \in P(\mathcal{U})$ and every $\mathcal{C}_2 \in P(\mathcal{V})$,

$$\begin{aligned}\limsup_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}(X,Y,\mathcal{C}_1,\mathcal{C}_2,&R + A_+(\mathcal{C}_1,\mathcal{C}_2) + \epsilon) \\ &\leq \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot \overline{\nu}_X(\mathcal{C}_1) \cdot \overline{\nu}_Y(\mathcal{C}_2) \cdot e^{hA_+(\mathcal{C}_1,\mathcal{C}_2)+h\epsilon}.\end{aligned} \tag{39}$$

From (38) and (39) we deduce that

$$\begin{aligned}&\limsup_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) \\ &\qquad\leq \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot e^{h\epsilon} \cdot \sum_{(\mathcal{C}_1,\mathcal{C}_2)\in P(\mathcal{U})\times P(\mathcal{V})} \cdot \overline{\nu}_X(\mathcal{C}_1) \cdot \overline{\nu}_Y(\mathcal{C}_2) \cdot e^{hA_+(\mathcal{C}_1,\mathcal{C}_2)}.\end{aligned}$$

Letting $\epsilon \to 0$ we deduce

$$\begin{aligned}&\limsup_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) \\ &\qquad\leq \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot \sum_{(\mathcal{C}_1,\mathcal{C}_2)\in P(\mathcal{U})\times P(\mathcal{V})} \cdot \overline{\nu}_X(\mathcal{C}_1) \cdot \overline{\nu}_Y(\mathcal{C}_2) \cdot e^{hA_+(\mathcal{C}_1,\mathcal{C}_2)}.\end{aligned}$$

Interpreting the right hand side of this inequality as a Riemann sum and letting the diameter, with respect to any continuous Riemannian metric on $\mathcal{PMF}_g$, of the partitions $P(\mathcal{U})$ and $P(\mathcal{V})$ go to zero we deduce

$$\limsup_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) \leq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)} \int_{\mathcal{U}\times\mathcal{V}} e^{hA([\eta],[\zeta])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\zeta]). \tag{40}$$

A similar argument shows that

$$\liminf_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) \geq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)} \int_{\mathcal{U}\times\mathcal{V}} e^{hA([\eta],[\zeta])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\zeta]). \tag{41}$$

Putting together (40) and (41) the desired conclusion follows. □

Let $\varsigma = \varsigma(g) \in (0,1)$ be the top Lyapunov exponent of $\mathbb{H}_g$ as introduced in §2. Given $X,Y \in \mathcal{T}_g$, measurable sets $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$, and $R > 0$ sufficiently large, endow $\mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R)$ with the uniform probability measure $\mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R}$. The following law of large numbers is a version of Theorem 3.33 driven by generalized distance functions; it follows by the same arguments introduced in the proof of Theorem 4.1 but using Theorem 3.33 in place of Theorem 3.32.

Theorem 4.2. *Let $X,Y \in \mathcal{T}_g$ be marked complex structures on S_g and let $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}(U), \overline{\nu}(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$. Suppose $A\colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is continuous and $D\colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a generalized distance function with adjustment A. Let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}^{-1}.v_0\|}{d(X,\mathbf{g}.Y)} \quad \textit{on} \quad (\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R), \mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R})$$

converge in distribution to the Dirac mass at ς as $R\to\infty$.

Let $V = V(g) > 0$ be the variance of $\mathbb{H}_g$ as introduced in §2. The following central limit theorem is a version of Theorem 3.34 driven by generalized distance functions; it follows by the same arguments introduced in the proof of Theorem 4.1 but using Theorem 3.34 in place of Theorem 3.32.

Theorem 4.3. *Let $X,Y \in \mathcal{T}_g$ be marked complex structures on S_g and let $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}(U), \overline{\nu}(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$. Suppose $A\colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is continuous and $D\colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a generalized distance function with adjustment A. Let $v_0 \in H_1(S_g;\mathbb{R})$ be a non-zero homology class, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|\mathbf{g}^{-1}.v_0\| - d_{\mathcal{T}}(X,\mathbf{g}.Y)\cdot\varsigma}{\sqrt{d_{\mathcal{T}}(X,\mathbf{g}.Y)}} \quad \textit{on} \quad (\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R), \mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R})$$

converge in distribution to a Gaussian of mean 0 and variance V as $R\to\infty$.

To proceed with the rest of our discussion we need versions of Theorems 4.2 and 4.3 that work for a slightly weaker notion of generalized distance function, which we now introduce. A subset $\mathfrak{M}^* \subseteq \mathrm{Mod}_g$ is said to have full density if there exists a pair $X,Y \in \mathcal{T}_g$ such that

$$\lim_{R\to\infty} e^{-hR} \cdot \#\{\mathbf{g} \in \mathrm{Mod}_g \setminus \mathfrak{M}^* \colon d(X,\mathbf{g}.Y) \leq R\} = 0. \tag{42}$$

Notice condition (42) holds for some pair $X,Y \in \mathcal{T}_g$ if and only if it holds for any such pair. Consider $X,Y \in \mathcal{T}_g$, $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$, and a continuous function $A\colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$. We say a function $D\colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a weakly generalized distance function with adjustment A if there exists a full density subset $\mathfrak{M}^* \subseteq \mathrm{Mod}_g$, a function $o\colon \mathfrak{M}^* \cap \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ with $o(\mathbf{g}) \to 0$ uniformly as $d(X,\mathbf{g}.Y) \to \infty$, and a bounded function $O\colon (\mathrm{Mod}_g \setminus \mathfrak{M}^*) \cap \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$, such that

(1) For every $\mathbf{g} \in \mathfrak{M}^* \cap \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V})$,
$$D(\mathbf{g}) = d_{\mathcal{T}}(X,\mathbf{g}.Y) - A([\Re(q_s(X,\mathbf{g}.Y))],[\Re(q_s(Y,\mathbf{g}^{-1}.X))]) + o(\mathbf{g}).$$
(2) For every $\mathbf{g} \in (\mathrm{Mod}_g \setminus \mathfrak{M}^*) \cap \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V})$,
$$D(\mathbf{g}) = d_{\mathcal{T}}(X,\mathbf{g}.Y) + O(\mathbf{g}).$$

Just as in (37), for every $R > 0$ denote

$$\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) := \{\mathbf{g} \in \mathcal{M}(X,Y,\mathcal{U},V) \colon D(\mathbf{g}) \leq R\}.$$

Theorem 4.4. *Let $X,Y \in \mathcal{T}_g$ be marked complex structures on S_g and $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ measurable with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$. Suppose $A\colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is a continuous function and $D\colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a weakly generalized distance function with adjustment A. Then,*

$$\lim_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R) = \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)} \int_{\mathcal{U}\times\mathcal{V}} e^{hA([\eta],[\zeta])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\zeta]).$$

Proof. Let $\mathfrak{M}^* \subseteq \mathrm{Mod}_g$ be the full density used in the definition of the weakly generalized distance function D. Define a distance function $D' \colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ as follows:

$$D'(\mathbf{g}) := \begin{cases} D(\mathbf{g}) & \text{if } \mathbf{g} \in \mathfrak{M}^*, \\ d(X,\mathbf{g}.Y) - A([\Re(q_s(X,\mathbf{g}.Y))],[\Re(q_s(Y,\mathbf{g}^{-1}.X))]) & \text{if } \mathbf{g} \in \mathrm{Mod}_g \setminus \mathfrak{M}^*. \end{cases}$$

Notice D' is a generalized distance function with adjustment A. In particular, by Theorem 4.1,

$$\lim_{R\to\infty} e^{-hR} \cdot \#\mathfrak{M}_{D'}(X,Y,\mathcal{U},\mathcal{V},R) = \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)} \int_{\mathcal{U}\times\mathcal{V}} e^{hA([\eta],[\zeta])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\zeta]). \tag{43}$$

For every $R > 0$ we can decompose

$$\mathfrak{M}_{D'}(X,Y,\mathcal{U},\mathcal{V},R) = (\mathfrak{M}^* \cap \mathfrak{M}_{D'}(X,Y,\mathcal{U},\mathcal{V},R)) \cup ((\mathrm{Mod}_g \setminus \mathfrak{M}^*) \cap \mathfrak{M}_{D'}(X,Y,\mathcal{U},\mathcal{V},R)). \tag{44}$$

Directly from the definition of D' it follows that, for every $R > 0$,

$$\mathfrak{M}^* \cap \mathfrak{M}_{D'}(X,Y,\mathcal{U},\mathcal{V},R) = \mathfrak{M}^* \cap \mathfrak{M}_{D}(X,Y,\mathcal{U},\mathcal{V},R) \tag{45}$$

Since A is continuous on $\overline{\mathcal{U}} \times \overline{\mathcal{V}}$, there exists a constant $C > 0$ such that for every $\mathbf{g} \in \mathrm{Mod}_g \setminus \mathfrak{M}^*$ if $D'(X,\mathbf{g}.Y) \leq R$ then $d(X,\mathbf{g}.Y) \leq R + C$. In particular, as $\mathfrak{M}^* \subseteq \mathrm{Mod}_g$ has full density,

$$\lim_{R\to\infty} e^{-hR} \cdot \#((\mathrm{Mod}_g \setminus \mathfrak{M}^*) \cap \mathfrak{M}_{D'}(X,Y,\mathcal{U},\mathcal{V},R)) = 0. \tag{46}$$

Similarly, for every $R > 0$ we can decompose

$$\mathfrak{M}_{D}(X,Y,\mathcal{U},\mathcal{V},R) = (\mathfrak{M}^* \cap \mathfrak{M}_{D}(X,Y,\mathcal{U},\mathcal{V},R)) \cup ((\mathrm{Mod}_g \setminus \mathfrak{M}^*) \cap \mathfrak{M}_{D}(X,Y,\mathcal{U},\mathcal{V},R)). \tag{47}$$

As the function O in the definition of the weakly generalized distance function D is bounded, there exists a constant $M > 0$ such that for every $\mathbf{g} \in \mathrm{Mod}_g \setminus \mathfrak{M}^*$ if $D(X,\mathbf{g}.Y) \leq R$ then $d(X,\mathbf{g}.Y) \leq R + M$. In particular, as $\mathfrak{M}^* \subseteq \mathrm{Mod}_g$ has full density,

$$\lim_{R\to\infty} e^{-hR} \cdot \#((\mathrm{Mod}_g \setminus \mathfrak{M}^*) \cap \mathfrak{M}_{D}(X,Y,\mathcal{U},\mathcal{V},R)) = 0. \tag{48}$$

Putting together (43), (44), (45), (46), (47), and (48) we obtained the desired conclusion. □

As above, given $X,Y \in \mathcal{T}_g$, measurable sets $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ with $\overline{\nu}_X(U), \overline{\nu}_Y(V) > 0$, and $R > 0$ sufficiently large, endow $\mathfrak{M}(X,Y,\mathcal{U},\mathcal{V},R)$ with the uniform probability measure $\mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R}$. The following law of large numbers is a version of Theorem 3.33 driven by weakly generalized distance functions; it follows by the same arguments introduced in the proof of Theorem 4.4 but using Theorem 4.2 in place of Theorem 4.1.

Theorem 4.5. *Let $X,Y \in \mathcal{T}_g$ be marked complex structures on S_g and let $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}(U), \overline{\nu}(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$. Suppose $A \colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is a continuous function and $D \colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a weakly generalized distance function with adjustment A. Let $v_0 \in H_1(S_g;\mathbb{R})$ non-zero, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|\mathbf{g}^{-1}.v_0\|}{d(X,\mathbf{g}.Y)} \quad \text{on} \quad (\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R), \mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R})$$

converge in distribution to the Dirac mass at ς as $R \to \infty$.

The following central limit theorem is a version of Theorem 3.34 driven by weakly generalized distance functions; it follows by the same arguments introduced in the proof of Theorem 4.4 but using Theorem 4.3 in place of Theorem 4.1.

Theorem 4.6. *Let $X,Y \in \mathcal{T}_g$ be marked complex structures on S_g and let $\mathcal{U},\mathcal{V} \subseteq \mathcal{PMF}_g$ be measurable sets with $\overline{\nu}(U), \overline{\nu}(V) > 0$ and $\overline{\nu}_X(\partial\mathcal{U}) = \overline{\nu}_Y(\partial\mathcal{V}) = 0$. Suppose $A \colon \overline{\mathcal{U}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is a continuous function and $D \colon \mathfrak{M}(X,Y,\mathcal{U},\mathcal{V}) \to \mathbb{R}$ is a weakly generalized distance function with adjustment A. Let $v_0 \in H_1(S_g;\mathbb{R})$ be non-zero, and let $\|\cdot\|$ be a norm on the homology group $H_1(S_g;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|\mathbf{g}^{-1}.v_0\| - d_{\mathcal{T}}(X,\mathbf{g}.Y)\cdot\varsigma}{\sqrt{d_{\mathcal{T}}(X,\mathbf{g}.Y)}} \quad \text{on} \quad (\mathfrak{M}_D(X,Y,\mathcal{U},\mathcal{V},R), \mathbb{P}_{X,Y,\mathcal{U},\mathcal{V},R})$$

converge in distribution to a Gaussian of mean 0 and variance V as $R \to \infty$.

The tracking principle. Aside from Theorems 4.5 and 4.6 above, the main technical tool we will need to prove Theorems 1.1 and 1.3 is the tracking principle introduced in [AH21b]. According to this principle, the action of the mapping class group on the space of closed curves of a closed, orientable surface tracks the corresponding action on Teichmüller space in the following sense: for all but a few mapping classes, the information of how a mapping class moves a given point of Teichmüller space determines, up to a small error term, how it changes the geometric intersection numbers of a given closed curve with respect to arbitrary closed curves. This principle will allow us to recast problems about the statistics of closed curves on surfaces as problems about the statistics of the mapping class group action on Teichmüller space, which we will then tackle using Theorems 4.5 and 4.6.

More precisely, recall that $\mathcal{C}_g$ denotes the space of geodesic currents on S_g and that $\mathcal{C}_g^* \subseteq \mathcal{C}_g$ denotes the subset of filling geodesic currents; see §2 for precise definitions. Recall that $\mathrm{Ext}_X(\eta) > 0$ denotes the extremal length with respect to $X \in \mathcal{T}_g$ of a singular measured foliation $\eta \in \mathcal{MF}_g$ and that extremal lengths scale quadratically with respect to transverse measures. Given $\alpha \in \mathcal{C}_g^*$ and a closed curve β on S_g, consider the function $A_{\alpha,\beta} \colon \mathcal{PMF}_g \times \mathcal{PMF}_g \to \mathbb{R}$ given by

$$A_{\alpha,\beta}([\eta],[\zeta]) = -\log i\left(\frac{\eta}{\sqrt{\mathrm{Ext}_X(\eta)}}, \beta\right) - \log i\left(\alpha, \frac{\zeta}{\sqrt{\mathrm{Ext}_Y(\zeta)}}\right).$$

Furthermore, consider the function $D_{\alpha,\beta} \colon \mathrm{Mod}_g \to \mathbb{R}$ given by

$$D_{\alpha,\beta}(\mathbf{g}) := \log i(\alpha, \mathbf{g}^{-1}.\beta).$$

The following is a non-effective reformulation of the tracking principle in [AH21a, Theorem 1.1].

Theorem 4.7. *Let β be a closed curve on S_g and $X, Y \in \mathcal{T}_g$. Then, there exists a full density subset $\mathfrak{M}^* = \mathfrak{M}^*(\beta, X, Y) \subseteq \mathrm{Mod}_g$ and a function $o \colon \mathfrak{M}^* \to \mathbb{R}$ with $o(\mathbf{g}) \to 0$ uniformly as $d(X, \mathbf{g}.Y) \to \infty$ such that for every filling geodesics current $\alpha \in \mathcal{C}_g^*$ and every $\mathbf{g} \in \mathfrak{M}^*$ we have $\mathbf{g}.Y \neq X$ and*

$$D_{\alpha,\beta}(\mathbf{g}) = d_{\mathcal{T}}(X, \mathbf{g}.Y) - A_{\alpha,\beta}([\Re(q_s(X,\mathbf{g}.Y))], [\Re(q_s(Y, \mathbf{g}^{-1}.X))]) + o(\mathbf{g}).$$

Non-filling closed curves. We now discuss some technical aspects that will need to be considered in the proofs of Theorems 1.1 and 1.3 in the case of non-filling closed curves. These aspects do not need to be taken into account in the case of filling closed curves; the reader is invited to consider this case in first instance as they read through the proofs of Theorems 1.1 and 1.3. Proofs of the technical results we now discuss can be found in [Hon24], where they were first introduced.

Let β be a closed curve on S_g. Denote by $\mathrm{Stab}(\beta) \subseteq \mathrm{Mod}_g$ the stabilizer of β, i.e., the set of mapping classes of S_g that preserve the homotopy class of β. Denote by $\mathrm{Stab}^*(\beta) \subseteq \mathrm{Stab}(\beta)$ the reduced stabilizer of β, i.e., the set of mapping classes that can be homotoped to homeomorphisms fixing β pointwise. The symmetry group of β is defined to be the finite group

$$\mathrm{Sym}(\beta) := \mathrm{Stab}(\beta)/\mathrm{Stab}^*(\beta).$$

The following result is a direct consequence of the definitions.

Lemma 4.8. *Let β be a closed curve on S_g. Then the map*

$$\mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g \to \mathrm{Mod}_g \cdot \beta, \quad \mathrm{Stab}^*(\beta)\mathbf{g} \mapsto \mathbf{g}^{-1}.\beta$$

is $\#\mathrm{Sym}(\beta)$-to-one.

Let β be a closed curve on S_g. Consider the set

$$\mathcal{MF}_g(\beta) := \{\eta \in \mathcal{MF}_g \colon \beta + \eta \text{ is filling}\}.$$

Denote by $\mathcal{PMF}_g(\beta) \subseteq \mathcal{PMF}_g$ the projectivization of $\mathcal{MF}_g(\beta)$. Following [ES22, Theorem 3.19], $\mathrm{Stab}^*(\beta)$ acts properly discontinuously on $\mathcal{PMF}_g(\beta)$. The difficulties introduced by the fact that $\mathrm{Stab}^*(\beta)$ is infinite for a non-filling closed curve β can be tackled by constructing a fundamental domain for the action of $\mathrm{Stab}^*(\beta)$ on $\mathcal{PMF}_g(\beta)$. Some control over the geometry of such a fundamental domain, e.g., in terms of train track coordinates, is also important. When β is a simple closed curve, such a fundamental domain was first constructed by Rafi and Souto; see [ES22, Proposition 8.4]. The case when β is an arbitrary closed curve is handled in [Hon24] using similar arguments. We now summarize the main properties of this construction.

Proposition 4.9. *Let β be a closed curve on S_g. Then there exists an open set $\mathcal{D}_\beta \subseteq \mathcal{MF}_g(\beta)$ which together with its relative closure $\overline{\mathcal{D}}_\beta \subseteq \mathcal{MF}_g(\beta)$ has the following properties:*

(1) $\mathbf{g}.\mathcal{D}_\beta \cap \mathcal{D}_\beta = \emptyset$ for every $\mathbf{g} \in \mathrm{Stab}^*(\beta)$.
(2) $\mathcal{MF}_g(\beta) = \bigcup_{\mathbf{g} \in \mathrm{Stab}^*(\beta)} \mathbf{g}.\overline{\mathcal{D}}_\beta$.
(3) The boundary $\partial D_\beta \subseteq \mathcal{PMF}_g$ is piecewise linear.
(4) $\overline{\nu}_X(\partial D_\beta) = 0$ for every $X \in \mathcal{T}_g$.

We are now ready to discuss the technical results in [Hon24] that will be needed in the proofs of Theorems 1.1 and 1.3. For concreteness, fix a set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$. These coordinates identify $\mathcal{MF}_g$ with Σ^{3g-3}, where $\Sigma := \mathbb{R}^2/\langle \pm 1\rangle$. We can then identify $\mathcal{PMF}_g$ with the L^1 unit sphere in Σ^{3g-3} and endow this space with the corresponding L^1 metric, which we denote by d_Ξ. Throughout the ensuing discussion, given a closed curve β on S_g, we consider the fundamental domain $\mathcal{D}_\beta \subseteq \mathcal{PMF}_g(\beta)$ and its relative closure $\overline{\mathcal{D}}_\beta \subseteq \mathcal{PMF}_g(\beta)$ as in Proposition 4.9. For any $\epsilon > 0$ define

$$\mathcal{D}^{+\epsilon}_{\beta,\Xi} := \{\eta \in \mathcal{PMF}_g \colon d_\Xi(\beta, \mathcal{D}_\beta) \leq \epsilon\}, \tag{49}$$

$$\mathcal{D}^{-\epsilon}_{\beta,\Xi} := \{\eta \in \mathcal{D}_\beta \colon d_\Xi(\beta, \partial\mathcal{D}_\beta) > \epsilon\}. \tag{50}$$

The following is the first technical result from [Hon24] we will need.

Proposition 4.10. *Let $\alpha \in \mathcal{C}_g^*$ be a filling geodesic current, β be a closed curve on S_g, Ξ be a set of Dehn-Thurston coordinates on $\mathcal{MF}_g$, and $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g. Then, there exists a constant $\epsilon_0 = \epsilon_0(\alpha,\beta,\Xi,X,Y) > 0$ and a bounded function $O\colon \mathfrak{M}(X,Y,\mathcal{D}^{+\epsilon_0}_{\beta,\Xi},\mathcal{PMF}_g) \to \mathbb{R}$ such that for every $\mathbf{g} \in \mathfrak{M}(X,Y,\mathcal{D}^{+\epsilon_0}_{\beta,\Xi},\mathcal{PMF}_g)$ we have*

$$D_{\alpha,\beta}(\mathbf{g}) = d_\mathcal{T}(X, \mathbf{g}.Y) + O(\mathbf{g}).$$

Theorem 4.7 together with Proposition 4.10 guarantees that the function $d_{\alpha,\beta}$ is a weakly generalized distance function with adjustment $A_{\alpha,\beta}$ in the sense described in the following corollary.

Corollary 4.11. *Let $\alpha \in \mathcal{C}_g^*$ be a filling geodesic current, β be a closed curve on S_g, Ξ be a set of Dehn-Thurston coordinates on $\mathcal{MF}_g$, and $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g. Then, there exists a constant $\epsilon_0 = \epsilon_0(\alpha,\beta,\Xi,X,Y) > 0$ such that the restriction of $D_{\alpha,\beta}$ to $\mathfrak{M}(X,Y,\mathcal{D}^{+\epsilon_0}_{\beta,\Xi},\mathcal{PMF}_g)$ is a weakly generalized distance function with adjustment the restriction of $A_{\alpha,\beta}$ to $\mathcal{D}^{+\epsilon_0}_{\beta,\Xi} \times \mathcal{PMF}_g$.*

The following technical result of [Hon24] controls overcounting in our setting by using the fundamental domain introduced in Propositon 4.9.

Proposition 4.12. *Let β be a closed curve on S_g and $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g. Then, there exists a full density subset $\mathfrak{M}^* = \mathfrak{M}^*(\beta, X, Y) \subseteq \mathrm{Mod}_g$ such that for every set of Dehn-Thurston coordinates Ξ on $\mathcal{MF}_g$ and every $\epsilon > 0$, there exists a constant $R_0 := R_0(\beta,X,Y,\Xi,\epsilon) > 0$ such that the following map is injective:*

$$(\mathfrak{M}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g) \setminus \mathfrak{M}(X,Y,R_0)) \cap \mathfrak{M}^* \to \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g, \quad \mathbf{g} \mapsto \mathrm{Stab}^*(\beta)\mathbf{g}.$$

The final issue that needs to be addressed is making sure that every coset in $\mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g$, or at least most of them, are being accounted for by our methods. To this end we use the so-called standard map introduced in [Hon24], whose main properties we now review. Let β be closed curve on S_g and $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g. Given this data there exists a map $\mathrm{std}\colon \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g \to \mathrm{Mod}_g$, called the standard map, which to every coset $\mathrm{Stab}^*(\beta)\mathbf{g} \in \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g$ assigns the, basically unique, element $\mathbf{g}' := \mathrm{std}(\mathrm{Stab}^*(\beta)\mathbf{g}) \in \mathrm{Stab}^*(\beta)\mathbf{g}$ such that $\mathbf{g}'.Y$ lies in or close to $\mathrm{Sect}_{\mathcal{D}_\beta}(X) \subseteq \mathcal{T}_g$; notice that any such map is automatically injective.

The main property of the standard map is that for most cosets $\mathrm{Stab}^*(\beta)\mathbf{g} \in \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g$, if $\mathbf{g}' := \mathrm{std}(\mathrm{Stab}^*(\beta)\mathbf{g}) \in \mathrm{Mod}_g$, then $\mathbf{g}'.Y \in \mathcal{T}_g$ belongs to the sector based at $X \in \mathcal{T}_g$ defined by an arbitrarily small thickening of the fundamental domain $\mathcal{D}_\beta$. More precisely, given a filling geodesic current $\alpha \in \mathcal{C}_g^*$, a closed curve β on S_g, and $L > 0$, denote

$$\mathfrak{C}(\alpha,\beta,L) := \{\mathrm{Stab}^*(\beta)\mathbf{g} \in \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g \colon i(\alpha, \mathbf{g}^{-1}.\beta) \leq L\}.$$

Furthermore, given a closed curve β on S_g, we say a subset $\mathfrak{C}^* \subseteq \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g$ has full density if for any filling geodesic current $\alpha \in \mathcal{C}_g^*$,

$$\lim_{L\to\infty} L^{-h} \cdot \#\{\mathrm{Stab}^*(\beta)\mathbf{g} \in (\mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g) \setminus \mathfrak{C}^* \colon i(\alpha, \mathbf{g}^{-1}.\beta) \leq L\} = 0.$$

Notice that condition (4) holds for some $\alpha \in \mathcal{C}_g^*$ if and only if it holds for any such α. With this terminology the main properties of the standard map can be described as follows.

Proposition 4.13. *Let β be a closed curve on S_g, Ξ be a set of Dehn-Thurston coordinates of $\mathcal{MF}_g$, and $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g. Then, there exists an injective map* $\mathrm{std}\colon \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g \to \mathrm{Mod}_g$ *with the following properties:*

(1) If $\mathbf{g}' = \mathrm{std}(\mathrm{Stab}^*(\beta)\mathbf{g})$ for some $\mathbf{g} \in \mathrm{Mod}_g$, then $\mathbf{g}' \in \mathrm{Stab}^*(\beta)\mathbf{g}$.

(2) There exists a full density subset $\mathfrak{C}^* = \mathfrak{C}^*(\beta,\Xi,X,Y) \subseteq \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g$ such that for every $\alpha \in \mathcal{C}_g^*$ and every $\epsilon > 0$, there exists $L_0 = L_0(\alpha,\beta,\Xi,X,Y,\epsilon) > 0$ such that

$$\mathrm{std}(\mathfrak{C}^* \setminus \mathfrak{C}(\alpha,\beta,L_0)) \subseteq \mathfrak{M}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g).$$

Integrals. In the course of the proofs of Theorems 1.1 and 1.3, it will be convenient to have alternative descriptions of the integrals featured in Theorems 4.1 and 4.4. Given $\alpha \in \mathcal{C}_g^*$ denote

$$c(\alpha) := \nu\left(\{\eta \in \mathcal{MF}_g \mid i(\alpha,\eta) \leq 1\}\right) > 0.$$

A similar constant can be defined for an arbitrary closed curve β on S_g as follows. Recall that

$$\mathcal{MF}_g(\beta) := \{\eta \in \mathcal{MF}_g \colon \beta + \eta \text{ is filling}\}.$$

Following [ES22, Theorem 3.19], we know that $\mathrm{Stab}^*(\beta)$ acts properly discontinuously on $\mathcal{MF}_g(\beta)$. Denote by ν_β the local pushforward to the quotient $\mathrm{Stab}^*(\beta)\backslash\mathcal{MF}_g(\beta)$ of the restriction of the Thurston measure ν to $\mathcal{MF}_g(\beta)$. In this context define

$$c^*(\beta) := \nu_\beta(\{\mathrm{Stab}^*(\beta)\eta \in \mathrm{Stab}^*(\beta)\backslash\mathcal{MF}_g(\beta) \colon i(\beta,\eta) \leq 1\}).$$

Corollary [ES22, Corollary 8.5] guarantees this constant is positive and finite. The next results follow directly from the definitions but we record them here explicitly for future reference.

Proposition 4.14. *Let $\alpha \in \mathcal{C}_g^*$ be a filling geodesic current. Then, for every $Y \in \mathcal{T}_g$,*

$$c(\alpha) = \int_{\mathcal{PMF}_g} \exp\left(-h \log i\left(\alpha, \frac{\xi}{\sqrt{\mathrm{Ext}_Y(\xi)}}\right)\right) d\overline{\nu}_Y([\xi]).$$

Proposition 4.15. *Let β be a closed curve on S_g and Ξ be a set of Dehn-Thurston coordinates of $\mathcal{MF}_g$. Then, for every marked complex structure $X \in \mathcal{T}_g$,*

$$c^*(\beta) = \int_{\mathcal{D}_\beta} \exp\left(-h \log i\left(\frac{\eta}{\sqrt{\mathrm{Ext}_X(\eta)}}, \beta\right)\right) d\overline{\nu}_X([\eta]).$$

Corollary 4.16. *Let $\alpha \in \mathcal{C}_g^*$ be a filling geodesic current, let β be a closed curve on S_g and Ξ be a set of Dehn-Thurston coordinates of $\mathcal{MF}_g$. Then, for every pair of marked complex structures $X, Y \in \mathcal{T}_g$,*

$$c(\alpha) \cdot c^*(\beta) = \int_{\mathcal{D}_\beta \times \mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\xi])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\xi]).$$

Counting. It will be convenient for us to have an asymptotic formula for the cardinality of $\mathfrak{C}(\alpha,\beta,L)$. The following result was originally proved by Mirzakhani [Mir16, Theorem 1.1] and Erlandsson and Souto [ES22, Theorem 8.1]; for effective versions see [AH21b, Theorem 1.1] and [Hon24].

Theorem 4.17. *Let $\alpha \in \mathcal{C}_g^*$ be a filling geodesic current and β be a closed curve on S_g. Then,*

$$\lim_{L \to \infty} L^{-h} \cdot \#\mathfrak{C}(\alpha,\beta,L) = \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot c(\alpha) \cdot c^*(\beta).$$

Statistics estimates. One last set of ingredient we need to prove Theorems 1.1 and 1.3 are a couple of elementary estimates on the statistics under consideration. Recall that if β is a closed curve on S_g then $[\beta]$ denotes its homology class. Given a homologically non-trivial closed curve β on S_g, $X, Y \in \mathcal{T}_g$, and a norm $\|\cdot\|$ on $H_1(S_g;\mathbb{R})$, for every $\mathbf{g} \in \mathrm{Mod}_d$ such that $\mathbf{g}.Y \neq X$ denote

$$\sigma_{X,Y,\beta}^{\|\cdot\|}(\mathbf{g}) := \frac{\log \|\mathbf{g}^{-1}.[\beta]\|}{d_\mathcal{T}(X, \mathbf{g}.Y)}.$$

Analogously, given a filling geodesic current $\alpha \in \mathcal{C}_g^*$, a homologically non-trivial closed curve β on S_g, and a norm $\|\cdot\|$ on $H_1(S_g;\mathbb{R})$, for every $\mathbf{g} \in \mathrm{Mod}_g$ denote

$$\sigma_{\alpha,\beta}^{\|\cdot\|}(\mathbf{g}) := \frac{\log \|\mathbf{g}^{-1}.[\beta]\|}{\log i(\alpha, \mathbf{g}^{-1}.\beta)}$$

With this notation, the first elementary estimate we need, which is concerned with the statistics of Theorem 1.1, can be stated as follows.

Lemma 4.18. *Let β be a homologically non-trivial closed curve on S_g, $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, and $\|\cdot\|$ be a norm on $H_1(S_g;\mathbb{R})$. Suppose $\alpha \in \mathcal{C}_g^*$, $M > 0$, and $\mathbf{g} \in \mathrm{Mod}_g$ are such that*

(1) $\mathbf{g}.Y \neq X$.
(2) $|D_{\alpha,\beta}(\mathbf{g}) - d_\mathcal{T}(X, \mathbf{g}.Y)| \leq M$.
(3) $\min\{|\sigma_{\alpha,\beta}^{\|\cdot\|}(\mathbf{g})|, |\sigma_{X,Y,\beta}^{\|\cdot\|}(\mathbf{g})|\} \leq M$.

Then, the following estimate holds,

$$|\sigma_{\alpha,\beta}^{\|\cdot\|}(\mathbf{g}) - \sigma_{X,Y,\beta}^{\|\cdot\|}(\mathbf{g})| \leq \frac{M^2}{\min\{|D_{\alpha,\beta}(\mathbf{g})|, d_\mathcal{T}(X, \mathbf{g}.Y)\}}.$$

Proof. Suppose for simplicity that $|\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})| \leq |\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})|$; the proof in the other case is analogous. This implies $d_{\mathcal{T}}(X, \mathbf{g}.Y) \leq |D_{\alpha,\beta}(\mathbf{g})|$. Furthermore, by condition (3), we have

$$\big|\log \|\mathbf{g}^{-1}.[\beta]\|\big| \leq M \cdot |D_{\alpha,\beta}(\mathbf{g})|. \tag{51}$$

Condition (2) ensures that

$$\left|\frac{1}{D_{\alpha,\beta}(\mathbf{g})} - \frac{1}{d_{\mathcal{T}}(X,\mathbf{g}.Y)}\right| \leq \frac{M}{|D_{\alpha,\beta}(\mathbf{g})| \cdot d_{\mathcal{T}}(X,\mathbf{g}.Y)}. \tag{52}$$

From (51) and (52) we conclude that

$$|\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}) - \sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})| \leq \frac{M^2}{d_{\mathcal{T}}(X,\mathbf{g}.Y)} \leq \frac{M^2}{\min\{|D_{\alpha,\beta}(\mathbf{g})|, d_{\mathcal{T}}(X,\mathbf{g}.Y)\}}. \qquad \square$$

Recall $\varsigma = \varsigma(g) \in (0,1)$ denotes the top Lyapunov exponent of $\mathbb{H}_g$ as introduced in §2. Given a homologically non-trivial closed curve β on S_g, marked complex structures $X, Y \in \mathcal{T}_g$, and a norm $\|\cdot\|$ on $H_1(S_g;\mathbb{R})$, for every $\mathbf{g} \in \mathrm{Mod}_d$ such that $\mathbf{g}.Y \neq X$ denote

$$\tau^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g}) := \frac{\log \|\mathbf{g}^{-1}.[\beta]\| - d_{\mathcal{T}}(X,\mathbf{g}.Y)\cdot \varsigma}{\sqrt{d_{\mathcal{T}}(X,\mathbf{g}.Y)}}.$$

Analogously, given a filling geodesic current $\alpha \in \mathcal{C}^*_g$, a homologically non-trivial closed curve β on S_g, and a norm $\|\cdot\|$ on $H_1(S_g;\mathbb{R})$, for every $\mathbf{g} \in \mathrm{Mod}_g$ such that $i(\alpha, \mathbf{g}^{-1}.\beta) > 1$ denote

$$\tau^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}) := \frac{\log \|\mathbf{g}^{-1}.[\beta]\| - \log i(\alpha,\mathbf{g}^{-1}.\beta)\cdot \varsigma}{\sqrt{\log i(\alpha,\mathbf{g}^{-1}.\beta)}}.$$

With this notation, the second elementary estimate we need, which is concerned with the statistics of Theorem 1.3, can be stated as follows; the proof is analogous to that of Lemma 4.18.

Lemma 4.19. *Let β be a homologically non-trivial closed curve on S_g, $X, Y \in \mathcal{T}_g$ be marked complex structures on S_g, and $\|\cdot\|$ be a norm on $H_1(S_g;\mathbb{R})$. Suppose $\alpha \in \mathcal{C}^*_g$, $M > 0$, and $\mathbf{g} \in \mathrm{Mod}_g$ are such that*

(1) $\mathbf{g}.Y \neq X$.
(2) $i(\alpha, \mathbf{g}^{-1}.Y) > 1$.
(3) $|D_{\alpha,\beta}(\mathbf{g}) - d_{\mathcal{T}}(X,\mathbf{g}.Y)| \leq M$.
(4) $\min\{|\tau^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})|, |\tau^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})| \leq M$.

Then, the following estimate holds,

$$|\tau^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}) - \tau^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})| \leq \frac{M\sqrt{M} + M}{\sqrt{\min\{D_{\alpha,\beta}(\mathbf{g}), d_{\mathcal{T}}(X,\mathbf{g}.Y)\}}}.$$

Proofs of the main results. We are now ready to prove Theorems 1.1 and 1.3. We will actually prove more general versions, for arbitrary geodesic currents, which we now state. Given a filling geodesic current $\alpha \in \mathcal{C}^*_g$, a closed curve β on S_g, and $L > 0$, consider the finite set $\mathfrak{G}(\alpha,\beta,L)$ of all homotopy classes of closed curves γ on X of the same topological type as β and satisfying $i(\alpha,\gamma) \leq L$; in other words, we require $\gamma \in \mathrm{Mod}_g \cdot \beta$. Endow this space with the uniform probability measure $\mathbb{P}_{\alpha,\beta,L}$. Recall that $[\gamma] \in H_1(S_g;\mathbb{R})$ denotes the homology class of a closed curve γ on S_g. Recall that $\varsigma = \varsigma(g) \in (0,1)$ denotes the top Lyapunov exponent of $\mathbb{H}_g$ as introduced in §2. The following is a generalization of Theorem 1.1; such result can be recovered by letting α be the Liouville current of the corresponding negatively curved Riemannian metric.

Theorem 4.20. *Let $\alpha \in \mathcal{C}^*_g$ be a filling geodesic current, β be a closed curve on S_g that is non-trivial in homology, and $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Then, the random variables*

$$\frac{\log \|[\gamma]\|}{\log i(\alpha,\gamma)} \quad \textit{on} \quad (\mathfrak{G}(\alpha,\beta,L), \mathbb{P}_{X,\gamma_0,L})$$

converge in distribution to the point mass at ς as $L \to \infty$.

Proof. Recall that Theorem 4.17 guarantees that

$$\lim_{L\to\infty} L^{-h} \cdot \#\mathfrak{C}(\alpha,\beta,L) = \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot c(\alpha) \cdot c^*(\beta).$$

In particular, to prove the desired result, it is enough to show that for every non-negative, compactly supported function with one continuous derivative $\xi \in \mathcal{C}^+_c(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$,

$$\lim_{L\to\infty} L^{-h} \cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g} \in \mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) = \frac{1}{h \cdot \mathbf{m}(\mathcal{M}_g)} \cdot c(\alpha) \cdot c^*(\beta) \cdot \xi(\varsigma).$$

For the rest of this discussion we fix $\xi \in \mathcal{C}_c^+(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and show that

$$\liminf_{L\to\infty} L^{-h} \cdot \sum_{\mathrm{Stab}(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \geq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)} \cdot c(\alpha)\cdot c^*(\beta)\cdot \xi(\varsigma), \tag{53}$$

$$\limsup_{L\to\infty} L^{-h} \cdot \sum_{\mathrm{Stab}(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \leq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)} \cdot c(\alpha)\cdot c^*(\beta)\cdot \xi(\varsigma). \tag{54}$$

Denote by $\mathcal{D}_\beta \subseteq \mathcal{PMF}_g(\beta)$ and $\overline{D}_\beta \subseteq \mathcal{PMF}_g(\beta)$ the fundamental domain and its relative closure introduced in Proposition 4.9. Fix a set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$. Let $\epsilon > 0$ be arbitrary. Recall the definitions of the sets $\mathcal{D}^{-\epsilon}_{\beta,\Xi} \subseteq \mathcal{PMF}_g(\beta)$ and $\mathcal{D}^{+\epsilon}_{\beta,\Xi} \subseteq \mathcal{PMF}_g$ in (49) and (50). Fix marked complex structures $X, Y \in \mathcal{T}_g$. By Proposition 4.12, there exists a full density subset $\mathfrak{M}^* = \mathfrak{M}^*(\beta, X, Y) \subseteq \mathrm{Mod}_g$ and $R_0 := R_0(\beta, X, Y, \Xi, \epsilon) > 0$ such that the following map is injective:

$$(\mathfrak{M}(X, Y, \mathcal{D}^{-\epsilon}_{\beta,\Xi}, \mathcal{PMF}_g) \setminus \mathfrak{M}(X, Y, R_0)) \cap \mathfrak{M}^* \to \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g, \quad \mathbf{g} \mapsto \mathrm{Stab}^*(\beta)\mathbf{g}.$$

In particular, for every $L > 0$, the following map is well defined and injective:

$$(\mathfrak{M}_{D_{\alpha,\beta}}(X, Y, \mathcal{D}^{-\epsilon}_{\beta,\Xi}, \mathcal{PMF}_g, \log L) \setminus \mathfrak{M}(X, Y, R_0)) \cap \mathfrak{M}^* \to \mathfrak{C}(\alpha, \beta, L), \quad \mathbf{g} \mapsto \mathrm{Stab}^*(\beta)\mathbf{g}$$

Thus, as ξ is non-negative, the following bound holds for every $L > 1$:

$$\sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \geq \sum_{\mathbf{g}\in(\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)\setminus\mathfrak{M}(X,Y,R_0))\cap\mathfrak{M}^*} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})). \tag{55}$$

Now let $\epsilon_0 = \epsilon_0(\alpha, \beta, \Xi, X, Y) > 0$ be the minimum of the corresponding constants in Proposition 4.10 and Corollary 4.11. Proposition 4.10 and fact that ξ has compact support guarantee the existence of $M = M(\alpha, \beta, \Xi, X, Y, \xi) > 0$ such that $\mathrm{supp}(\xi) \subseteq [-M, M]$ and for every $\mathbf{g} \in \mathfrak{M}(X, Y, D^{+\epsilon_0}_{\beta,\Xi}, \mathcal{PMF}_g)$,

$$|D_{\alpha,\beta}(\mathbf{g}) - d_{\mathcal{T}}(X, \mathbf{g}.Y)| \leq M. \tag{56}$$

In particular, by Lemma 4.18, if $\mathbf{g} \in \mathfrak{M}(X, Y, D^{+\epsilon_0}_{\beta,\Xi}, \mathcal{PMF}_g)$ satisfies $\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g}) \in \mathrm{supp}(\xi)$, then

$$|\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}) - \sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})| \leq M^2/d_{\mathcal{T}}(X, \mathbf{g}.Y).$$

As ξ is Lipschitz, we deduce that, under the same conditions,

$$|\xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) - \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g}))| \leq \|\xi\|_{\mathcal{C}^1} \cdot M^2/d_{\mathcal{T}}(X, \mathbf{g}.Y).$$

In particular, by Theorem 3.9 and (56), as ξ is non-negative,

$$\begin{aligned} &\sum_{\mathbf{g}\in(\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)\setminus\mathfrak{M}(X,Y,R_0))\cap\mathfrak{M}^*} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \\ &\geq \sum_{\mathbf{g}\in(\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)\setminus\mathfrak{M}(X,Y,R_0))\cap\mathfrak{M}^*} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})) + O_{X,Y,M,\xi}(L^h \cdot (\log L)^{-1}). \end{aligned} \tag{57}$$

Putting together (55) and (57) we deduce

$$\begin{aligned} &\sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \\ &\geq \sum_{\mathbf{g}\in(\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)\setminus\mathfrak{M}(X,Y,R_0))\cap\mathfrak{M}^*} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})) + O_{X,Y,M,\xi}(L^h \cdot (\log L)^{-1}). \end{aligned}$$

In particular, taking $L \to \infty$,

$$\begin{aligned} &\liminf_{L\to\infty} L^{-h} \cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \\ &\geq \liminf_{L\to\infty} L^{-h} \cdot \sum_{\mathbf{g}\in(\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)\setminus\mathfrak{M}(X,Y,R_0))\cap\mathfrak{M}^*} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})). \end{aligned} \tag{58}$$

Notice that condition (3) in Proposition 4.9 guarantees $\overline{\nu}_X(\partial\mathcal{D}^{-\epsilon}_{\beta,\Xi}) = 0$. We can thus apply Theorems 3.9, 4.4, and 4.5, and the fact that $\mathfrak{M}^* \subseteq \mathrm{Mod}_g$ has full density, to deduce

$$\begin{aligned} &\lim_{L\to\infty} L^{-h} \sum_{\mathbf{g}\in(\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{-\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)\setminus\mathfrak{M}(X,Y,R_0))\cap\mathfrak{M}^*} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})) \\ &= \frac{1}{h\cdot\mathbf{m}(\mathcal{M}_g)} \cdot \xi(\varsigma) \cdot \int_{D^{-\epsilon}_{\beta,\Xi}\times\mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\zeta])}\, d(\overline{\nu}_X \otimes \overline{\nu}_Y)([\eta],[\zeta]). \end{aligned} \tag{59}$$

Putting together (58) and (59) we obtain

$$\begin{aligned}&\liminf_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))\\ &\geq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)}\cdot \xi(\varsigma)\cdot \int_{D^{-\epsilon}_{\beta,\Xi}\times \mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\zeta])}\, d(\overline{\nu}_X\otimes\overline{\nu}_Y)([\eta],[\zeta])\end{aligned}$$

Letting $\epsilon\to 0$ yields

$$\begin{aligned}&\liminf_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))\\ &\geq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)}\cdot \xi(\varsigma)\cdot \int_{D_{\beta}\times \mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\zeta])}\, d(\overline{\nu}_X\otimes\overline{\nu}_Y)([\eta],[\zeta])\end{aligned}$$

Finally, Corollary 4.16 allows us to conclude the proof of (53):

$$\liminf_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \geq \frac{1}{h\cdot \mathbf{m}(\mathcal{M}_g)}\cdot c(\alpha)\cdot c^*(\beta)\cdot\xi(\varsigma).$$

We now prove (54). Again, fix a set of Dehn-Thurston coordinates Ξ of $\mathcal{MF}_g$ and marked complex structures $X,Y\in\mathcal{T}_g$. Let $\epsilon_0=\epsilon_0(\alpha,\beta,\Xi,X,Y)>0$ be the minimum of the corresponding constants in Proposition 4.10 and Corollary 4.11. Consider the standard map $\mathrm{std}\colon \mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g\to\mathrm{Mod}_g$ and the corresponding full density subset $\mathfrak{C}^*=\mathfrak{C}^*(\beta,\Xi,X,Y)\subseteq\mathrm{Stab}^*(\beta)$ introduced in Proposition 4.13. Fix $0<\epsilon<\epsilon_0$ and let $L_0=L_0(\alpha,\beta,\Xi,X,Y,\epsilon)>0$ be as in Proposition 4.13. Thus,

$$\mathrm{std}(\mathfrak{C}^*\setminus\mathfrak{C}(\alpha,\beta,L_0))\subseteq \mathfrak{M}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g),$$

In particular, for every $L>0$ we get an injection,

$$\mathrm{std}\colon (\mathfrak{C}(\alpha,\beta,L)\setminus\mathfrak{C}(\alpha,\beta,L_0))\cap\mathfrak{C}^*\to \mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L). \tag{60}$$

As $\mathfrak{C}^*\subseteq\mathrm{Stab}^*(\beta)\backslash\mathrm{Mod}_g$ has full density,

$$\begin{aligned}&\limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))\\ &=\limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\in(\mathfrak{C}(\alpha,\beta,L)\setminus\mathfrak{C}(\alpha,\beta,L_0))\cap\mathfrak{C}^*} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})).\end{aligned}\tag{61}$$

As the map in (60) is injective and a ξ is non-negative,

$$\sum_{\mathrm{Stab}^*(\beta)\in(\mathfrak{C}(\alpha,\beta,L)\setminus\mathfrak{C}(\alpha,\beta,L_0))\cap\mathfrak{C}^*} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))\leq \sum_{\mathbf{g}\in\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \tag{62}$$

Now, Proposition 4.10 and fact that ξ has compact support guarantee the existence of a constant $M=M(\alpha,\beta,\Xi,X,Y,\xi)>0$ such that $\mathrm{supp}(\xi)\subseteq[-M,M]$ and for every $\mathbf{g}\in\mathfrak{M}(X,Y,D^{+\epsilon_0}_{\beta,\Xi},\mathcal{PMF}_g)$,

$$|D_{\alpha,\beta}(\mathbf{g})-d_{\mathcal{T}}(X,\mathbf{g}.Y)|\leq M. \tag{63}$$

The same arguments introduced above, in particular, Lemma 4.18, allow one to deduce that, if $\mathbf{g}\in\mathfrak{M}(X,Y,D^{+\epsilon_0}_{\beta,\Xi},\mathcal{PMF}_g)$ satisfies $\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})\in\mathrm{supp}(\xi)$, then

$$|\xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))-\xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g}))|\leq \|\xi\|_{\mathcal{C}^1}\cdot M^2/d_{\mathcal{T}}(X,\mathbf{g}.Y).$$

In particular, by Theorem 3.9 and (63), as ξ is non-negative,

$$\begin{aligned}&\sum_{\mathbf{g}\in\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))\\ &\leq \sum_{\mathbf{g}\in\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g}))+O_{X,Y,M,\xi}(L^h\cdot(\log L)^{-1})\end{aligned}$$

Taking $L\to\infty$ we deduce

$$\begin{aligned}&\limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathbf{g}\in\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))\\ &\leq \limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathbf{g}\in\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g})).\end{aligned}\tag{64}$$

Notice that condition (3) in Proposition 4.9 guarantees $\overline{\nu}_X(\partial\mathcal{D}^{+\epsilon}_{\beta,\Xi}) = 0$. We can thus apply Theorems 4.4 and 4.5 to deduce the following:

$$\lim_{L\to\infty} L^{-h} \sum_{\mathbf{g}\in\mathfrak{M}_{D_{\alpha,\beta}}(X,Y,\mathcal{D}^{+\epsilon}_{\beta,\Xi},\mathcal{PMF}_g,\log L)} \xi(\sigma^{\|\cdot\|}_{X,Y,\beta}(\mathbf{g}))$$

$$= \frac{1}{h\cdot\mathbf{m}(\mathcal{M}_g)}\cdot\xi(\varsigma)\cdot\int_{\mathcal{D}^{+\epsilon}_{\beta,\Xi}\times\mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\zeta])}\, d(\overline{\nu}_X\otimes\overline{\nu}_Y)([\eta],[\zeta]). \tag{65}$$

Putting together (61), (62), (64), and (65) we obtain

$$\limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))$$

$$\leq \frac{1}{h\cdot\mathbf{m}(\mathcal{M}_g)}\cdot\xi(\varsigma)\cdot\int_{\mathcal{D}^{+\epsilon}_{\beta,\Xi}\times\mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\zeta])}\, d(\overline{\nu}_X\otimes\overline{\nu}_Y)([\eta],[\zeta]).$$

Letting $\epsilon\to 0$ yields

$$\limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}^*(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g}))$$

$$\leq \frac{1}{h\cdot\mathbf{m}(\mathcal{M}_g)}\cdot\xi(\varsigma)\cdot\int_{\overline{\mathcal{D}_\beta}\times\mathcal{PMF}_g} e^{hA_{\alpha,\beta}([\eta],[\zeta])}\, d(\overline{\nu}_X\otimes\overline{\nu}_Y)([\eta],[\zeta])$$

Finally, condition (4) in Proposition 4.9 and Corollary 4.16 allow us to conclude the proof of (54):

$$\limsup_{L\to\infty} L^{-h}\cdot \sum_{\mathrm{Stab}(\beta)\mathbf{g}\in\mathfrak{C}(\alpha,\beta,L)} \xi(\sigma^{\|\cdot\|}_{\alpha,\beta}(\mathbf{g})) \leq \frac{1}{h\cdot\mathbf{m}(\mathcal{M}_g)}\cdot c(\alpha)\cdot c^*(\beta)\cdot\xi(\varsigma). \qquad \square$$

Recall $V = V(g) > 0$ denotes the variance of $\mathbb{H}_g$ as introduced in §2. The following result is a generalization of Theorem 1.3; such result can be recovered by letting α be the Liouville current of the corresponding negatively curved Riemannian metric. The proof follows the same arguments as Theorem 4.20 but uses Theorem 4.6 in place of Theorem 4.5 and Lemma 4.19 in place of Lemma 4.18.

Theorem 4.21. *Let $\alpha\in\mathcal{C}^*_g$ be a filling geodesic current, β be a closed curve on S_g that is non-trivial in homology, and $\|\cdot\|$ be a norm on the homology group $H_1(X;\mathbb{R})$. Then, the random variables*

$$\frac{\log\|[\gamma]\| - \log i(\alpha,\gamma)\cdot\varsigma}{\sqrt{\log i(\alpha,\gamma)}} \quad \text{on} \quad (\mathfrak{G}(\alpha,\beta,L), \mathbb{P}_{X,\gamma_0,L})$$

converge in distribution to a Gaussian of mean zero and variance V as $L\to\infty$.

References

[ABEM12a] J. Athreya, A. Bufetov, A. Eskin, and M. Mirzakhani. Lattice point asymptotics and volume growth on Teichmüller space. *Duke Math. J.*, 161(6):1055–1111, 2012.

[ABEM12b] Jayadev Athreya, Alexander Bufetov, Alex Eskin, and Maryam Mirzakhani. Lattice point asymptotics and volume growth on Teichmüller space. *Duke Math. J.*, 161(6):1055–1111, 2012.

[AF24] Francisco Arana-Herrera and Giovanni Forni. From limit theorems to mixing limit theorems. *arXiv e-prints*, page arXiv:2405.16825, May 2024.

[AH21a] Francisco Arana-Herrera. Effective mapping class group dynamics ii: Geometric intersection numbers, 2021.

[AH21b] Francisco Arana-Herrera. Effective mapping class group dynamics iii: Counting filling closed curves on surfaces, 2021.

[AH23] Francisco Arana-Herrera. Effective mapping class group dynamics, I: Counting lattice points in Teichmüller space. *Duke Mathematical Journal*, pages 1 – 93, 2023.

[ASF22] Hamid Al-Saqban and Giovanni Forni. A central limit theorem for the kontsevich-zorich cocycle, 2022.

[Ath06] Jayadev S. Athreya. Quantitative recurrence and large deviations for Teichmuller geodesic flow. *Geom. Dedicata*, 119:121–140, 2006.

[BDG+21] Mark Bell, Vincent Delecroix, Vaibhav Gadre, Rodolfo Gutiérrez-Romo, and Saul Schleimer. The flow group of rooted abelian or quadratic differentials, 2021.

[BL98] Martine Babillot and François Ledrappier. Lalley's theorem on periodic orbits of hyperbolic flows. *Ergodic Theory Dynam. Systems*, 18(1):17–39, 1998.

[BL18] Anja Bankovic and Christopher J. Leininger. Marked-length-spectral rigidity for flat metrics. *Trans. Amer. Math. Soc.*, 370(3):1867–1884, 2018.

[Bon88] F. Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.

[DLR10] Moon Duchin, Christopher J. Leininger, and Kasra Rafi. Length spectra and degeneration of flat metrics. *Invent. Math.*, 182(2):231–277, 2010.

[DN20] Dmitry Dolgopyat and Péter Nándori. On mixing and the local central limit theorem for hyperbolic flows. *Ergodic Theory Dynam. Systems*, 40(1):142–174, 2020.

[Dum15] David Dumas. Skinning maps are finite-to-one. *Acta Math.*, 215(1):55–126, 2015.

[EMR19] Alex Eskin, Maryam Mirzakhani, and Kasra Rafi. Counting closed geodesics in strata. *Invent. Math.*, 215(2):535–607, 2019.

[Eps87] Charles L. Epstein. Asymptotics for closed geodesics in a homology class, the finite volume case. *Duke Math. J.*, 55(4):717–757, 1987.

[ES22] Viveka Erlandsson and Juan Souto. *Mirzakhani's curve counting and geodesic currents*, volume 345 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, [2022] ©2022.

[Fil17] Simion Filip. Zero Lyapunov exponents and monodromy of the Kontsevich–Zorich cocycle. *Duke Mathematical Journal*, 166(4):657 – 706, 2017.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[For02] Giovanni Forni. Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. *Ann. of Math. (2)*, 155(1):1–103, 2002.

[Fou20] Charles Fougeron. Diffusion rate of windtree models and Lyapunov exponents. *Bull. Soc. Math. France*, 148(1):25–49, 2020.

[Gar84] Frederick P. Gardiner. Measured foliations and the minimal norm property for quadratic differentials. *Acta Math.*, 152(1-2):57–76, 1984.

[GM91] F. P. Gardiner and H. Masur. Extremal length geometry of Teichmüller space. *Complex Variables Theory Appl.*, 16(2-3):209–237, 1991.

[HM79] J. Hubbard and H. Masur. Quadratic differentials and foliations. *Acta Math.*, 142(3-4):221–274, 1979.

[Hon24] Pouya Honaryar. Effective counts for closed curves of fixed type. *In Progress*, 2024.

[HP97] Sa'ar Hersonsky and Frédéric Paulin. On the rigidity of discrete isometry groups of negatively curved spaces. *Comment. Math. Helv.*, 72(3):349–388, 1997.

[Hub61] Heinz Huber. Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. II. *Math. Ann.*, 143:463—464, 1961.

[Jen57] J. A. Jenkins. On the existence of certain general extremal metrics. *Ann. of Math. (2)*, 66:440–453, 1957.

[Ker80] Steven P. Kerckhoff. The asymptotic geometry of Teichmüller space. *Topology*, 19(1):23–41, 1980.

[Lal89] Steven P. Lalley. Closed geodesics in homology classes on surfaces of variable negative curvature. *Duke Math. J.*, 58(3):795–821, 1989.

[Lev83] G. Levitt. Foliations and laminations on hyperbolic surfaces. *Topology*, 22(2):119–135, 1983.

[Mar70] G. A. Margulis. *On some aspects of the theory of Anosov systems*. Ph.D. Thesis, 1970. Springer-Verlag, Berlin, 2003.

[Mas82] Howard Masur. Interval exchange transformations and measured foliations. *Ann. of Math. (2)*, 115(1):169–200, 1982.

[Mir08a] Maryam Mirzakhani. Ergodic theory of the earthquake flow. *Int. Math. Res. Not. IMRN*, (3):Art. ID rnm116, 39, 2008.

[Mir08b] Maryam Mirzakhani. Growth of the number of simple closed geodesics on hyperbolic surfaces. *Ann. of Math. (2)*, 168(1):97–125, 2008.

[Mir16] M. Mirzakhani. Counting Mapping Class group orbits on hyperbolic surfaces. Preprint, arXiv:1601.03342, 2016.

[MR95a] Greg McShane and Igor Rivin. A norm on homology of surfaces and counting simple geodesics. *Internat. Math. Res. Notices*, (2):61–69, 1995.

[MR95b] Greg McShane and Igor Rivin. Simple curves on hyperbolic tori. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(12):1523–1528, 1995.

[MT02] Howard Masur and Serge Tabachnikov. Rational billiards and flat structures. In *Handbook of dynamical systems, Vol. 1A*, pages 1015–1089. North-Holland, Amsterdam, 2002.

[Ota90] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Ann. of Math. (2)*, 131(1):151–162, 1990.

[PH92] R. C. Penner and J. L. Harer. *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992.

[Pol91] Mark Pollicott. Homology and closed geodesics in a compact negatively curved surface. *Amer. J. Math.*, 113(3):379–385, 1991.

[PS87] Ralph Phillips and Peter Sarnak. Geodesics in homology classes. *Duke Math. J.*, 55(2):287–297, 1987.

[RS19] K. Rafi and J. Souto. Geodesic currents and counting problems. *Geom. Funct. Anal.*, 29(3):871–889, 2019.

[Sha04] Richard Sharp. A local limit theorem for closed geodesics and homology. *Trans. Amer. Math. Soc.*, 356(12):4897–4908, 2004.

[Str66] K. Strebel. Über quadratische Differentiale mit geschlossenen Trajektorien und extremale quasikonforme Abbildungen. In *Festband 70. Geburtstag R. Nevanlinna*, pages 105–127. Springer, Berlin, 1966.

[Str75] K. Strebel. On quadratic differentials and extremal quasi-conformal mappings. In *Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 2*, pages 223–227. Canad. Math. Congress, Montreal, Que., 1975.

[Str76] K. Strebel. On the existence of extremal Teichmueller mappings. *J. Analyse Math.*, 30:464–480, 1976.

[Vee82] William A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)*, 115(1):201–242, 1982.

Email address: `farana@umd.edu`

Department of Mathematics, University of Maryland, 4176 Campus Drive, College Park, MD 20742, USA.

Email address: `pouya.honaryar@mail.utoronto.ca`

Department of Mathematics, University of Zurich, 190, Winterthurerstrasse, 8057 Zürich, Switzerland.