COUNTING HYPERBOLIC MULTI-GEODESICS WITH RESPECT TO THE LENGTHS OF INDIVIDUAL COMPONENTS AND ASYMPTOTICS OF WEIL-PETERSSON VOLUMES

FRANCISCO ARANA–HERRERA

ABSTRACT. Given a connected, oriented, complete, finite area hyperbolic surface X of genus g with n punctures, Mirzakhani showed that the number of simple closed multi-geodesics on X of a prescribed topological type and total hyperbolic length $\leq L$ is asymptotic to a polynomial in L of degree $6g-6+2n$ as $L \to \infty$. We establish asymptotics of the same kind for countings of simple closed multi-geodesics that keep track of the hyperbolic length of individual components rather than just the total hyperbolic length, proving a conjecture of Wolpert. The leading terms of these asymptotics are related to limits of Weil-Petersson volumes of expanding subsets of quotients of Teichmüller space. We introduce a framework for computing limits of this kind in terms of purely topological information. We provide two further applications of this framework to countings of square-tiled surfaces and countings of filling closed multi-geodesics on hyperbolic surfaces.

Contents

1. Introduction

Let X be a connected, oriented, complete, finite area hyperbolic surface of genus g with n punctures. In [Mir08b], Mirzakhani showed that the number of simple closed multi-geodesics on X of a prescribed topological type and total hyperbolic length $\leq L$ is asymptotic to a polynomial in L of degree $6g-6+2n$ as $L \to \infty$. Wolpert conjectured that analogous results should hold for countings of simple closed multi-geodesics that keep track of the hyperbolic length of individual components rather than just the total hyperbolic length.

For instance, let X be a connected, oriented, complete, finite area hyperbolic surface of genus g with n punctures and $\gamma := (\gamma_1, \ldots, \gamma_k)$ be an ordered simple closed multi-curve on X with $1 \leq k \leq 3g-3+n$ components. For every $L > 0$

consider the counting function

$$m(X,\gamma,L) \tag{1.1}$$

$$:= \# \left\{ \begin{array}{l} \text{ordered simple closed multi-geodesics } \alpha := (\alpha_1,\ldots,\alpha_k) \text{ on } X \\ \text{of the same topological type as } \gamma \text{ with } \max_{i=1,\ldots,k} \ell_{\alpha_i}(X) \leq L \end{array} \right\},$$

where $\ell_{\alpha_i}(X)$ denotes the hyperbolic length of the geodesic α_i on X. In this paper we show that $m(X,\gamma,L)$ is asymptotic to a polynomial in L of degree $6g-6+2n$ as $L \to \infty$. Wolpert's more general conjecture, introduced below as Theorem 1.9, is also proved in this paper.

Mirzakhani's results and techniques in [Mir08b] can be used to establish asymptotics for countings of simple closed multi-curves with respect to length functions much more general than total hyperbolic length. Recent generalizations due to several authors also establish asymptotics for countings of objects much more general than simple closed multi-curves [ES16, EPS20, RS19]. Wolpert's conjecture does not fit into this framework; see Remark 1.8 below.

Instead, our proof of Wolpert's conjecture draws inspiration from ideas introduced by Margulis in his thesis [Mar70]. Using general averaging and unfolding techniques for parametrized countings, we reduce the proof of this conjecture to an application of equidistribution results for analogues of expanding horoballs on moduli spaces of hyperbolic surfaces. A first version of these results was established by Mirzakhani in [Mir07a] and was later generalized by the author in [Ara20b].

As described in Theorem 1.9, the leading terms of the asymptotics in Wolpert's conjecture are related to limits of Weil-Petersson volumes of expanding subsets of quotients of Teichmüller space. In this paper we introduce a framework for computing limits of this kind in terms of purely topological information; see Theorem 1.16. We provide two further applications of this framework to countings of square-tiled surfaces and countings of filling closed multi-geodesics on hyperbolic surfaces; see Theorems 1.18 and 1.21 for precise statements.

The centerpiece of this framework is Proposition 4.4, which shows that an appropriate renormalization of the Weil-Petersson measure on Teichmüller space converges to the Thurston measure on the space of measured geodesic laminations as one lets the curvature of the metrics diverge to $-\infty$. The main tool used in the proof of Proposition 4.4 is the correspondence of the Weil-Petersson measure and the Thurston measure through Thurston's shear coordinates [Thu86, PP93, SB01].

The rest of this section is devoted to setting up notation and providing precise statements of the aforementioned results.

Notation. For the rest of this paper we fix a pair of non-negative integers $g,n \geq 0$ satisfying $2-2g-n<0$ and a connected, oriented surface $S_{g,n}$ of genus g with n punctures (and negative Euler characteristic).

Denote by $\mathcal{T}_{g,n}$ the Teichmüller space of marked, oriented, complete, finite area hyperbolic structures on $S_{g,n}$ up to isotopy, by $\mathrm{Mod}_{g,n}$ the mapping class group of $S_{g,n}$, and by $\mathcal{M}_{g,n} := \mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}$ the moduli space of oriented, complete, finite area hyperbolic structures on $S_{g,n}$.

Let $\alpha := (\alpha_1,\ldots,\alpha_k)$ with $k \geq 1$ be an ordered tuple of pairwise non-isotopic essential closed curves on $S_{g,n}$, an *ordered closed multi-curve* for short. For every $X \in \mathcal{T}_{g,n}$, the *hyperbolic length vector* of α with respect to X is given by

$$\vec{\ell}_\alpha(X) := (\ell_{\alpha_1}(X),\ldots,\ell_{\alpha_k}(X)) \in (\mathbf{R}_{>0})^k,$$

where, for every $i \in \{1, \ldots, k\}$, $\ell_{\alpha_i}(X) > 0$ denotes the hyperbolic length of the unique geodesic representative of α_i on X. Given a vector $\mathbf{a} := (a_1, \ldots, a_k) \in (\mathbf{R}_{>0})^k$ of positive weights on the components of α, consider the *weighted closed multi-curve* on $S_{g,n}$ given by

$$\mathbf{a} \cdot \alpha := a_1\alpha_1 + \cdots + a_k\alpha_k. \tag{1.2}$$

The *total hyperbolic length* of $\mathbf{a} \cdot \alpha$ with respect to X is given by

$$\ell_{\mathbf{a}\cdot\alpha}(X) := \mathbf{a} \cdot \vec{\ell}_\alpha(X) = a_1\ell_{\alpha_1}(X) + \cdots + a_k\ell_{\alpha_k}(X) \in \mathbf{R}_{>0}$$

Unless otherwise specified, the term *length* will always refer to *hyperbolic length*.

Mirzakhani's asymptotic counting formula. Let $\gamma := (\gamma_1, \ldots, \gamma_k)$ with $k \geq 1$ be an ordered closed multi-curve on $S_{g,n}$, $\mathbf{a} := (a_1, \ldots, a_k) \in (\mathbf{R}_{>0})^k$ be a vector of positive weights on the components of γ, and $X \in \mathcal{T}_{g,n}$. For every $L > 0$ consider the counting function

$$t(X, \gamma, \mathbf{a}, L) := \#\{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell_{\mathbf{a}\cdot\alpha}(X) \leq L\}. \tag{1.3}$$

In words, $t(X, \gamma, \mathbf{a}, L)$ is the number of ordered multi-geodesics on X of the same topological type as γ whose total hyperbolic length with respect to the weights $\mathbf{a}$ is $\leq L$. This function does not depend on the marking of $X \in \mathcal{T}_{g,n}$ but only on the subjacent hyperbolic structure $X \in \mathcal{M}_{g,n}$. In [Mir08b], Mirzakhani's described the asymptotics of $t(X, \gamma, \mathbf{a}, L)$ as $L \to \infty$ when γ is *simple*, i.e., when the components of γ are simple and pairwise disjoint.

To give a precise statement of Mirzakhani's asymptotic counting formula, we first introduce some notation. Consider the subgroup

$$\mathrm{Stab}(\gamma) = \bigcap_{i=1}^{k} \mathrm{Stab}(\gamma_i) \subseteq \mathrm{Mod}_{g,n}$$

of mapping classes of $S_{g,n}$ that fix every component of γ up to isotopy. Denote by μ_{wp} the Weil-Petersson measure on $\mathcal{T}_{g,n}$ and by $\widetilde{\mu}_{\mathrm{wp}}^{\gamma}$ the local pushforward of μ_{wp} to $\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)$. In [Mir08b], Mirzakhani showed that, if γ is simple, the following limit, known as the *frequency* of the weighted simple closed multi-curve $\mathbf{a}\cdot\gamma$, exists,

$$r(\gamma, \mathbf{a}) := \lim_{L\to\infty} \frac{\widetilde{\mu}_{\mathrm{wp}}^{\gamma}(\{Y \in \mathcal{T}_{g,n}/\mathrm{Stab}(\gamma) \mid \ell_{\mathbf{a}\cdot\gamma}(Y) \leq L\})}{L^{6g-6+2n}}. \tag{1.4}$$

Furthermore, Mirzakhani provided an explicit formula for computing $r(\gamma, \mathbf{a})$. More precisely, letting $\mathbf{x} := (x_1, \ldots, x_k)$ be the standard coordinates of $(\mathbf{R}_{\geq 0})^k$ and $d\mathbf{x} := dx_1 \cdots dx_k$ be the standard measure of $(\mathbf{R}_{\geq 0})^k$, there exists an explicit polynomial $W_{g,n}(\gamma, \mathbf{x})$ of degree $6g - 6 + 2n - k$ on the $\mathbf{x}$ variables, all of whose non-zero monomials are of top degree, with non-negative rational coefficients, and which has $x_1 \cdots x_k$ as a factor, such that the following result holds.

Proposition 1.1. [Mir08b, Proposition 5.1] *Let $\gamma := (\gamma_1, \ldots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g - 3 + n$ components and $\mathbf{a} := (a_1, \ldots, a_k) \in (\mathbf{R}_{>0})^k$ be a vector of positive weights. Then,*

$$r(\gamma, \mathbf{a}) = \int_{\mathbf{a}\cdot\mathbf{x}\leq 1} W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}.$$

Remark 1.2. Up to a constant, $W_{g,n}(\gamma,\mathbf{x})$ is equal to $x_1\cdots x_k$ times the sum of the top degree monomials of the product of the Weil-Petersson volume polynomials of the moduli spaces of bordered Riemann surfaces associated to the components of the surface obtained by cutting $S_{g,n}$ along γ. See §2 for a precise definition.

Denote by $\mathcal{ML}_{g,n}$ the space of measured geodesic laminations on $S_{g,n}$ and by μ_{Thu} be the Thurston measure on $\mathcal{ML}_{g,n}$. Consider the function $B\colon \mathcal{M}_{g,n}\to\mathbf{R}_{>0}$ which to every $X\in\mathcal{M}_{g,n}$ assigns the value

$$B(X) := \mu_{\mathrm{Thu}}(\{\lambda\in\mathcal{ML}_{g,n} \mid \ell_\lambda(X)\leq 1\}), \tag{1.5}$$

where $\ell_\lambda(X)>0$ denotes the hyperbolic length of λ with respect to X. We refer to this function as the *Mirzakhani function*. Roughly speaking, $B(X)$ measures the shortness of simple closed geodesics on X. Denote by $\widehat{\mu}_{\mathrm{wp}}$ the local pushforward of the Weil-Petersson measure μ_{wp} on $\mathcal{T}_{g,n}$ to $\mathcal{M}_{g,n}:=\mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}$. By work of Mirzakhani, B is continuous, proper, and integrable with respect to $\widehat{\mu}_{\mathrm{wp}}$ [Mir08b, Proposition 3.2, Theorem 3.3]. Define

$$b_{g,n} := \int_{\mathcal{M}_{g,n}} B(X)\, d\widehat{\mu}_{\mathrm{wp}}(X) < +\infty. \tag{1.6}$$

Remark 1.3. In [AA20], upper and lower bounds of the same order describing the behavior of B near the cusp of $\mathcal{M}_{g,n}$ are established. In particular, it is proved that B is square-integrable with respect to $\widehat{\mu}_{\mathrm{wp}}$.

The following theorem due to Mirzakhani describes the asymptotics of the counting functions $t(X,\gamma,\mathbf{a},L)$ as $L\to\infty$ when γ is simple.

Theorem 1.4. [Mir08b, Theorem 6.1] *Let $X\in\mathcal{M}_{g,n}$, $\gamma:=(\gamma_1,\dots,\gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1\leq k\leq 3g-3+n$ components, and $\mathbf{a}:=(a_1,\dots,a_k)\in(\mathbf{R}_{>0})^k$ be a vector of positive weights. Then,*

$$\lim_{L\to\infty}\frac{t(X,\gamma,\mathbf{a},L)}{L^{6g-6+2n}} = \frac{B(X)\cdot r(\gamma,\mathbf{a})}{b_{g,n}}.$$

Remark 1.5. In [EMM19], Eskin, Mirzakhani, and Mohammadi improved Theorem 1.4 by obtaining a power saving error term for the asymptotics of $t(X,\gamma,\mathbf{a},L)$. Their methods are very different from the ones in [Mir08b] and rely on the exponential mixing rate of the Teichmüller geodesic flow.

Wolpert conjectured that results analogous to Theorem 1.4 should hold for countings of simple closed multi-geodesics that keep track of the the hyperbolic length of individual components rather than just the total hyperbolic length. To give a precise statement of this conjecture we first introduce some terminology.

Length spectra of ordered closed multi-curves. Let $\gamma:=(\gamma_1,\dots,\gamma_k)$ be an ordered closed multi-curve on $S_{g,n}$ with $k\geq 1$ components and $X\in\mathcal{T}_{g,n}$. One can record, with multiplicity, the hyperbolic length vector with respect to X of every ordered closed multi-curve in the mapping class group orbit of γ by considering the counting measure on $(\mathbf{R}_{\geq 0})^k$ given by

$$\mu_{\gamma,X} := \sum_{\alpha\in\mathrm{Mod}_{g,n}\cdot\gamma}\delta_{\vec{\ell}_\alpha(X)}.$$

This measure does not depend on the marking of $X \in \mathcal{T}_{g,n}$ but only on the subjacent hyperbolic structure $X \in \mathcal{M}_{g,n}$. We refer to this measure as the *length spectrum* of γ with respect to X.

To study the asymptotic behavior of $\mu_{\gamma,X}$ we consider the rescaled counting measures $\{\mu_{\gamma,X}^L\}_{L>0}$ on $(\mathbf{R}_{\geq 0})^k$ given by

$$\mu_{\gamma,X}^L := \sum_{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma} \delta_{\frac{1}{L} \cdot \vec{\ell}_\alpha(X)}.$$

Asymptotics of length spectra of ordered simple closed multi-curves. The first main result of this paper is the following theorem, which describes the behavior near infinity of the length spectrum of ordered simple closed multi-curves with respect to complete, finite area hyperbolic structures.

Theorem 1.6. *Let $\gamma := (\gamma_1, \dots, \gamma_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$ and $X \in \mathcal{M}_{g,n}$. Then,*

$$\lim_{L \to \infty} \frac{\mu_{\gamma,X}^L}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}$$

in the weak-$\star$ topology for measures on $(\mathbf{R}_{\geq 0})^k$.

Remark 1.7. Theorem 1.4 can be deduced directly from Theorem 1.6 and Portmanteau's theorem. This provides an alternative proof of Theorem 1.4 independent of Mirzakhani's original work in [Mir08b].

Remark 1.8. Theorem 1.6 is not a direct consequence of Theorem 1.4. Indeed, simplices of $(\mathbf{R}_{\geq 0})^k$ of the form

$$\Delta_{\mathbf{a}} := \{(x_1, \dots, x_k) \in (\mathbf{R}_{\geq 0})^k \mid a_1 x_1 + \cdots + a_k x_k \leq 1\}$$

with $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{R}_{>0})^k$ arbitrary do not generate the σ-algebra of Borel measurable subsets of $(\mathbf{R}_{\geq 0})^k$. Furthermore, Mirzakhani's more general counting results in [Mir08b] do not directly imply Theorem 1.6 as the notion of *hyperbolic length of individual components* does not extend continuously from the dense subset of rationally weighted simple closed multi-curves to all $\mathcal{ML}_{g,n}$.

Let $\gamma := (\gamma_1, \dots, \gamma_k)$ be an ordered closed multi-curve on $S_{g,n}$ with $k \geq 1$ components, $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$ be a vector of positive scaling parameters, and $X \in \mathcal{T}_{g,n}$. For every $L > 0$ consider the counting function

$$\begin{aligned} (1.7) \qquad & c(X, \gamma, \mathbf{b}, L) \\ & := \#\{\alpha := (\alpha_1, \dots, \alpha_k) \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell_{\alpha_i}(X) \leq b_i L, \ \forall i = 1, \dots, k\}. \end{aligned}$$

In words, $c(X, \gamma, \mathbf{b}, L)$ is the of number of ordered closed multi-geodesics on X of the same topological type as γ whose i-th component has hyperbolic length $\leq b_i L$. This function does not depend on the marking of $X \in \mathcal{T}_{g,n}$ but only on the underlying hyperbolic structure $X \in \mathcal{M}_{g,n}$. The following theorem corresponds to Wolpert's conjecture. It is a direct consequence of Theorem 1.6 and Portmanteau's theorem.

Theorem 1.9. *Let $X \in \mathcal{M}_{g,n}$, $\gamma := (\gamma_1, \dots, \gamma_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$, and $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$. Then,*

$$\lim_{L \to \infty} \frac{c(X, \gamma, \mathbf{b}, L)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot \int_{\prod_{i=1}^k [0, b_i]} W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}.$$

Remark 1.10. A version of Theorem 1.9 for pair of pants decompositions was proved by Mirzakhani [Mir16, Theorem 1.2]. A result closely related to Theorem 1.9 was independently established by Liu [Liu19, Theorem 1.2]. Other results related to Theorem 1.9 are discussed in forthcoming work of Erlandsson and Souto [ES20].

Remark 1.11. Letting $\mathbf{b} := (1, \ldots, 1) \in (\mathbf{R}_{>0})^k$ in Theorem 1.9 gives asymptotics for the counting functions $m(X, \gamma, L)$ introduced in (1.1).

Remark 1.12. Wolpert has used Theorem 1.9 to prove asymptotic formulas for counting functions of proper, bi-infinite, simple complete geodesics on connected, oriented, complete, finite area hyperbolic surfaces with punctures.

Remark 1.13. Theorems 1.6 and 1.9 can be strengthened to obtain asymptotics of countings that also keep track of the equivalence class of rationally weighted simple closed multi-geodesics in the space of projective measured geodesic laminations; see Theorems 3.5 and 3.7 for precise statements.

Outline of the proof of Theorem 1.6. To prove Theorem 1.6 we consider the following equivalent reformulation. Let $X \in \mathcal{M}_{g,n}$, $\gamma := (\gamma_1, \ldots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g - 3 + n$ components, and $f \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be an arbitrary non-negative, continuous, compactly supported function. For every $L > 0$ consider the counting function

$$c(X, \gamma, f, L) := \int_{\mathbf{R}^k} f(\mathbf{x}) \ d\mu_{\gamma,X}^L(\mathbf{x}) = \sum_{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma} f\left(\tfrac{1}{L} \cdot \vec{\ell}_\alpha(X)\right). \tag{1.8}$$

This function does not depend on the marking of $X \in \mathcal{T}_{g,n}$ but only on the hyperbolic structure $X \in \mathcal{M}_{g,n}$. Notice that, for $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$, if

$$f(\mathbf{x}) := \prod_{i=1}^k \mathbb{1}_{[0,b_i]}(x_i),$$

then $c(X, \gamma, f, L) = c(X, \gamma, \mathbf{b}, L)$. By the definition of weak convergence of measures, Theorem 1.6 is equivalent to the following result.

Theorem 1.14. *Let $X \in \mathcal{M}_{g,n}$, $\gamma := (\gamma_1, \ldots, \gamma_k)$ with $1 \leq k \leq 3g - 3 + n$ be an ordered simple closed multi-curve on $S_{g,n}$, and $f \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a non-negative, continuous, compactly supported function. Then,*

$$\lim_{L \to \infty} \frac{c(X, \gamma, f, L)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot \int_{\mathbf{R}^k} f(\mathbf{x}) \cdot W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}.$$

Our proof of Theorem 1.14 is inspired by ideas introduced by Margulis in his thesis [Mar70]. The upshot of the proof is the following: approaching these countings directly for a particular hyperbolic structure X is rather hard but averaging them over nearby points in $\mathcal{M}_{g,n}$ should make them more tractable. After suitably averaging the countings over nearby points, unfolding such averages on an appropriate intermediate cover reduces the proof of Theorem 1.14 to the question of whether certain analogues of expanding horoballs on $\mathcal{M}_{g,n}$ equidistribute. Such equidistribution results were established by the author in [Ara20b] building on ideas introduced by Mirzakhani in [Mir07a].

Remark 1.15. If the analogues of expanding horoballs on $\mathcal{M}_{g,n}$ alluded to in the previous paragraph equidistributed at a polynomial rate, see Remark 3.4 for a precise statement of this condition, the methods in our proof would yield an effective version of Theorem 1.14 with a power saving error term.

Asymptotics of Weil-Petersson volumes. Let $\gamma := (\gamma_1, \dots, \gamma_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$. According to Theorem 1.6, the asymptotic length spectrum of γ with respect to any $X \in \mathcal{T}_{g,n}$ has a factor

$$W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}$$

which depends only on γ and not on X. This factor can be described in terms of limits of Weil-Petersson volumes of expanding subsets of quotients of $\mathcal{T}_{g,n}$; see Remark 1.2. The second goal of this paper is to introduce a framework for computing limits of this kind in terms of purely topological information.

Let $i \colon \mathcal{ML}_{g,n} \times \mathcal{ML}_{g,n} \to \mathbf{R}_{\geq 0}$ denote the geometric intersection number of measured geodesic laminations on $S_{g,n}$. For every $\mu \in \mathcal{ML}_{g,n}$ denote by $\mathcal{ML}_{g,n}(\mu) \subseteq \mathcal{ML}_{g,n}$ the open, dense, full measure subset of measured geodesic laminations that together with μ fill $S_{g,n}$. More precisely,

$$\mathcal{ML}_{g,n}(\mu) := \{\lambda \in \mathcal{ML}_{g,n} \mid i(\lambda, \eta) + i(\mu, \eta) > 0, \ \forall \eta \in \mathcal{ML}_{g,n}\}. \tag{1.9}$$

Denote by $\mathcal{ML}_{g,n}(\gamma) \subseteq \mathcal{ML}_{g,n}$ the corresponding subset when γ is endowed with an arbitrary transverse measure of full support. The stabilizer $\mathrm{Stab}(\gamma) \subseteq \mathrm{Mod}_{g,n}$ acts properly discontinuously on $\mathcal{ML}_{g,n}(\gamma)$; see Proposition 4.5. Consider the measure $\mu_{\mathrm{Thu}}^{\gamma} := \mu_{\mathrm{Thu}}|_{\mathcal{ML}_{g,n}(\gamma)}$ on $\mathcal{ML}_{g,n}(\gamma)$ and denote by $\widetilde{\mu}_{\mathrm{Thu}}^{\gamma}$ its local pushforward to $\mathcal{ML}_{g,n}(\gamma)/\mathrm{Stab}(\gamma)$. Let

$$I_\gamma \colon \mathcal{ML}_{g,n}(\gamma) \to (\mathbf{R}_{\geq 0})^k$$

be the map which to every $\lambda \in \mathcal{ML}_{g,n}(\gamma)$ assigns the vector

$$I_\gamma(\lambda) := (i(\gamma_1, \lambda), \dots, i(\gamma_k, \lambda)) \in (\mathbf{R}_{\geq 0})^k$$

and let

$$\widetilde{I}_\gamma \colon \mathcal{ML}_{g,n}(\gamma)/\mathrm{Stab}(\gamma) \to (\mathbf{R}_{\geq 0})^k$$

be the induced map on $\mathcal{ML}_{g,n}(\gamma)/\mathrm{Stab}(\gamma)$. In this paper we prove the following.

Theorem 1.16. *Let $\gamma := (\gamma_1, \dots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components. Then,*

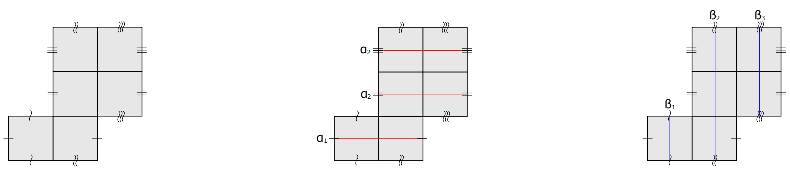
$$W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x} = (\widetilde{I}_\gamma)_*(\widetilde{\mu}_{Thu}^{\gamma}).$$

Remark 1.17. For a pair of pants decomposition of $S_{g,n}$, Theorem 1.16 can be proved directly using Wolpert's magic formula [Wol85, Theorem 1.3] and an explicit computation in Dehn-Thurston coordinates; see [Ara20a, §4] for details.

The main tool used in the proof of Theorem 1.16 is the correspondence of the Weil-Petersson measure on $\mathcal{T}_{g,n}$ and the Thurston measure on $\mathcal{ML}_{g,n}$ through Thurston's shear coordinates [Thu86, PP93, SB01]. Using this correspondence we show that an appropriate renormalization of the Weil-Petersson measure converges to the Thurston measure as one lets the curvature of the metrics diverge to $-\infty$; see Proposition 4.4 for a precise statement. The characterization of the subset $\mathcal{ML}_{g,n}(\gamma) \subseteq \mathcal{ML}_{g,n}$ provided by Proposition 4.8 will play an important role when dealing with issues of non-compactness that arise in the course of the proof.

We provide two further applications of the framework developed in the proof of Theorem 1.16 to countings of square-tiled surfaces and countings of filling closed multi-geodesics on hyperbolic surfaces. We now describe these applications.

Counting square-tiled surfaces. A *square-tiled surface* is a surface constructed from finitely many disjoint unit area squares on the complex plane, with sides parallel to the real and imaginary axes, by identifying pairs of sides by translation and/or $180°$ rotation. Points of the surface with cone angle π are considered as punctures. The *horizontal core multi-curve* of a square tiled-surface is the integrally weighted simple closed multi-curve obtained by concatenating the horizontal segments running through the middle of each square. The *vertical core multi-curve* of a square tiled-surface is defined in an analogous way. See Figure 1 for an example.



(a) Square-tiled surface. (b) Horizontal core multi-curve. (c) Vertical core multi-curve.

FIGURE 1. Example of a square-tiled surface of genus 2 with no punctures. The horizontal core multi-curve is $\alpha_1 + 2\alpha_2$. The vertical core multi-curve is $\beta_1 + \beta_2 + \beta_3$.

Two integrally weighted simple closed multi-curves on homeomorphic surfaces have the same *topological type* if there exists a homeomorphism between the surfaces mapping one multi-curve to the other preserving the weights. Let $\alpha := (\alpha_1, \dots, \alpha_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Z}_{>0})^k$ be vector of positive integral weights on the components of α, and $\mathbf{a} \cdot \alpha$ be as in (1.2). For every $L > 0$ consider the counting function

$$s(\mathbf{a} \cdot \alpha, L) := \# \left\{ \begin{array}{c} \text{square-tiled surfaces with horizontal core multi-curve} \\ \text{of the same topological type as } \mathbf{a} \cdot \alpha \text{ and } \leq L \text{ squares} \end{array} \right\} / \sim,$$

where $\sim$ denotes the equivalence relation induced by cut and paste operations. We are interested in the asymptotic behavior of $s(\mathbf{a} \cdot \alpha, L)$ as $L \to \infty$.

Denote by

$$\epsilon_{g,n} := \begin{cases} 4 & \text{if} \quad (g,n) = (0,4), \\ 2 & \text{if} \quad (g,n) \in \{(1,1),(1,2),(2,0)\}, \\ 1 & \text{if} \quad (g,n) \notin \{(0,4),(1,1),(1,2),(2,0)\}, \end{cases} \tag{1.10}$$

the number of automorphisms of a generic square-tiled surface of genus g with n punctures. Recall the definition of $r(\alpha, \mathbf{a})$ in (1.4). In this paper we combine results and techniques from [Ara20a] with Theorem 1.16 to prove the following.

Theorem 1.18. *Let $\alpha := (\alpha_1, \dots, \alpha_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$ and $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Z}_{>0})^k$. Then,*

$$\lim_{L \to \infty} \frac{s(\mathbf{a} \cdot \alpha, L)}{L^{6g-6+2n}} = \frac{\epsilon_{g,n} \cdot r(\alpha, \mathbf{a})}{2^{2g-3+n}}.$$

Remark 1.19. Theorem 1.18 was originally proved by Delecroix, Goujard, Zograf, and Zorich using algebro-geometric methods [DGZZ19]. In [Ara20a] we gave a different proof of Theorem 1.18 using results of Mirzakhani [Mir08b]. The proof in this paper establishes an explicit connection between countings of square-tiled surfaces and asymptotics of Weil-Petersson volumes of expanding subsets of quotients of Teichmüller space through their relation with Thurston volumes of subsets of quotients of $\mathcal{ML}_{g,n}$.

Asymptotics of length spectrum of ordered filling closed multi-curves. An ordered closed multi-curve on $S_{g,n}$ is said to be filling if it cuts the surface into polygons with at most one puncture in their interior. Recall the definition of the counting functions with respect to total hyperbolic length $t(X,\gamma,\mathbf{a},L)$ in (1.3). In [Mir16], Mirzakhani showed that, if $\gamma := (\gamma_1,\ldots,\gamma_k)$ is a filling ordered closed multi-curve on $S_{g,n}$ with $k \geq 1$ components and $\mathbf{a} := (a_1,\ldots,a_k) \in (\mathbf{R}_{>0})^k$ is a vector of positive weights, then the counting function $t(X,\gamma,\mathbf{a},L)$ is asymptotic to a polynomial in L of degree $6g-6+2n$ as $L \to \infty$.

In this paper we combine results and techniques of Mirzakhani in [Mir16] with the framework developed in the proof of Theorem 1.16 to describe the behavior near infinity of the length spectra of ordered filling closed multi-curves with respect to complete, finite area hyperbolic structures. More precisely, we prove the following.

Theorem 1.20. *Let $\gamma := (\gamma_1,\ldots,\gamma_k)$ with $k \geq 1$ be an ordered filling closed multi-curve on $S_{g,n}$ and $X \in \mathcal{M}_{g,n}$. Then,*

$$\lim_{L\to\infty} \frac{\mu^L_{\gamma,X}}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot (\widetilde{I}_\gamma)_* \, (\widetilde{\mu}^\gamma_{Thu})$$

in the weak-$\star$ topology for measures on $(\mathbf{R}_{\geq 0})^k$.

Recall the definition of the counting functions $c(X,\gamma,\mathbf{b},L)$ introduced above. The following theorem is an analogue of Wolpert's conjecture for filling closed multi-geodesics. It is a direct consequence of Theorem 1.20 and Portmanteau's theorem.

Theorem 1.21. *Let $X \in \mathcal{M}_{g,n}$, $\gamma := (\gamma_1,\ldots,\gamma_k)$ with $k \geq 1$ be an ordered filling closed multi-curve on $S_{g,n}$, and $\mathbf{b} := (b_1,\ldots,b_k) \in (\mathbf{R}_{>0})^k$. Then,*

$$\lim_{L\to\infty} \frac{c(X,\gamma,\mathbf{b},L)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot \widetilde{\mu}^\gamma_{Thu}\left(\{\lambda \in \mathcal{ML}_{g,n}(\gamma)/Stab(\gamma) \mid i(\lambda,\gamma_i) \leq b_i\}\right).$$

Remark 1.22. As highlighted by Mirzakhani in [Mir16], applying the methods in the proof of Theorem 1.21 to get an effective version of the same theorem with a power saving error term seems rather hard.

Organization of the paper. In §2 we introduce the preliminaries needed to understand the proofs of the main results. In §3 we prove Theorem 1.14 and discuss how refining the ideas in this proof leads to the stronger version alluded to in Remark 1.13. In §4 we prove Theorem 1.16 and develop the aforementioned general framework for computing limits of Weil-Petersson volumes of expanding subsets of quotients of Teichmüller space. In §5 we review the techniques introduced by the author in [Ara20a] and prove Theorem 1.18. In §6 we review the techniques introduced by Mirzakhani in [Mir16] and prove Theorem 1.20.

Acknowledgments. The author is very grateful to Alex Wright and Steve Kerckhoff for their invaluable advice, patience, and encouragement. The author would also like to thank the anonymous referee whose detailed comments and suggestions greatly helped improve this paper.

2. Preliminaries

Outline of this section. In this section we introduce the preliminaries needed to understand the proofs of the main results of this paper. Of particular importance for later will be the definitions of the horoball segment measures $\mu_\gamma^{f,L}$ and $\nu_{\gamma,\mathbf{a}}^{f,L}$ as well as Theorems 2.3 and 2.4.

The Thurston measure. Train track coordinates induce a $6g-6+2n$ dimensional piecewise integral linear (PIL) structure on the space $\mathcal{ML}_{g,n}$ of measured geodesic laminations on $S_{g,n}$; see [PH92, §3.1] for details. By work of Masur, there exists a unique (up to scaling) non-zero, locally finite, $\text{Mod}_{g,n}$-invariant, Lebesgue class measure on $\mathcal{ML}_{g,n}$ [Mas85, Theorem 2]. Several different definitions of such measure (equal up to scaling) can be found in the literature. We will consider the definition coming from the symplectic structure of $\mathcal{ML}_{g,n}$.

More precisely, consider the $\text{Mod}_{g,n}$-invariant symplectic form ω_{Thu} on the PIL manifold $\mathcal{ML}_{g,n}$ induced by train track coordinates; see [PH92, §3.2] for an explicit definition. This symplectic form is known as the *Thurston symplectic form*. The top exterior power $v_{\text{Thu}} := \frac{1}{(3g-3+n)!} \bigwedge_{i=1}^{3g-3+n} \omega_{\text{Thu}}$ induces a non-zero, locally finite, $\text{Mod}_{g,n}$-invariant, Lebesgue class measure μ_{Thu} on $\mathcal{ML}_{g,n}$. We refer to this measure as the *Thurston measure*.

This measure satisfies the following scaling property:

$$\mu_{\text{Thu}}(t \cdot A) = t^{6g-6+2n} \cdot \mu_{\text{Thu}}(A) \tag{2.1}$$

for every Borel measurable subset $A \subseteq \mathcal{ML}_{g,n}$ and every $t > 0$. In particular, the following lemma applies; see [EU18, Page 24] for a proof.

Lemma 2.1. *Let Ω be a topological space endowed with a continuous $(\mathbf{R}_{>0})$-action and μ be a measure on Ω such that the following property holds for some $k > 0$:*

$$\mu(t \cdot A) = t^k \cdot \mu(A)$$

for every Borel measurable subset $A \subseteq \Omega$ and every $t > 0$. Let $f\colon \Omega \to \mathbf{R}_{\geq 0}$ be a non-negative, homogeneous, continuous function. Then, for every $c > 0$,

$$\mu(f^{-1}(\{c\})) = 0.$$

Dehn-Thurston coordinates. Let $\mathcal{P} := (\gamma_1, \dots, \gamma_{3g-3+n})$ be a pair of pants decomposition of $S_{g,n}$. The following theorem, originally due to Dehn in the case of integral multi-curves and later extended by Thurston to the case of general measured geodesic laminations, gives an explicit parametrization of $\mathcal{ML}_{g,n}$ in terms of intersection numbers $m_i \in \mathbf{R}_{\geq 0}$ and twisting numbers $t_i \in \mathbf{R}$ with respect to the components of $\mathcal{P}$; see [PH92, §1.2] and [Mar16, §8.3.9] for details.

Theorem 2.2. *Let $\mathcal{P} := (\gamma_1, \dots, \gamma_{3g-3+n})$ be a pair of pants decomposition of $S_{g,n}$. Any set of intersection and twisting numbers $(m_i, t_i)_{i=1}^{3g-3+n}$ of $\mathcal{ML}_{g,n}$ with respect to the components of $\mathcal{P}$ gives a parametrization of $\mathcal{ML}_{g,n}$ by the set*

$$\Theta := \left\{ (m_i, t_i) \in (\mathbf{R}_{\geq 0} \times \mathbf{R})^{3g-3+n} \mid m_i = 0 \Rightarrow t_i \geq 0, \ \forall i = 1, \dots, 3g-3+n \right\}.$$

We refer to any parametrization as in Theorem 2.2 as a set of *Dehn-Thurston coordinates* of $\mathcal{ML}_{g,n}$ adapted to $\mathcal{P}$ and to the set Θ as the *parameter space* of such parametrization. The Thurston measure μ_{Thu} on $\mathcal{ML}_{g,n}$ corresponds (up to scaling) to the Lebesgue measure on Θ.

The Mirzakhani measure. Consider the bundle $P^1\mathcal{T}_{g,n}$ of unit length measured geodesic laminations over $\mathcal{T}_{g,n}$. More precisely,

$$P^1\mathcal{T}_{g,n} := \{(X,\lambda) \in \mathcal{T}_{g,n} \times \mathcal{ML}_{g,n} \mid \ell_\lambda(X) = 1\}.$$

For every $X \in \mathcal{T}_{g,n}$ consider the measure μ_{Thu}^X on the fiber $P_X^1\mathcal{T}_{g,n}$ of $P^1\mathcal{T}_{g,n}$ above X which to every Borel measurable subset $A \subseteq P_X^1\mathcal{T}_{g,n}$ assigns the value

$$\mu_{\mathrm{Thu}}^X(A) := \mu_{\mathrm{Thu}}([0,1]\cdot A). \tag{2.2}$$

The *Mirzakhani measure* ν_{Mir} on $P^1\mathcal{T}_{g,n}$ is defined by the disintegration formula

$$d\nu_{\mathrm{Mir}}(X,\lambda) := d\mu_{\mathrm{Thu}}^X(\lambda)\, d\mu_{\mathrm{wp}}(X).$$

The mapping class group $\mathrm{Mod}_{g,n}$ acts diagonally on $P^1\mathcal{T}_{g,n}$ in a properly discontinuous way preserving ν_{Mir}. The quotient $P^1\mathcal{M}_{g,n} := P^1\mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}$ is the bundle of unit length measured geodesic laminations over $\mathcal{M}_{g,n}$. Locally pushing forward ν_{Mir} through the quotient map $P^1\mathcal{T}_{g,n} \to P^1\mathcal{M}_{g,n}$ yields a measure $\widehat{\nu}_{\mathrm{Mir}}$ on $P^1\mathcal{M}_{g,n}$, also called the *Mirzakhani measure*. The pushforward of $\widehat{\nu}_{\mathrm{Mir}}$ under the bundle map $\pi\colon P^1\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$ is given by

$$d\pi_*(\widehat{\nu}_{\mathrm{Mir}})(X) = B(X)\, d\widehat{\mu}_{\mathrm{wp}}(X),$$

where $B\colon \mathcal{M}_{g,n} \to \mathbf{R}_{>0}$ is the Mirzakhani function defined in (1.5). The total mass of $P^1\mathcal{M}_{g,n}$ with respect to $\widehat{\nu}_{\mathrm{Mir}}$ is given by

$$\widehat{\nu}_{\mathrm{Mir}}(P^1\mathcal{M}_{g,n}) = \int_{\mathcal{M}_{g,n}} B(X)\, d\widehat{\mu}_{\mathrm{wp}}(X) = b_{g,n}.$$

In particular, by (1.6), the measure $\widehat{\nu}_{\mathrm{Mir}}$ is finite.

Horoball segment measures. Let $\gamma := (\gamma_1,\dots,\gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components and $f\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a non-negative, bounded, compactly supported, Borel measurable function that is not almost everywhere zero with respect to the Lebesgue measure class. For every $L > 0$ consider the *horoball segment* $B_\gamma^{f,L} \subseteq \mathcal{T}_{g,n}$ given by

$$B_\gamma^{f,L} := \{X \in \mathcal{T}_{g,n} \mid \vec{\ell}_\gamma(X) \in L\cdot \mathrm{supp}(f)\}.$$

Every such horoball segment supports a *horoball segment measure* $\mu_\gamma^{f,L}$ defined as

$$d\mu_\gamma^{f,L}(X) := f\left(\tfrac{1}{L}\cdot \vec{\ell}_\gamma(X)\right) d\mu_{\mathrm{wp}}(X). \tag{2.3}$$

This measure is $\mathrm{Stab}(\gamma)$-invariant. To get a locally finite horoball segment measure on $\mathcal{M}_{g,n}$ one needs to get rid of the redundancies that arise when taking pushforwards. For this purpose consider the intermediate cover

$$\mathcal{T}_{g,n} \to \mathcal{T}_{g,n}/\mathrm{Stab}(\gamma) \to \mathcal{M}_{g,n}.$$

Let $\widetilde{\mu}_\gamma^{f,L}$ be the local pushforward of $\mu_\gamma^{f,L}$ to $\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)$ and $\widehat{\mu}_\gamma^{f,L}$ be the pushforward of $\widetilde{\mu}_\gamma^{f,L}$ to $\mathcal{M}_{g,n}$.

Let $\mathbf{a} := (a_1,\dots,a_k) \in (\mathbf{Q}_{>0})^k$ be a vector of positive rational weights on the components of γ. The rationally weighted simple closed multi-curve $\mathbf{a}\cdot\gamma$ defined as in (1.2) belongs to $\mathcal{ML}_{g,n}$. The horoball segment measures $\mu_\gamma^{f,L}$ on $\mathcal{T}_{g,n}$ also give rise to *horoball segment measures* $\nu_{\gamma,\mathbf{a}}^{f,L}$ on the bundle $P^1\mathcal{T}_{g,n}$ by considering the disintegration formula

$$d\nu_{\gamma,\mathbf{a}}^{f,L}(X,\lambda) := d\delta_{\mathbf{a}\cdot\gamma/\ell_{\mathbf{a}\cdot\gamma}(X)}(\lambda)\, d\mu_\gamma^{f,L}(X),$$

where δ denotes point masses. This measure is $\text{Stab}(\gamma)$-invariant. In analogy with the case above, to get locally finite horoball segment measures on $P^1\mathcal{M}_{g,n}$ we consider the intermediate cover

$$P^1\mathcal{T}_{g,n} \to P^1\mathcal{T}_{g,n}/\text{Stab}(\gamma) \to P^1\mathcal{M}_{g,n}.$$

Let $\widetilde{\nu}_{\gamma,\mathbf{a}}^{f,L}$ be the local pushforward of $\nu_{\gamma,\mathbf{a}}^{f,L}$ to $P^1\mathcal{T}_{g,n}/\text{Stab}(\gamma)$ and $\widehat{\nu}_{\gamma,\mathbf{a}}^{f,L}$ be the pushforward of $\widetilde{\nu}_{\gamma,\mathbf{a}}^{f,L}$ to $P^1\mathcal{M}_{g,n}$.

One can check, see Proposition 2.9 below, that the measures $\widehat{\mu}_{\gamma}^{f,L}$ and $\widehat{\nu}_{\gamma,\mathbf{a}}^{f,L}$ are finite. We denote by $m_{\gamma}^{f,L}$ the total mass of the measures $\widehat{\mu}_{\gamma}^{f,L}$ and $\widehat{\nu}_{\gamma,\mathbf{a}}^{f,L}$, i.e.,

$$m_{\gamma}^{f,L} := \widehat{\mu}_{\gamma}^{f,L}(\mathcal{M}_{g,n}) = \widehat{\nu}_{\gamma,\mathbf{a}}^{f,L}(P^1\mathcal{M}_{g,n}) < +\infty.$$

The main tool used in the proof of Theorem 1.6 is the following result, which shows that horoball segment measures on $P^1\mathcal{M}_{g,n}$ equidistribute with respect to $\widehat{\nu}_{\text{Mir}}$. This result is an analogue of the classical equidistribution theorem for expanding horoballs on homogeneous spaces; see [KM96] for instance. This result is proved in [Ara20b], expanding on ideas introduced by Mirzakhani in [Mir07a].

Theorem 2.3. *In the weak-$\star$ topology for measures on $P^1\mathcal{M}_{g,n}$,*

$$\lim_{L\to\infty} \frac{\widehat{\nu}_{\gamma,\mathbf{a}}^{f,L}}{m_{\gamma}^{f,L}} = \frac{\widehat{\nu}_{Mir}}{b_{g,n}}.$$

From Theorem 2.3, taking pushforwards through the map $P^1\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$, we deduce the following result, which shows that horoball segment measures on $\mathcal{M}_{g,n}$ equidistribute with respect to $B(X)\cdot d\widehat{\mu}_{\text{wp}}(X)$. The proof of Theorem 1.6 will directly use this result

Theorem 2.4. *In the weak-$\star$ topology for measures on $\mathcal{M}_{g,n}$,*

$$\lim_{L\to\infty} \frac{\widehat{\mu}_{\gamma}^{f,L}}{m_{\gamma}^{f,L}} = \frac{B(X)\cdot d\widehat{\mu}_{wp}(X)}{b_{g,n}}.$$

Teichmüller and moduli spaces of hyperbolic surfaces with boundary. Let $g,n,b \geq 0$ be a triple of non-negative integers satisfying $2-2g-n-b<0$ and $S_{g,n}^b$ be a connected, oriented surface of genus g with n punctures and b labeled boundary components $\beta_1,\dots,\beta_b$. Let $\mathbf{L} := (L_i)_{i=1}^b \in (\mathbf{R}_{>0})^b$ be a vector of positive real numbers.

Denote by $\mathcal{T}_{g,n}^b(\mathbf{L})$ the *Teichmüller space* of marked, oriented, complete, finite area hyperbolic structures on $S_{g,n}^b$ with labeled geodesic boundary components whose lengths are given by $\mathbf{L}$, up to isotopy fixing each boundary component setwise. The *mapping class group* of $S_{g,n}^b$, denoted $\text{Mod}_{g,n}^b$, is the group of orientation preserving diffeomorphisms of $S_{g,n}^b$ that setwise fix each boundary component, up to isotopy fixing each boundary component setwise. The quotient $\mathcal{M}_{g,n}^b(\mathbf{L}) := \mathcal{T}_{g,n}^b(\mathbf{L})/\text{Mod}_{g,n}^b$ is the *moduli space* of oriented, complete, finite area hyperbolic structures on $S_{g,n}^b$ with labeled geodesic boundary components whose lengths are given by $\mathbf{L}$. We warn the reader that these definitions of $\mathcal{T}_{g,n}^b(\mathbf{L})$, $\text{Mod}_{g,n}^b$, and $\mathcal{M}_{g,n}^b(\mathbf{L})$ might differ from others found in the literature.

Denote the *total Weil-Petersson volume* of the moduli space $\mathcal{M}_{g,n}^b(\mathbf{L})$ by

$$V_{g,n}^b(\mathbf{L}) := \text{Vol}_{\text{wp}}(\mathcal{M}_{g,n}^b(\mathbf{L})).$$

The following remarkable theorem due to Mirzakhani shows that $V_{g,n}^b(\mathbf{L})$ is a polynomial on the $\mathbf{L}$ variables.

Theorem 2.5. [Mir07b, Theorem 6.1] [Mir07c, Theorem 1.1] *Let $g,n,b \geq 0$ be non-negative integers with $2-2g-n-b<0$. The total Weil-Petersson volume*

$$V_{g,n}^b(L_1,\dots,L_b)$$

is a polynomial of degree $3g-3+n+b$ on the variables $L_1^2,\dots,L_b^2$. Moreover, if

$$V_{g,n}^b(L_1,\dots,L_b) = \sum_{\substack{\alpha\in(\mathbf{Z}_{\geq 0})^b,\\ |\alpha|\leq 3g-3+n+b}} c_\alpha \cdot L_1^{2\alpha_1}\cdots L_b^{2\alpha_b},$$

where $|\alpha| := \alpha_1+\dots+\alpha_b$ for every $\alpha \in (\mathbf{Z}_{\geq 0})^b$, then $c_\alpha \in \mathbf{Q}_{>0}\cdot \pi^{6g-6+2n+2b-2|\alpha|}$. In particular, the leading coefficients of $V_{g,n}^b(L_1,\dots,L_b)$ belong to $\mathbf{Q}_{>0}$.

Remark 2.6. If the surface $S_{g,n}^b$ is a pair of pants, i.e., if $g=0$ and $n+b=3$, then, for any $\mathbf{L} := (L_i)_{i=1}^b \in (\mathbf{R}_{>0})^b$, the moduli space $\mathcal{M}_{g,n}^b(\mathbf{L})$ has exactly one point. In this case we adopt the convention

$$V_{g,n}^b(\mathbf{L}) := 1.$$

The polynomials $W_{g,n}(\gamma,\mathbf{x})$. Given a simple closed curve α on $S_{g,n}$, let

$$\mathrm{Stab}_0(\alpha)\subseteq \mathrm{Mod}_{g,n}$$

be the subgroup of all mapping classes of $S_{g,n}$ that fix α up to isotopy together with its orientations. Although α is unoriented, it admits two possible orientations. We require mapping classes in $\mathrm{Stab}_0(\alpha)$ to fix these orientations. More generally, given an ordered simple closed multi-curve $\gamma := (\gamma_1,\dots,\gamma_k)$ on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, let

$$\mathrm{Stab}_0(\gamma) := \bigcap_{i=1}^k \mathrm{Stab}_0(\gamma_i)\subseteq \mathrm{Mod}_{g,n}$$

be the subgroup of all mapping classes of $S_{g,n}$ that fix each component of γ up to isotopy together with their respective orientations.

For the rest of this discussion fix an ordered simple closed multi-curve $\gamma := (\gamma_1,\dots,\gamma_k)$ on $S_{g,n}$ with $1\leq k\leq 3g-3+n$ components. Let $S_{g,n}(\gamma)$ be the (potentially disconnected) oriented surface with boundary obtained by cutting $S_{g,n}$ along the components of γ. Let $c \in \mathbf{Z}_{>0}$ be the number of components of $S_{g,n}(\gamma)$ and $\{\Sigma_j\}_{j=1}^c$ be the components of $S_{g,n}(\gamma)$. For every $j\in\{1,\dots,c\}$ let $g_j,n_j,b_j \in \mathbf{Z}_{\geq 0}$ be the triple of non-negative integers satisfying $2-2g_j-n_j-b_j<0$ such that Σ_j is homeomorphic to $S_{g_j,n_j}^{b_j}$. Given a vector $\mathbf{x} := (x_i)_{i=1}^k \in (\mathbf{R}_{>0})^k$, for every $j \in \{1,\dots,c\}$, let $\mathbf{x}_j \in (\mathbf{R}_{>0})^{b_j}$ be the subvector of $\mathbf{x}$ whose entries correspond to the boundary components of Σ_j.

Let $\rho_{g,n}(\gamma)$ be the number of components of γ that bound (on any of its sides) a component of $S_{g,n}(\gamma)$ which is a torus with one boundary component. Let $\sigma_{g,n}(\gamma)\in \mathbf{Q}_{>0}$ be the rational number given by

$$\sigma_{g,n}(\gamma) := \frac{\prod_{j=1}^c |K_{g_j,n_j}^{b_j}|}{|\mathrm{Stab}_0(\gamma)\cap K_{g,n}|},$$

where $K_{g_j,n_j}^{b_j} \lhd \mathrm{Mod}_{g_j,n_j}^{b_j}$ is the kernel of the mapping class group action on $\mathcal{T}_{g_j,n_j}^{b_j}$ and $K_{g,n}\lhd \mathrm{Mod}_{g,n}$ is the kernel of the mapping class group action on $\mathcal{T}_{g,n}$. These kernels are non-trivial only in the low complexity cases where special involutions arise. For example, if $g = 2$, $n = 0$, and γ is a separating simple closed curve on $S_{2,0}$, then $\sigma_{2,0}(\gamma) = 4/2 = 2$.

For vectors $\mathbf{x} := (x_i)_{i=1}^k \in (\mathbf{R}_{>0})^k$ consider the polynomial

$$V_{g,n}(\gamma, \mathbf{x}) := \frac{1}{[\mathrm{Stab}(\gamma) : \mathrm{Stab}_0(\gamma)]} \cdot \sigma_{g,n}(\gamma) \cdot 2^{-\rho_{g,n}(\gamma)} \cdot \prod_{j=1}^{c} V_{g_j,n_j}^{b_j}(\mathbf{x}_j) \cdot x_1 \cdots x_k.$$

By Theorem 2.5, $V_{g,n}(\gamma, \mathbf{x})$ is a polynomial of degree $6g - 6 + 2n - k$ with non-negative coefficients and rational leading coefficients. This polynomial represents the total Weil-Petersson volume of the moduli spaces associated to the (potentially disconnected) oriented surface with boundary $S_{g,n}(\gamma)$. Denote by

$$W_{g,n}(\gamma, \mathbf{x}) := V_{g,n}^{\mathrm{top}}(\gamma, \mathbf{x}) \tag{2.4}$$

the polynomial obtained by adding up all the leading (maximal degree) monomials of $V_{g,n}(\gamma, \mathbf{x})$. This polynomial depends only on g, n, and the $\mathrm{Mod}_{g,n}$-orbit of γ.

Example 2.7. Table 1 contains the polynomials $W_{2,0}(\gamma, x_1, \ldots, x_k)$ for all $\mathrm{Mod}_{2,0}$-orbits of ordered simple closed multi-curves $\gamma := (\gamma_1, \ldots, \gamma_k)$ on $S_{2,0}$. These polynomial were computed using (2.4) and the tables in [Do13, §B].

Example 2.8. For every pair of pants decomposition $\mathcal{P} := (\gamma_1, \ldots, \gamma_{3g-3+n})$ of $S_{g,n}$ there exists $k_{\mathcal{P}} \in \mathbf{Z}_{\geq 0}$ such that

$$W_{g,n}(\mathcal{P}, x_1, \ldots, x_{3g-3+n}) = 2^{-k_{\mathcal{P}}} \cdot x_1 \cdots x_{3g-3+n}.$$

Total mass of horoball segment measures. Let $\gamma := (\gamma_1, \ldots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g - 3 + n$ components and $f \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a non-negative, bounded, compactly supported, Borel measurable function that is not almost everywhere zero with respect to the Lebesgue measure class. As mentioned above, the horoball segment measures $\widehat{\mu}_{\gamma}^{f,L}$ on $\mathcal{M}_{g,n}$ and $\widehat{\nu}_{\gamma,\mathbf{a}}^{f,L}$ on $P^1\mathcal{M}_{g,n}$ are finite. One can actually compute explicit formulas for their total mass $m_{\gamma}^{f,L}$ in terms of the polynomial $V_{g,n}(\gamma, \mathbf{x})$ and use them to describe the asymptotics of $m_{\gamma}^{f,L}$ as $L \to \infty$ in terms of the polynomial $W_{g,n}(\gamma, \mathbf{x})$. More concretely, in [Ara20b] we prove the following.

Proposition 2.9. [Ara20b, Proposition 3.1] *For every $L > 0$,*

$$m_{\gamma}^{f,L} = \int_{\mathbf{R}^k} f(\mathbf{x}) \cdot V_{g,n}(\gamma, L \cdot \mathbf{x}) \cdot L^k \cdot d\mathbf{x},$$

where $d\mathbf{x} := dx_1 \cdots dx_k$. In particular,

$$\lim_{L \to \infty} \frac{m_{\gamma}^{f,L}}{L^{6g-6+2n}} = \int_{\mathbf{R}^k} f(\mathbf{x}) \cdot W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}.$$

The symmetric Thurston metric. Consider the *asymmetric Thurston metric* d'_{Thu} on $\mathcal{T}_{g,n}$ which to every pair $X, Y \in \mathcal{T}_{g,n}$ assigns the distance

$$d'_{\mathrm{Thu}}(X, Y) := \sup_{\lambda \in \mathcal{ML}_{g,n}} \log \left(\frac{\ell_\lambda(Y)}{\ell_\lambda(X)} \right).$$

As this metric is asymmetric, it is convenient to consider the *symmetric Thurston metric* d_{Thu} on $\mathcal{T}_{g,n}$ which to every pair $X, Y \in \mathcal{T}_{g,n}$ assigns the distance

$$d_{\mathrm{Thu}}(X, Y) := \max\{d'_{\mathrm{Thu}}(X, Y), d'_{\mathrm{Thu}}(Y, X)\}.$$

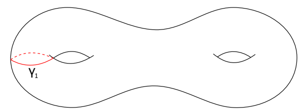
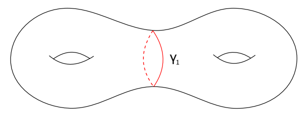
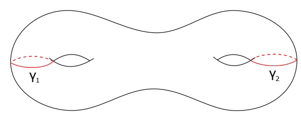
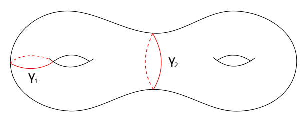
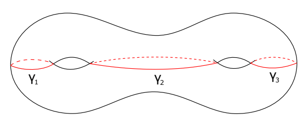
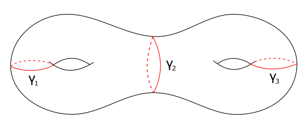
$\gamma := (\gamma_1, \ldots, \gamma_k)$	$W_{2,0}(\gamma, x_1, \ldots, x_k)$
	$\frac{1}{96} x_1^5$
	$\frac{1}{4608} x_1^5$
	$\frac{1}{4} x_1^3 x_2 + \frac{1}{4} x_1 x_2^3$
	$\frac{1}{96} x_1 x_2^3$
	$\frac{1}{2} x_1 x_2 x_3$
	$\frac{1}{4} x_1 x_2 x_3$

TABLE 1. Polynomials $W_{2,0}(\gamma, x_1, \ldots, x_k)$ for all $\mathrm{Mod}_{2,0}$-orbits of ordered simple closed multi-curves $\gamma := (\gamma_1, \ldots, \gamma_k)$ on $S_{2,0}$.

A pair $X, Y \in \mathcal{T}_{g,n}$ satisfies $d_{\mathrm{Thu}}(X,Y) \leq \epsilon$ for some $\epsilon > 0$ precisely when

$$e^{-\epsilon} \ell_\lambda(X) \leq \ell_\lambda(Y) \leq e^{\epsilon} \ell_\lambda(X), \ \forall \lambda \in \mathcal{ML}_{g,n}. \tag{2.5}$$

The metric d_{Thu} induces the usual topology on $\mathcal{T}_{g,n}$. We denote by $U_X(\epsilon) \subseteq \mathcal{T}_{g,n}$ the closed ball of radius $\epsilon > 0$ centered at $X \in \mathcal{T}_{g,n}$ with respect to d_{Thu}. For more details on the theory of the Thurston metrics see [Thu86] and [PS15].

Measure	Description
μ_{wp}	Weil-Petersson measure on $\mathcal{T}_{g,n}$
$\widetilde{\mu}_{\text{wp}}^{\gamma}$	Local pushforward of μ_{wp} to $\mathcal{T}_{g,n}/\text{Stab}(\gamma)$
$\widehat{\mu}_{\text{wp}}$	Local pushforward of μ_{wp} to $\mathcal{M}_{g,n}$
μ_{Thu}	Thurston measure on $\mathcal{ML}_{g,n}$
$\mu_{\text{Thu}}^{\gamma}$	Restriction of μ_{Thu} to $\mathcal{ML}_{g,n}(\gamma)$
$\widetilde{\mu}_{\text{Thu}}^{\gamma}$	Local pushforward of $\mu_{\text{Thu}}^{\gamma}$ to $\mathcal{ML}_{g,n}(\gamma)/\text{Stab}(\gamma)$
$\mu_{\gamma}^{f,L}$	Horoball segment measure on $\mathcal{T}_{g,n}$
$\widetilde{\mu}_{\gamma}^{f,L}$	Local pushforward of $\mu_{\gamma}^{f,L}$ to $\mathcal{T}_{g,n}/\text{Stab}(\gamma)$
$\widehat{\mu}_{\gamma}^{f,L}$	Pushforward of $\widetilde{\mu}_{\gamma}^{f,L}$ to $\mathcal{M}_{g,n}$
$\nu_{\gamma,\mathbf{a}}^{f,L}$	Horoball segment measure on $\mathcal{P}^1\mathcal{T}_{g,n}$
$\widetilde{\nu}_{\gamma,\mathbf{a}}^{f,L}$	Local pushforward of $\nu_{\gamma,\mathbf{a}}^{f,L}$ to $\mathcal{P}^1\mathcal{T}_{g,n}/\text{Stab}(\gamma)$
$\nu_{\gamma,\mathbf{a}}^{f,L}$	Pushforward of $\widetilde{\nu}_{\gamma,\mathbf{a}}^{f,L}$ to $\mathcal{P}^1\mathcal{M}_{g,n}$

TABLE 2. Measures introduced in §1 and §2.

Table of measures. As a guide to the reader, Table 2 contains brief descriptions of the measures introduced in §1 and §2 that will appear in the rest of this paper.

3. COUNTING SIMPLE CLOSED HYPERBOLIC MULTI-GEODESICS

Outline of this section. In this section we prove Theorem 1.14. As explained in §1, Theorem 1.6 is equivalent to Theorem 1.14. We also prove a stronger version of Theorem 1.6, introduced below as Theorem 3.5. This version allows one to consider countings that also keep track of the equivalence class of simple closed multi-geodesics in the space of projective measured geodesic laminations; see Theorems 3.7 and 3.8.

Setting. For the rest of this section, let $\gamma := (\gamma_1, \dots, \gamma_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$, $X \in \mathcal{T}_{g,n}$ be a marked, oriented, complete, finite area hyperbolic structure on $S_{g,n}$, and $f\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a non-negative, continuous, compactly supported function. We refer the reader back to §1, in particular to the definition of the counting functions $c(X,\gamma,f,L)$ in (1.8), for the notation that will be used in the following discussion.

Outline of the proof of Theorem 1.14. As explained in §1, to prove Theorem 1.14, we proceed in two steps. First, considering X as an element of $\mathcal{M}_{g,n}$, we average the counting functions $c(X,\gamma,f,L)$ over points $Y \in \mathcal{M}_{g,n}$ near X. Second, we unfold these averages over a suitable intermediate cover, reducing the proof of Theorem 1.14 to an application of Theorem 2.4.

Averaging counting functions. With the purpose of describing how the countings $c(Y,\gamma,f,L)$ vary as we move Y in a small neighborhood of $X \in \mathcal{M}_{g,n}$, we introduce functions $f_\epsilon^{\max}$ and $f_\epsilon^{\min}$ that closely bound f above and below. Given $\mathbf{x} := (x_i)_{i=1}^k \in (\mathbf{R}_{\geq 0})^k$ and $\epsilon > 0$, let $N_\epsilon(\mathbf{x}) \subseteq (\mathbf{R}_{\geq 0})^k$ be the subset

$$N_\epsilon(\mathbf{x}) := \{\mathbf{y} := (y_i)_{i=1}^k \in (\mathbf{R}_{\geq 0})^k \mid e^{-\epsilon} x_i \leq y_i \leq e^{\epsilon} x_i, \ \forall i = 1, \dots, k\}.$$

For every $\epsilon > 0$ consider the functions $f_\epsilon^{\max}, f_\epsilon^{\min} \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ which to every $\mathbf{x} \in (\mathbf{R}_{\geq 0})^k$ assign the value

$$f_\epsilon^{\max}(\mathbf{x}) := \max_{\mathbf{y} \in N_\epsilon(\mathbf{x})} f(\mathbf{y}), \quad f_\epsilon^{\min}(\mathbf{x}) := \min_{\mathbf{y} \in N_\epsilon(\mathbf{x})} f(\mathbf{y}). \tag{3.1}$$

As $f \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ is continuous and compactly supported,

$$\lim_{\epsilon \to 0} f_\epsilon^{\max}(\mathbf{x}) = f(\mathbf{x}), \quad \lim_{\epsilon \to 0} f_\epsilon^{\min}(\mathbf{x}) = f(\mathbf{x})$$

uniformly over all $\mathbf{x} \in (\mathbf{R}_{\geq 0})^k$. Recall that d_{Thu} denotes the symmetric Thurston metric on $\mathcal{T}_{g,n}$.

Proposition 3.1. *Let $Y \in \mathcal{T}_{g,n}$ and $\epsilon > 0$ be such that $d_{Thu}(X,Y) \leq \epsilon$. Then, for every $L > 0$,*

$$c(Y, \gamma, f_\epsilon^{\min}, L) \leq c(X, \gamma, f, L) \leq c(Y, \gamma, f_\epsilon^{\max}, L).$$

Proof. Fix $\epsilon > 0$. As highlighted in (2.5), if $Y \in \mathcal{T}_{g,n}$ satisfies $d_{\text{Thu}}(X,Y) \leq \epsilon$, then

$$e^{-\epsilon} \ell_\lambda(X) \leq \ell_\lambda(Y) \leq e^\epsilon \ell_\lambda(X), \ \forall \lambda \in \mathcal{ML}_{g,n}.$$

In particular, directly from (1.8), one deduces that, for every $L > 0$,

$$c(Y, \gamma, f_\epsilon^{\min}, L) \leq c(X, \gamma, f, L) \leq c(Y, \gamma, f_\epsilon^{\max}, L). \qquad \square$$

Recall that $U_X(\epsilon) \subseteq \mathcal{T}_{g,n}$ denotes the closed ball of radius $\epsilon > 0$ centered at $X \in \mathcal{T}_{g,n}$ with respect to d_{Thu}. Denote by $\pi \colon \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ be the quotient map. Recall that $\widehat{\mu}_{\text{wp}}$ denotes the Weil-Petersson measure on $\mathcal{M}_{g,n}$. For every $\epsilon > 0$ let $\eta_\epsilon \colon \mathcal{M}_{g,n} \to \mathbf{R}_{\geq 0}$ be a bump function of total $\widehat{\mu}$-mass 1 with support in $\pi(U_X(\epsilon))$. Directly from Proposition 3.1 we deduce the following corollary, which concludes the averaging step of the proof of Theorem 1.14.

Corollary 3.2. *For every $\epsilon > 0$ and every $L > 0$,*

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y) \cdot c(Y, \gamma, f_\epsilon^{\min}, L) \, d\widehat{\mu}_{wp}(Y) \leq c(X, \gamma, f, L), \tag{3.2}$$

$$c(X, \gamma, f, L) \leq \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y) \cdot c(Y, \gamma, f_\epsilon^{\max}, L) \, d\widehat{\mu}_{wp}(Y). \tag{3.3}$$

Unfolding counting averages. Consider the intermediate cover

$$\mathcal{T}_{g,n} \to \mathcal{T}_{g,n}/\text{Stab}(\gamma) \to \mathcal{M}_{g,n}.$$

Unfolding the integrals in (3.2) and (3.3) over $\mathcal{T}_{g,n}/\text{Stab}(\gamma)$ and pushing them back down to $\mathcal{M}_{g,n}$ in a suitable way will reduce the proof of Theorem 1.14 to an applicaton of Theorem 2.4. The following proposition describes this principle; see §2 for the definition of the horoball segment measures $\widehat{\mu}_\gamma^{h,L}$.

Proposition 3.3. *Let $h \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a non-negative, continuous, compactly supported function. Then, for every $\epsilon > 0$ and every $L > 0$,*

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y) \cdot c(Y, \gamma, h, L) \, d\widehat{\mu}_{wp}(Y) = \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y) \, d\widehat{\mu}_\gamma^{h,L}(Y). \tag{3.4}$$

Proof. Let $\epsilon > 0$ and $L > 0$ be arbitrary. For every $Y \in \mathcal{M}_{g,n}$ one can rewrite the counting function $c(Y, \gamma, h, L)$ as follows:

$$\begin{aligned} c(Y,\gamma,h,L) &= \sum_{\alpha \in \mathrm{Mod}_{g,n}\cdot\gamma} h\left(\tfrac{1}{L}\cdot \vec{\ell}_\alpha(Y)\right) \\ &= \sum_{[\phi]\in \mathrm{Mod}_{g,n}/\mathrm{Stab}(\gamma)} h\left(\tfrac{1}{L}\cdot \vec{\ell}_{\phi\cdot\gamma}(Y)\right) \\ &= \sum_{[\phi]\in \mathrm{Mod}_{g,n}/\mathrm{Stab}(\gamma)} h\left(\tfrac{1}{L}\cdot \vec{\ell}_{\gamma}(\phi^{-1}\cdot Y)\right) \\ &= \sum_{[\phi]\in \mathrm{Stab}(\gamma)\backslash \mathrm{Mod}_{g,n}} h\left(\tfrac{1}{L}\cdot \vec{\ell}_{\gamma}(\phi\cdot Y)\right). \end{aligned}$$

Let us record this as

$$c(X,\gamma,h,L) = \sum_{[\phi]\in \mathrm{Stab}(\gamma)\backslash \mathrm{Mod}_{g,n}} h\left(\tfrac{1}{L}\cdot \vec{\ell}_{\alpha}(\phi\cdot X)\right). \tag{3.5}$$

Let $p_\gamma \colon \mathcal{T}_{g,n}/\mathrm{Stab}(\gamma) \to \mathcal{M}_{g,n}$ be the quotient map and $\widetilde{\eta}_\epsilon^\gamma \colon \mathcal{T}_{g,n}/\mathrm{Stab}(\gamma) \to \mathbf{R}_{\geq 0}$ be the lift of η_ϵ given by $\widetilde{\eta}_\epsilon^\gamma := \eta_\epsilon \circ p_\gamma$. Recall that $\widetilde{\mu}_{\mathrm{wp}}^\gamma$ denotes the local pushforward of the Weil-Petersson measure μ_{wp} on $\mathcal{T}_{g,n}$ to the quotient $\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)$. Unfolding the integral on the left hand side of (3.4) using (3.5) it follows that

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\cdot c(Y,\gamma,h,L)\, d\widehat{\mu}_{\mathrm{wp}}(Y) = \int_{\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)} \widetilde{\eta}_\epsilon^\gamma(Y)\cdot h\left(\tfrac{1}{L}\cdot \vec{\ell}_\gamma(Y)\right) d\widetilde{\mu}_{\mathrm{wp}}^\gamma(Y).$$

By definition, see (2.3), the measure $\mu_\gamma^{h,L}$ on $\mathcal{T}_{g,n}$ is given by

$$d\mu_\gamma^{h,L}(Y) := h\left(\tfrac{1}{L}\cdot \vec{\ell}_\gamma(Y)\right) d\mu_{\mathrm{wp}}(Y).$$

Taking local pushforwards to $\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)$ we deduce

$$d\widetilde{\mu}_\gamma^{h,L}(Y) = h\left(\tfrac{1}{L}\cdot \vec{\ell}_\gamma(Y)\right) d\widetilde{\mu}_{\mathrm{wp}}^\gamma(Y).$$

It follows that

$$\int_{\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)} \widetilde{\eta}_\epsilon^\gamma(Y)\cdot h\left(\tfrac{1}{L}\cdot \vec{\ell}_\gamma(Y)\right) d\widetilde{\mu}_{\mathrm{wp}}^\gamma(Y) = \int_{\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)} \widetilde{\eta}_\epsilon^\gamma(Y)\, d\widetilde{\mu}_\gamma^{h,L}(Y).$$

As $\widehat{\mu}_\gamma^{h,L}$ is the pushforward of $\widetilde{\mu}_\gamma^{h,L}$ to $\mathcal{M}_{g,n}$,

$$\int_{\mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)} \widetilde{\eta}_\epsilon^\gamma(Y)\, d\widetilde{\mu}_\gamma^{h,L}(Y,\alpha) = \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, d\widehat{\mu}_\gamma^{h,L}(Y).$$

Putting everything together we conclude

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\cdot c(Y,\gamma,h,L)\, d\widehat{\mu}_{\mathrm{wp}}(Y) = \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, d\widehat{\mu}_\gamma^{h,L}(Y). \qquad \square$$

Application of Theorem 2.4. We are now ready to prove Theorem 1.14. Theorem 2.4 and Proposition 2.9 will play a fundamental role in the proof.

Proof of Theorem 1.14. By Proposition 2.9, given any non-negative, continuous, compactly supported function $h\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$,

$$r(\gamma,h) := \lim_{L\to\infty} \frac{m_\gamma^{h,L}}{L^{6g-6+2n}} = \int_{\mathbf{R}^k} h(\mathbf{x}) \cdot W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x}. \tag{3.6}$$

Proving Theorem 1.14 is then equivalent to showing that

$$r(\gamma,f) \cdot \frac{B(X)}{b_{g,n}} \leq \liminf_{L\to\infty} \frac{c(X,\gamma,f,L)}{L^{6g-6+2n}}, \tag{3.7}$$

$$\limsup_{L\to\infty} \frac{c(X,\gamma,f,L)}{L^{6g-6+2n}} \leq r(\gamma,f) \cdot \frac{B(X)}{b_{g,n}}. \tag{3.8}$$

We first verify (3.7). Let $\epsilon > 0$ and $L > 0$ be arbitrary. Consider $h := f_\epsilon^{\min}$. By Proposition 3.3 and (3.2),

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, d\widehat{\mu}_\gamma^{h,L}(Y) \leq c(X,\gamma,f,L).$$

Dividing this inequality by $m_\gamma^{h,L} > 0$ we get

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, \frac{d\widehat{\mu}_\gamma^{h,L}(Y)}{m_\gamma^{h,L}} \leq \frac{c(X,\gamma,f,L)}{m_\gamma^{h,L}}.$$

Taking $\liminf_{L\to\infty}$ on both sides of this inequalty and using Theorem 2.4 we deduce

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, \frac{B(Y)\cdot d\widehat{\mu}_{\mathrm{wp}}(Y)}{b_{g,n}} \leq \liminf_{L\to\infty} \frac{c(X,\gamma,f,L)}{m_\gamma^{h,L}}.$$

As

$$r(\gamma,f_\epsilon^{\min}) = r(\gamma,h) = \lim_{L\to\infty} \frac{m_\gamma^{h,L}}{L^{6g-6+2n}},$$

it follows that

$$r(\gamma,f_\epsilon^{\min}) \cdot \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, \frac{B(Y)\cdot d\widehat{\mu}_{\mathrm{wp}}(Y)}{b_{g,n}} \leq \liminf_{L\to\infty} \frac{c(X,\gamma,f,L)}{L^{6g-6+2n}}. \tag{3.9}$$

Recall that $f_\epsilon^{\min} \to f$ uniformly on $(\mathbf{R}_{\geq 0})^k$ as $\epsilon \to 0$. In particular,

$$\begin{aligned}\lim_{\epsilon\to 0} r(\gamma,f_\epsilon^{\min}) &= \lim_{\epsilon\to 0} \int_{\mathbf{R}^k} f_\epsilon^{\min}(\mathbf{x}) \cdot W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x}\\ &= \int_{\mathbf{R}^k} f(\mathbf{x}) \cdot W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x}\\ &= r(f,\gamma).\end{aligned}$$

Using the properties of the functions $\eta_\epsilon\colon \mathcal{M}_{g,n} \to \mathbf{R}_{\geq 0}$ one can check that

$$\lim_{\epsilon\to 0} \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)\, \frac{B(Y)\cdot d\widehat{\mu}_{\mathrm{wp}}(Y)}{b_{g,n}} = \frac{B(X)}{b_{g,n}}.$$

Taking $\epsilon \to 0$ in (3.9) we deduce

$$r(\gamma,f) \cdot \frac{B(X)}{b_{g,n}} \leq \liminf_{L\to\infty} \frac{c(X,\gamma,f,L)}{L^{6g-6+2n}}.$$

This finishes the proof of (3.7).

Analogous arguments using $f_\epsilon^{\max}$ instead of $f_\epsilon^{\min}$ and (3.3) instead of (3.2) yield a proof of (3.8). This finishes the proof of Theorem 1.14. □

Remark 3.4. Let $\|\cdot\|_{\mathcal{C}^1}$ denote the $\mathcal{C}^1$ norm for real valued, smooth, compactly supported functions on $\mathcal{M}_{g,n}$. Carefully following the steps of the proof of Theorem 1.14, one can check that the same methods would yield an effective version of the theorem with a power saving error term under the following polynomial equidistribution condition: There exist constants $C > 0$, $\kappa > 0$, and $\epsilon_0 > 0$ such that for every smooth, compactly supported function $\eta\colon \mathcal{M}_{g,n} \to \mathbf{R}_{\geq 0}$ and every $L > 0$,

$$\left| \int_{\mathcal{M}_{g,n}} \eta(Y) \, \frac{d\widehat{\mu}_\gamma^{h,L}(Y)}{m_\gamma^{h,L}} - \int_{\mathcal{M}_{g,n}} \eta(Y) \, \frac{B(Y) \cdot d\widehat{\mu}_{\mathrm{wp}}(Y)}{b_{g,n}} \right| \leq C \cdot \|\eta\|_{\mathcal{C}^1} \cdot L^{-\kappa},$$

where h ranges over all the functions $f_\epsilon^{\min}, f_\epsilon^{\max}$ with $0 < \epsilon < \epsilon_0$.

As explained in §1, this finishes the proof of Theorem 1.14.

Length and projective class spectra of ordered simple closed multi-curves. To state the stronger version of Theorem 1.6 alluded to at the beginning of this section, we first introduced some notation. Denote by $P\mathcal{ML}_{g,n} := \mathcal{ML}_{g,n}/\mathbf{R}_{>0}$ the space of projective measured geodesic laminations on $S_{g,n}$. The projective class of a measured geodesic laminations $\lambda \in \mathcal{ML}_{g,n}$ will be denoted by $\overline{\lambda} \in P\mathcal{ML}_{g,n}$.

Let $\gamma := (\gamma_1, \dots, \gamma_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$, $X \in \mathcal{T}_{g,n}$, and $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Q}_{>0})^k$ be a vector of positive rational weights. One can record, with multiplicity, the hyperbolic length vector with respect to X and the projective class in $P\mathcal{ML}_{g,n}$ with respect to the weights $\mathbf{a}$ of every ordered closed multi-curve in the mapping class group orbit of γ by considering the counting measure on $(\mathbf{R}_{\geq 0})^k \times P\mathcal{ML}_{g,n}$ given by

$$\nu_{\gamma,X,\mathbf{a}} := \sum_{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma} \delta_{\vec{\ell}_\alpha(X)} \otimes \delta_{\overline{\mathbf{a}\cdot\alpha}}.$$

This measure depends on the marking of $X \in \mathcal{T}_{g,n}$. We refer to this measure as the *length and projective class spectrum* of γ with respect to X and $\mathbf{a}$.

To study the asymptotic behavior of $\nu_{\gamma,X,\mathbf{a}}$ we consider the family of rescaled counting measures $\{\nu_{\gamma,X,\mathbf{a}}^L\}_{L>0}$ on $(\mathbf{R}_{\geq 0})^k \times P\mathcal{ML}_{g,n}$ given by

$$\nu_{\gamma,X,\mathbf{a}}^L := \sum_{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma} \delta_{\frac{1}{L} \cdot \vec{\ell}_\alpha(X)} \otimes \delta_{\overline{\mathbf{a}\cdot\alpha}}.$$

Asymptotics of length and projective class spectra of ordered simple closed multi-curves. Given $X \in \mathcal{T}_{g,n}$, denote by μ_{Thu}^X the measure on $P\mathcal{ML}_{g,n}$ which to every Borel measurable subset $V \subseteq P\mathcal{ML}_{g,n}$ assigns the value

$$\mu_{\mathrm{Thu}}^X(V) := \mu_{\mathrm{Thu}}(\{\lambda \in \mathcal{ML}_{g,n} \mid \ell_\lambda(X) \leq 1, \ \overline{\lambda} \in V\}).$$

Under the natural identification of $P\mathcal{ML}_{g,n}$ with the fiber $P_X^1\mathcal{T}_{g,n}$ of the bundle $P^1\mathcal{T}_{g,n}$ above X, this definition is equivalent to the one in (2.2). A refinement of the ideas in the proof of Theorem 1.6 yields the following stronger result, which describes the behavior near infinity of the length and projective class spectra of ordered simple closed multi-curves with respect to complete, finite area hyperbolic structures and positive rational weights.

Theorem 3.5. *Let $\gamma := (\gamma_1, \dots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, $X \in \mathcal{T}_{g,n}$, and $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Q}_{>0})^k$ be a vector of positive rational weights. Then,*

$$\lim_{L \to \infty} \frac{\nu^L_{\gamma,X,\mathbf{a}}}{L^{6g-6+2n}} = \frac{1}{b_{g,n}} \cdot W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x} \otimes \mu^X_{Thu}$$

in the weak-$\star$ topology for measures on $(\mathbf{R}_{\geq 0})^k \times P\mathcal{ML}_{g,n}$.

Remark 3.6. Theorem 1.6 can be deduced from Theorem 3.5 by taking pushforwards under the map $(\mathbf{R}_{\geq 0})^k \times P\mathcal{ML}_{g,n} \to (\mathbf{R}_{\geq 0})^k$ which projects to the first coordinate.

Let $X \in \mathcal{T}_{g,n}$, $\gamma := (\gamma_1, \dots, \gamma_k)$ with $1 \leq k \leq 3g-3+n$ be an ordered simple closed multi-curve on $S_{g,n}$, $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$, $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Q}_{>0})^k$, and $V \subseteq P\mathcal{ML}_{g,n}$ be a continuity subset of the Thurston measure class, i.e., V is a Borel measurable subset satisfying

$$\mu_{\mathrm{Thu}}(\{\lambda \in \mathcal{ML}_{g,n} \mid \overline{\lambda} \in \partial V\}) = 0.$$

For every $L > 0$ consider the counting function

$$\begin{aligned} &c(X, \gamma, \mathbf{b}, L, \mathbf{a}, V) \\ &:= \# \left\{ \alpha := (\alpha_i)_{i=1}^k \in \mathrm{Mod}_{g,n} \cdot \gamma \;\middle|\; \begin{array}{l} \ell_{\alpha_i}(X) \leq b_i L, \ \forall i = 1, \dots, k, \\ \overline{\mathbf{a} \cdot \alpha} \in V. \end{array} \right\}. \end{aligned}$$

Compared to $c(X, \gamma, \mathbf{b}, L)$, the counting function $c(X, \gamma, \mathbf{b}, L, \mathbf{a}, V)$ imposes the additional restriction that the weighted simple closed multi-curves $\mathbf{a} \cdot \alpha \in \mathcal{ML}_{g,n}$ must belong to the cone in $\mathcal{ML}_{g,n}$ corresponding to $V \subseteq \mathcal{PML}_{g,n}$. This function depends on marking of $X \in \mathcal{T}_{g,n}$. The following strengthening of Theorem 1.9 is a direct consequence of Theorem 3.5 and Portmanteau's theorem.

Theorem 3.7. *Let $X \in \mathcal{T}_{g,n}$, $\gamma := (\gamma_1, \dots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$, $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Q}_{>0})^k$, and $V \subseteq P\mathcal{ML}_{g,n}$ be a continuity subset of the Thurston measure class. Then,*

$$\lim_{L \to \infty} \frac{c(X, \gamma, \mathbf{b}, L, \mathbf{a}, V)}{L^{6g-6+2n}} = \frac{\mu^X_{Thu}(V)}{b_{g,n}} \cdot \int_{\prod_{i=1}^k [0, b_i]} W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}.$$

Proof of Theorem 3.5. We now explain how to adapt the arguments in the proofs of Theorems 1.6 and 1.14 to obtain a proof of Theorem 3.5.

Let $X \in \mathcal{T}_{g,n}$, $\gamma := (\gamma_1, \dots, \gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, and $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Q}_{>0})^k$. Let $f \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ and $g \colon P\mathcal{ML}_{g,n} \to \mathbf{R}_{\geq 0}$ be non-negative, continuous, compactly supported functions. For every $L > 0$ consider the counting function

$$\begin{aligned} c(X, \gamma, f, L, \mathbf{a}, g) &:= \int_{\mathbf{R}^k \times P\mathcal{ML}_{g,n}} f(\mathbf{x}) \cdot g\left(\overline{\lambda}\right) \, d\nu^L_{\gamma,X,\mathbf{a}}\left(\mathbf{x}, \overline{\lambda}\right) \\ &= \sum_{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma} f\left(\tfrac{1}{L} \cdot \vec{\ell}_\alpha(X)\right) \cdot g\left(\overline{\mathbf{a} \cdot \alpha}\right). \end{aligned} \tag{3.10}$$

This function depends on the marking of $X \in \mathcal{T}_{g,n}$. Notice that, for any $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$ and any $V \subseteq \mathcal{PML}_{g,n}$, if

$$f(\mathbf{x}) := \prod_{i=1}^k \mathbb{1}_{[0,b_i]}(x_i), \quad g(\overline{\lambda}) := \mathbb{1}_V(\overline{\lambda}),$$

then $c(X,\gamma,f,L,\mathbf{a},g) = c(X,\gamma,\mathbf{b},L,\mathbf{a},V)$. By the definition of weak convergence of measures and the Stone-Weierstrass theorem, Theorem 3.5 is equivalent to the following analogue of Theorem 1.14.

Theorem 3.8. *Let $X \in \mathcal{T}_{g,n}$, $\gamma := (\gamma_1,\dots,\gamma_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, and $\mathbf{a} := (a_1,\dots,a_k) \in (\mathbf{Q}_{>0})^k$. Let $f\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ and $g\colon P\mathcal{ML}_{g,n} \to \mathbf{R}_{\geq 0}$ be non-negative, continuous, compactly supported functions. Then,*

$$\lim_{L\to\infty} \frac{c(X,\gamma,f,L,\mathbf{a},g)}{L^{6g-6+2n}} = \frac{1}{b_{g,n}} \cdot \int_{\mathbf{R}^k} f(\mathbf{x}) \cdot W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x} \cdot \int_{P\mathcal{ML}_{g,n}} g\left(\overline{\lambda}\right)\, d\mu_{Thu}^X\left(\overline{\lambda}\right).$$

We now explain how to adapt the arguments in the proof Theorem 1.14 to prove Theorem 3.8 . For the rest of this discussion we fix $X \in \mathcal{T}_{g,n}$, $\gamma := (\gamma_1,\dots,\gamma_k)$ an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, $\mathbf{a} := (a_1,\dots,a_k) \in (\mathbf{Q}_{>0})^k$, and a pair of non-negative, continuous, compactly supported functions $f\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ and $g\colon P\mathcal{ML}_{g,n} \to \mathbf{R}_{\geq 0}$. Identify

$$\begin{aligned} P^1\mathcal{T}_{g,n} &= \mathcal{T}_{g,n} \times P\mathcal{ML}_{g,n}, \\ P^1\mathcal{M}_{g,n} &= (\mathcal{T}_{g,n} \times P\mathcal{ML}_{g,n})/\mathrm{Mod}_{g,n}. \end{aligned}$$

It will be important to make a clear distinction between points $Y \in \mathcal{T}_{g,n}$ and their images $[Y] := \pi(Y) \in \mathcal{M}_{g,n}$ under the quotient map $\pi\colon \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$, as well as between points $(Y,\overline{\lambda}) \in P^1\mathcal{T}_{g,n}$ and their images $[Y,\overline{\lambda}] \in P^1\mathcal{M}_{g,n}$ under the quotient map $\Pi\colon P^1\mathcal{T}_{g,n} \to P^1\mathcal{M}_{g,n}$.

To deal with the fact that the counting functions defined in (3.10) depend on the marking of $Y \in \mathcal{T}_{g,n}$, we introduce a local averaging procedure that yields well defined counting functions on $\mathcal{M}_{g,n}$. This procedure also deals with the orbifold issues that arise from working on $\mathcal{M}_{g,n}$. Using the proper discontinuity of the action of $\mathrm{Mod}_{g,n}$ on $\mathcal{T}_{g,n}$, one can find a neighborhood $W_X \subseteq \mathcal{T}_{g,n}$ of X such that

(1) W_X is $\mathrm{Stab}(X)$-invariant,
(2) $\phi \cdot W_X \cap W_X = \emptyset$ for every $\phi \in \mathrm{Mod}_{g,n} \setminus \mathrm{Stab}(X)$.

For every every $[Y] \in \pi(W_X)$, every non-negative, continuous, compactly supported function $h\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$, and every $L > 0$, consider the counting function

$$c'([Y],\gamma,h,L,\mathbf{a},g) := \frac{1}{|\mathrm{Stab}(X)|} \cdot \sum_{\phi\in\mathrm{Stab(X)}} c(\phi\cdot Y,\gamma,h,L,\mathbf{a},g).$$

Notice that

$$c'([X],\gamma,h,L,\mathbf{a},g) = c(X,\gamma,h,L,\mathbf{a},g).$$

For the rest of this discussion let $\epsilon_0 := \epsilon_0(X) > 0$ be small enough so that $U_X(\epsilon) \subseteq W_X$ for every $0 < \epsilon < \epsilon_0$. Recall the definition of the functions $f_\epsilon^{\min}, f_\epsilon^{\max}$ in (3.1). Equation (2.5) ensures the following analogue of Proposition 3.1 holds.

Proposition 3.9. *Let $Y \in \mathcal{T}_{g,n}$ and $0 < \epsilon < \epsilon_0$ be such that $d_{Thu}(X,Y) \leq \epsilon$. Then, for every $L > 0$,*

$$c'\left([Y],\gamma,f_\epsilon^{\min},L,\mathbf{a},g\right) \leq c(X,\gamma,f,L,\mathbf{a},g) \leq c'\left([Y],\gamma,f_\epsilon^{\max},L,\mathbf{a},g\right).$$

As in the proof of Theorem 1.14, for every $\epsilon > 0$ consider a bump function $\eta_\epsilon\colon \mathcal{M}_{g,n} \to \mathbf{R}_{\geq 0}$ of total $\widehat{\mu}$-mass 1 with support in $\pi(U_X(\epsilon))$. Directly from Proposition 3.9 we deduce the following analogue of Corollary 3.2.

Corollary 3.10. *For every $0 < \epsilon < \epsilon_0$ and every $L > 0$,*

$$\int_{\mathcal{M}_{g,n}} \eta_\epsilon([Y]) \cdot c'\left([Y], \gamma, f_\epsilon^{\min}, L, \mathbf{a}, g\right) \, d\widehat{\mu}_{wp}([Y]) \leq c(X, \gamma, f, L, \mathbf{a}, g), \tag{3.11}$$

$$c(X, \gamma, f, L, \mathbf{a}, g) \leq \int_{\mathcal{M}_{g,n}} \eta_\epsilon([Y]) \cdot c([Y], \gamma, f_\epsilon^{\max}, L, \mathbf{a}, g) \, d\widehat{\mu}_{wp}([Y]). \tag{3.12}$$

Let $p \colon P^1\mathcal{T}_{g,n} = \mathcal{T}_{g,n} \times P\mathcal{ML}_{g,n} \to P\mathcal{ML}_{g,n}$ be the map that projects to the second coordinate. Consider the function $g' \colon P^1\mathcal{M}_{g,n} \to \mathbf{R}_{\geq 0}$ which to every $[Y, \overline{\lambda}] \in P^1\mathcal{M}_{g,n}$ assigns the value

$$g'\left([Y, \overline{\lambda}]\right) := \mathbb{1}_{\pi(W_X)}([Y]) \cdot \frac{1}{|\mathrm{Stab}(X)|} \cdot \sum_{\phi \in \mathrm{Stab}(X)} g\left(\phi \cdot p\left(\Pi|^{-1}_{W_X \times P\mathcal{ML}_{g,n}}\left([Y, \overline{\lambda}]\right)\right)\right),$$

where $\Pi|^{-1}_{W_X \times P\mathcal{ML}_{g,n}}([Y, \overline{\lambda}]) \in W_X \times P\mathcal{ML}_{g,n}$ denotes any of the finitely many preimages of $[Y, \overline{\lambda}]$ under the restriction $\Pi|_{W_X \times P\mathcal{ML}_{g,n}}$. This function averages the value of g over the second coodinate of every $\mathrm{Stab}(X)$-orbit in W_X. The following analogue of Proposition 3.3 can be proved using a similar unfolding argument; see §2 for the definition of the horoball segment measures $\widehat{\nu}_{\gamma,\mathbf{a}}^{h,L}$.

Proposition 3.11. *Let $h \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a non-negative, continuous, compactly supported function. Then, for every $0 < \epsilon < \epsilon_0$ and every $L > 0$,*

$$\begin{aligned} &\int_{\mathcal{M}_{g,n}} \eta_\epsilon([Y]) \cdot c'([Y], \gamma, h, L, \mathbf{a}, g) \, d\widehat{\mu}_{wp}([Y]) \\ &= \int_{P^1\mathcal{M}_{g,n}} \eta_\epsilon([Y]) \cdot g'\left([Y, \overline{\lambda}]\right) \, d\widehat{\nu}_{\gamma,\mathbf{a}}^{h,L}\left([Y, \overline{\lambda}]\right). \end{aligned}$$

Theorem 3.8 can now be proved by mimicking the proof of Theorem 1.14 above: the inequalities (3.11) and (3.12) are used in place of the inequalities (3.2) and (3.3), Proposition 3.11 is used in place of Proposition 3.3, and Theorem 2.3 is used in place of Theorem 2.4.

Remark 3.12. A polynomial equidistribution condition analogous to the one introduced in Remark 3.4 but for horoball segment measures on $P^1\mathcal{M}_{g,n}$ would yield an effective version of Theorem 3.8 with a power saving error term.

4. Asymptotics of Weil-Petersson volumes

Outline of this section. The leading terms in the asymptotic formulas of Theorems 1.6 and 3.5 include a factor $W_{g,n}(\gamma, \mathbf{x}) \cdot d\mathbf{x}$ which can be described in terms of limits of Weil-Petersson volumes of expanding subsets of quotients of $\mathcal{T}_{g,n}$. The purpose of this section is to prove Theorem 1.16, which gives a purely topological description of this factor. In the course of the proof we develop a framework for computing general limits of this kind in terms of purely topological information.

Setting. For the rest of this section, let $\gamma := (\gamma_1, \ldots, \gamma_k)$ be a fixed ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g - 3 + n$ components.

Outline of the proof of Theorem 1.16. The main tool used in the proof of Theorem 1.16 is the correspondence of the Weil-Petersson measure on $\mathcal{T}_{g,n}$ and the Thurston measure on $\mathcal{ML}_{g,n}$ through Thurston's shear coordinates [Thu86, PP93, SB01]. We begin with an elementary reduction of the proof of Theorem 1.16 to a more concrete statement, which we introduce below as Theorem 4.1. We then focus on proving this statement for the rest of this section. After introducing Thurston's shear coordinates and the correspondence of measures alluded to above, we turn to an analysis of how an appropriate renormalization of the Weil-Petersson measure converges to the Thurston measure as one lets the curvature of the hyperbolic metrics diverge to $-\infty$; see Proposition 4.4 for a precise statement. We then prove Theorem 4.1 by passing to a suitable quotient and using an intuitive but rather technical no escape of mass argument; see Proposition 4.7. The last part of this section is devoted to the technical aspects of the proof of Proposition 4.7.

A first reduction of the proof of Theorem 1.16. We refer the reader back to the statement of Theorem 1.16 for the notation that will be used in the following discussion. By Carathéodory's extension theorem, to prove Theorem 1.16, it is enough to show that the measures $W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x}$ and $(\widetilde{I}_\gamma)_*(\widetilde{\mu}^{\gamma}_{\text{Thu}})$ on $(\mathbf{R}_{\geq 0})^k$ coincide on the generating semi-ring of boxes

$$B_{\mathbf{a},\mathbf{b}} := \prod_{i=1}^{k} [a_i, b_i)$$

with $\mathbf{a} := (a_i)_{i=1}^k, \mathbf{b} := (b_i)_{i=1}^k \in (\mathbf{R}_{\geq 0})^k$ arbitrary. By the inclusion-exclusion principle and Lemma 2.1, to prove this, it is enough to show that the measures $W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x}$ and $(\widetilde{I}_\gamma)_*(\widetilde{\mu}^{\gamma}_{\text{Thu}})$ coincide on the set of closed boxes

$$B_{\mathbf{b}} := \prod_{i=1}^{k} [0, b_i]$$

with $\mathbf{b} := (b_i)_{i=1}^k \in (\mathbf{R}_{>0})^k$ arbitrary. By Proposition 2.9,

$$\int_{B_{\mathbf{b}}} W_{g,n}(\gamma,\mathbf{x}) \cdot d\mathbf{x} = \lim_{L\to\infty} \frac{m_\gamma^{f_{\mathbf{b}},L}}{L^{6g-6+2n}},$$

where $f_{\mathbf{b}} \colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ is the function which to every $\mathbf{x} := (x_i)_{i=1}^k \in (\mathbf{R}_{\geq 0})^k$ assigns the value

$$f_{\mathbf{b}}(\mathbf{x}) := \prod_{i=1}^{k} \mathbb{1}_{[0,b_i]}(x_i).$$

By definition,

$$m_\gamma^{f_{\mathbf{b}},L} := \widehat{\mu}_\gamma^{f_{\mathbf{b}},L}(\mathcal{M}_{g,n}).$$

As $\widehat{\mu}_\gamma^{f_{\mathbf{b}},L}$ is the pushforward to $\mathcal{M}_{g,n}$ of the measure $\widetilde{\mu}_\gamma^{f_{\mathbf{b}},L}$ on $\mathcal{T}_{g,n}/\text{Stab}(\gamma)$,

$$\widehat{\mu}_\gamma^{f_{\mathbf{b}},L}(\mathcal{M}_{g,n}) = \widetilde{\mu}_\gamma^{f_{\mathbf{b}},L}(\mathcal{T}_{g,n}/\text{Stab}(\gamma)).$$

Denote by $\widetilde{\mu}^{\gamma}_{\text{wp}}$ the local pushforward of the Weil-Petersson measure μ_{wp} on $\mathcal{T}_{g,n}$ to $\mathcal{T}_{g,n}/\text{Stab}(\gamma)$. Notice that

$$d\widetilde{\mu}_\gamma^{f_{\mathbf{b}},L}(X) = f_{\mathbf{b}}\left(\tfrac{1}{L} \cdot \vec{\ell}_\gamma(X)\right) d\widetilde{\mu}^{\gamma}_{\text{wp}}(X).$$

In particular,

$$m_\gamma^{f_{\mathbf{b}},L} = \widetilde{\mu}^{\gamma}_{\text{wp}}\left(\{X \in \mathcal{T}_{g,n}/\text{Stab}(\gamma) \mid \ell_{\gamma_i}(X) \leq b_i L, \ \forall i = 1,\dots,k\}\right).$$

It follows that, to prove Theorem 1.16, it is enough to prove the following result.

Theorem 4.1. *For any* $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$,

$$\lim_{L\to\infty} \frac{\widetilde{\mu}^{\gamma}_{wp}\left(\{X \in \mathcal{T}_{g,n}/Stab(\gamma) \mid \ell_{\gamma_i}(X) \leq b_i L,\ \forall i = 1, \ldots, k\}\right)}{L^{6g-6+2n}}$$
$$= \widetilde{\mu}^{\gamma}_{Thu}\left(\{\lambda \in \mathcal{ML}_{g,n}(\gamma)/Stab(\gamma) \mid i(\lambda, \gamma_i) \leq b_i,\ \forall i = 1, \ldots, k\}\right).$$

The rest of this section is devoted to the proof of Theorem 4.1. Some of the arguments in our proof are closely related to ideas in the proofs of [Mir04, Theorem 5.17] and [RS19, Theorem 3.3].

The Yamabe space. Denote by $\mathcal{Y}_{g,n}$ the *Yamabe space* of all complete, finite area, constant negative curvature metrics on $S_{g,n}$ up to isotopy. One can identify

$$\mathcal{Y}_{g,n} = (\mathbf{R}_{>0}) \times \mathcal{T}_{g,n},$$

where $(t, X) \in (\mathbf{R}_{>0}) \times \mathcal{T}_{g,n}$ corresponds to the scaling $t \cdot X \in \mathcal{Y}_{g,n}$ of the hyperbolic metric $X \in \mathcal{T}_{g,n}$ which scales lengths by $t > 0$. Denote by $\overline{\mathcal{Y}_{g,n}}$ the *enlarged Yamabe space* obtained by adjoining a copy of $\mathcal{ML}_{g,n}$ to $\overline{\mathcal{Y}_{g,n}}$,

$$\overline{\mathcal{Y}_{g,n}} := \mathcal{Y}_{g,n} \sqcup \mathcal{ML}_{g,n}.$$

Consider the pairing $i\colon \overline{\mathcal{Y}_{g,n}} \times \mathcal{ML}_{g,n} \to \mathbf{R}_{\geq 0}$ which to every $(\alpha, \mu) \in \overline{\mathcal{Y}_{g,n}} \times \mathcal{ML}_{g,n}$ assigns the value

$$i(\alpha, \mu) := \begin{cases} t \cdot \ell_\mu(X) & \text{if} \quad \alpha := (t, X) \in \mathcal{Y}_{g,n}, \\ i(\lambda, \mu) & \text{if} \quad \alpha := \lambda \in \mathcal{ML}_{g,n}. \end{cases}$$

This pairing is homogenous with respect to the natural $\mathbf{R}_{>0}$ actions on each coordinate. On $\overline{\mathcal{Y}_{g,n}}$ consider the weakest topology making this pairing continuous. With this topology $\mathcal{T}_{g,n} = \{1\} \times \mathcal{T}_{g,n} \subseteq \mathcal{Y}_{g,n}$ and $\mathcal{ML}_{g,n} \subseteq \overline{\mathcal{Y}_{g,n}}$ are embedded. By work of Thurston, see for instance [FLP12, Theorem 8.7], $\overline{\mathcal{Y}_{g,n}}$ is projectively compact, that is, $P\overline{\mathcal{Y}_{g,n}} := \overline{\mathcal{Y}_{g,n}}/\mathbf{R}_{>0}$ is compact. The natural action of $\mathrm{Mod}_{g,n}$ on $\overline{\mathcal{Y}_{g,n}}$ is continuous.

Thurston's shear coordinates. Let μ be a maximal geodesic lamination on $S_{g,n}$. In the following discussion it is not required that μ supports an invariant transverse measure. For instance, μ could be an ideal geodesic triangulation (if $n > 0$) or a maximal completion of a geodesic pair of pants decomposition. Denote by $\mathrm{Stab}(\mu) \subseteq \mathrm{Mod}_{g,n}$ the subgroup of mapping classes of $S_{g,n}$ that stabilize μ. In [Thu86], Thurston introduced a $\mathrm{Stab}(\mu)$-equivariant global parametrization of $\mathcal{T}_{g,n}$,

$$F_\mu\colon \mathcal{T}_{g,n} \to \mathcal{ML}_{g,n},$$

called the *shear coordinates* of $\mathcal{T}_{g,n}$ with respect to μ. The map F_μ is a homeomorphism onto its image; below we describe its image in certain cases of interest. Roughly speaking, this map sends $X \in \mathcal{T}_{g,n}$ to the *transverse horocyclic foliation* $F_\mu(X)$ of μ on X. The $F_\mu(X)$-measure of a subarc of μ is given by its hyperbolic length on X. In particular, given any $X \in \mathcal{T}_{g,n}$ and any $\lambda \in \mathcal{ML}_{g,n}$,

$$i(F_\mu(X), \lambda) \leq \ell_\lambda(X). \tag{4.1}$$

Moreover, if one of the leaves of μ is a simple closed curve γ, then

$$i(F_\mu(X), \gamma) = \ell_\gamma(X). \tag{4.2}$$

As explained in [PP93, §4], if $n > 0$ and μ is an ideal geodesic triangulation, F_μ surjects onto $\mathcal{ML}_{g,n}$. By work of Mirzakahani [Mir08a, Theorem 1.3], if $n = 0$ and μ is a maximal measured geodesic lamination, $\mathrm{Im}(F_\mu) = \mathcal{ML}_{g,n}(\mu)$, where $\mathcal{ML}_{g,n}(\mu)$ is as in (1.9). As explained in [Mir08a, §7], if μ is a maximal completion of a geodesic pair of pants decomposition $\mathcal{P}$, $\mathrm{Im}(F_\mu) = \mathcal{ML}_{g,n}(\mathcal{P})$. Given an arbitrary maximal geodesic lamination μ on $S_{g,n}$, denote $\mathcal{ML}_{g,n}(\mu) := \mathrm{Im}(F_\mu) \subseteq \mathcal{ML}_{g,n}$; this notation is consistent with (1.9).

By work of Papadopoulos and Penner [PP93, Corollary 4.2] and of Bonahon and Sözen [SB01, Theorem 1], if $n > 0$ and μ is an ideal geodesic triangulation, or if $n = 0$ and μ is a maximal measured geodesic lamination, the shear coordinates

$$F_\mu \colon \mathcal{T}_{g,n} \to \mathcal{ML}_{g,n}(\mu)$$

pull back the the restriction of Thurston symplectic form on $\mathcal{ML}_{g,n}(\mu)$ to the Weil-Petersson symplectic form on $\mathcal{T}_{g,n}$. As a direct consequence of these results we deduce the following.

Theorem 4.2. *Suppose that $n > 0$ and μ is an ideal geodesic triangulation of $S_{g,n}$, or that $n = 0$ and μ is a maximal measured geodesic lamination on $S_{g,n}$. Then, the shear coordinates*

$$F_\mu \colon \mathcal{T}_{g,n} \to \mathcal{ML}_{g,n}(\mu)$$

pull back the restriction of the Thurston measure μ_{Thu} on $\mathcal{ML}_{g,n}(\mu)$ to the Weil-Petersson measure μ_{wp} on $\mathcal{T}_{g,n}$.

By work of Papadopoulos [Pap88, Pap91], the behavior of Thurston's shear coordinates along sequence in $\mathcal{T}_{g,n}$ approaching the Thurston boundary $P\mathcal{ML}_{g,n}$ is well understood. More precisely, the following holds.

Proposition 4.3. [Pap88, Proposition 3.1] [Pap91, Lemma 4.9] *Suppose that $n > 0$ and μ is an ideal geodesic triangulation of $S_{g,n}$, or that $n = 0$ and μ is a maximal measured geodesic lamination on $S_{g,n}$. Let $(X_k)_{k\in\mathbf{N}}$ be a sequence of points in $\mathcal{T}_{g,n}$ converging to a projective measured geodesic lamination on the Thurston boundary $P\mathcal{ML}_{g,n}$. Then, for every simple closed curve α on $S_{g,n}$ there exists a constant $C_\alpha > 0$ such that for every $k \in \mathbf{N}$,*

$$i(F_\mu(X_k), \alpha) \leq \ell_\alpha(X_k) \leq i(F_\mu(X_k), \alpha) + C_\alpha.$$

Shear coordinates of the enlarged Yamabe space. Suppose that $n > 0$ and μ is an ideal geodesic triangulation of $S_{g,n}$, or that $n = 0$ and μ is a maximal measured geodesic lamination on $S_{g,n}$. Denote by $F_\mu \colon \mathcal{T}_{g,n} \to \mathcal{ML}_{g,n}(\mu)$ the corresponding shear coordinates. Consider the map $\Phi_\mu \colon \mathcal{Y}_{g,n} \to (0,\infty) \times \mathcal{ML}_{g,n}(\mu)$ given by

$$\Phi_\mu(t, X) := (t, t \cdot F_\mu(X))$$

for every $t > 0$ and every $X \in \mathcal{T}_{g,n}$. Using Proposition 4.3, one can check that this map extends to a homeomorphism

$$\overline{\Phi_\mu} \colon \overline{\mathcal{Y}_{g,n}} \to ((0,\infty) \times \mathcal{ML}_{g,n}(\mu)) \sqcup (\{0\} \times \mathcal{ML}_{g,n}),$$

where the topology on the target is the one coming from its natural embedding into $[0,\infty) \times \mathcal{ML}_{g,n}$. This map satisfies the following property:

$$\overline{\Phi}_\mu(\lambda) = (0, \lambda), \ \forall \lambda \in \mathcal{ML}_{g,n}. \tag{4.3}$$

We refer to this map as the *shear coordinates* of $\overline{\mathcal{Y}_{g,n}}$ with respect to μ.

Asymptotics of the Weil-Petersson measure. Given $t > 0$, denote by μ^t_{wp} the pushforward to $\{t\} \times \mathcal{T}_{g,n} \subseteq \overline{\mathcal{Y}_{g,n}}$ of the Weil-Petersson measure μ_{wp} on $\mathcal{T}_{g,n}$ with respect to the map

$$\mathcal{T}_{g,n} \to \{t\} \times \mathcal{T}_{g,n}, \quad X \mapsto (t, X).$$

We will also denote by μ^t_{wp} the extension by zero of this measure to $\overline{\mathcal{Y}_{g,n}}$ and by μ_{Thu} the extension by zero of the Thurston measure on $\mathcal{ML}_{g,n} \subseteq \overline{\mathcal{Y}_{g,n}}$ to $\overline{\mathcal{Y}_{g,n}}$. The following proposition describes the asymptotic behavior of the measures μ^t_{wp} on $\overline{\mathcal{Y}_{g,n}}$ as $t \to 0$. Roughly speaking, this proposition shows that an appropriate renormalization of the Weil-Petersson measure converges to the Thurston measure as one lets the curvature of the hyperbolic metrics diverge to $-\infty$.

Proposition 4.4. *In the weak-$\star$ topology for measures on $\overline{\mathcal{Y}_{g,n}}$,*

$$\lim_{t \to 0} t^{6g-6+2n} \cdot \mu^t_{wp} = \mu_{Thu}.$$

Proof. If $n > 0$ let μ is an ideal geodesic triangulation of $S_{g,n}$, and if $n = 0$ let μ is a maximal measured geodesic lamination on $S_{g,n}$. Denote by

$$\overline{\Phi_\mu} \colon \overline{\mathcal{Y}_{g,n}} \to ((0,\infty) \times \mathcal{ML}_{g,n}(\mu)) \sqcup (\{0\} \times \mathcal{ML}_{g,n})$$

be the shear coordinates of $\overline{\mathcal{Y}_{g,n}}$ with respect to μ. For every $t \geq 0$ consider the measure μ^t_{Thu} on

$$((0,\infty) \times \mathcal{ML}_{g,n}(\mu)) \sqcup (\{0\} \times \mathcal{ML}_{g,n})$$

given by

$$\mu^t_{\mathrm{Thu}} := \delta_t \otimes \mu_{\mathrm{Thu}}|_{\mathcal{ML}_{g,n}(\mu)}.$$

Notice that

$$\lim_{t \to 0} \mu^t_{\mathrm{Thu}} = \mu^0_{\mathrm{Thu}}$$

in the weak-$\star$ topology. Using Theorem 4.2 and the scaling property (2.1) of the Thurston measure, one can check that, for every $t > 0$,

$$\left(\overline{\Phi_\mu}\right)_* \mu^t_{\mathrm{wp}} = t^{-(6g-6+2n)} \cdot \mu^t_{\mathrm{Thu}}.$$

As the subset $\mathcal{ML}_{g,n}(\mu) \subseteq \mathcal{ML}_{g,n}$ has full measure,

$$\mu^0_{\mathrm{Thu}} = \delta_0 \otimes \mu_{\mathrm{Thu}}.$$

This together with (4.3) imply

$$\left(\overline{\Phi_\mu}\right)_* \mu_{\mathrm{Thu}} = \mu^0_{\mathrm{Thu}}.$$

Putting everything together we conclude

$$\lim_{t \to 0} t^{6g-6+2n} \cdot \mu^t_{\mathrm{wp}} = \mu_{\mathrm{Thu}}. \qquad \square$$

Properly discontinuous stabilizer actions. Consider the subset

$$\overline{\mathcal{Y}_{g,n}}(\gamma) := \mathcal{Y}_{g,n} \cup \mathcal{ML}_{g,n}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}.$$

If $n = 0$, the following result is a direct consequence of [EM18, Proposition 4.1]; the same arguments can be adapted to obtain a proof in the case $n > 0$.

Proposition 4.5. *The group Stab(γ) acts properly discontinuously on $\overline{\mathcal{Y}_{g,n}}(\gamma)$.*

Proposition 4.5 implies in particular that $\mathrm{Stab}(\gamma)$ acts properly discontinuously on $\mathcal{ML}_{g,n}(\gamma)$. It follows that, as was mentioned in §1, $\widetilde{\mu}^\gamma_{\mathrm{Thu}}$, the local pushforward of the measure $\mu^\gamma_{\mathrm{Thu}} := \mu_{\mathrm{Thu}}|_{\mathcal{ML}_{g,n}(\gamma)}$ on $\mathcal{ML}_{g,n}(\gamma)$ to the quotient $\mathcal{ML}_{g,n}(\gamma)/\mathrm{Stab}(\gamma)$, is well defined.

Towards a proof of Theorem 4.1. By Proposition 4.5, the subgroup $\text{Stab}(\gamma) \subseteq \text{Mod}_{g,n}$ acts properly discontinuously on

$$\overline{\mathcal{Y}_{g,n}}(\gamma) := \mathcal{Y}_{g,n} \sqcup \mathcal{ML}_{g,n}(\gamma).$$

Denote by $\widetilde{\mu}_{\text{wp}}^{\gamma,t}$ and $\widetilde{\mu}_{\text{Thu}}^{\gamma}$ the local pushforwards of the measures μ_{wp}^t and $\mu_{\text{Thu}}^{\gamma} := \mu_{\text{Thu}}|_{\mathcal{ML}_{g,n}(\gamma)}$ on $\overline{\mathcal{Y}_{g,n}}(\gamma)$ to the quotient $\overline{\mathcal{Y}_{g,n}}(\gamma)/\text{Stab}(\gamma)$. Directly from Proposition 4.4 we deduce the following corollary.

Corollary 4.6. *In the weak-$\star$ topology for measures on $\overline{\mathcal{Y}_{g,n}}(\gamma)/Stab(\gamma)$,*

$$\lim_{t\to 0} t^{6g-6+2n} \cdot \widetilde{\mu}_{wp}^{\gamma,t} = \widetilde{\mu}_{Thu}^{\gamma}.$$

Consider the subsets

$$\begin{aligned} \mathcal{Y}_{g,n}^1 &:= (0,1] \cdot \mathcal{T}_{g,n} \subseteq \mathcal{Y}_{g,n}, \\ \overline{\mathcal{Y}_{g,n}^1} &:= \mathcal{Y}_{g,n}^1 \cup \mathcal{ML}_{g,n} \subseteq \overline{\mathcal{Y}_{g,n}}, \\ \overline{\mathcal{Y}_{g,n}^1}(\gamma) &:= \mathcal{Y}_{g,n}^1 \cup \mathcal{ML}_{g,n}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma). \end{aligned}$$

Notice that $\text{Stab}(\gamma)$ preserves $\overline{\mathcal{Y}_{g,n}^1}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)$. Consider the embedded quotient

$$\overline{\mathcal{Y}_{g,n}^1}(\gamma)/\text{Stab}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/\text{Stab}(\gamma).$$

Given $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$, let $\widetilde{B}_{\mathbf{b}}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/\text{Stab}(\gamma)$ denote the subset

$$\widetilde{B}_{\mathbf{b}}(\gamma) := \{\alpha \in \overline{\mathcal{Y}_{g,n}^1}(\gamma)/\text{Stab}(\gamma) \mid i(\alpha, \gamma_i) < b_i, \ \forall i = 1, \dots, k\}.$$

One would like to use Corollary 4.6 together with Portmanteau's theorem to deduce

$$\lim_{t\to 0} t^{6g-6+2n} \cdot \widetilde{\mu}_{\text{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma)\right) = \widetilde{\mu}_{\text{Thu}}^{\gamma} \left(\widetilde{B}_{\mathbf{b}}(\gamma)\right). \tag{4.4}$$

Notice that, for every $0 < t \leq 1$,

$$\widetilde{\mu}_{\text{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma)\right) = \widetilde{\mu}_{\text{wp}}^{\gamma} \left(\{X \in \mathcal{T}_{g,n}/\text{Stab}(\gamma) \mid \ell_{\alpha_i}(X) < b_i/t, \ \forall i = 1, \dots, k\}\right),$$

and that

$$\widetilde{\mu}_{\text{Thu}}^{\gamma} \left(\widetilde{B}_{\mathbf{b}}(\gamma)\right) = \widetilde{\mu}_{\text{Thu}}^{\gamma} \left(\{\lambda \in \mathcal{ML}_{g,n}/\text{Stab}(\gamma) \mid i(\lambda, \gamma_i) < b_i, \ \forall i = 1, \dots, k\}\right).$$

Letting $t = 1/L$ with $0 < L \leq 1$ and taking $L \searrow 0$ would prove Theorem 4.1. But the hypothesis of Portmanteau's theorem are not verified by the subset $\widetilde{B}_{\mathbf{b}}(\gamma) \subseteq \overline{\mathcal{Y}}_{g,n}(\gamma)/\text{Stab}(\gamma)$ as it does not have compact closure. Such non-compactness comes from the fact that $\mathcal{ML}_{g,n}(\gamma) \subseteq \mathcal{ML}_{g,n}$ is open. To overcome this difficulty we will prove the following no escape of mass result.

Proposition 4.7. *Let $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$. For every $\epsilon > 0$ there exists a compact subset $\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \subseteq \widetilde{B}_{\mathbf{b}}(\gamma)$ with the following properties:*

(1) $\widetilde{\mu}_{Thu}^{\gamma} \left(\partial \widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma)\right) = 0$,
(2) $\widetilde{\mu}_{Thu}^{\gamma} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma)\right) < \epsilon$,
(3) $t^{6g-6+2n} \cdot \widetilde{\mu}_{wp}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma)\right) < \epsilon$ *for all small enough $t > 0$.*

Let us prove Theorem 4.1 assuming Proposition 4.7 holds.

Proof of Theorem 4.1. Following the discussion above, it remains to verify (4.4). Fix $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$ and let $\epsilon > 0$ be arbitrary. Consider the compact subset $\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \subseteq \widetilde{B}_{\mathbf{b}}(\gamma)$ given by Proposition 4.7. As $\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \subseteq \overline{\mathcal{Y}}_{g,n}(\gamma)/\mathrm{Stab}(\gamma)$ is compact and satisfies $\widetilde{\mu}_{\mathrm{Thu}}^{\gamma}(\partial \widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma)) = 0$, Corollary 4.6 together with Portmanteau's theorem imply

$$\lim_{t \to 0} t^{6g-6+2n} \cdot \widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \right) = \widetilde{\mu}_{\mathrm{Thu}}^{\gamma} \left(\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \right).$$

Let $t_0 > 0$ be small enough so that

$$\left| t^{6g-6+2n} \cdot \widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \right) - \widetilde{\mu}_{\mathrm{Thu}}^{\gamma} \left(\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \right) \right| < \epsilon$$

and

$$t^{6g-6+2n} \cdot \widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \right) < \epsilon$$

for every $0 < t < t_0$. As $\widetilde{\mu}_{\mathrm{Thu}}^{\gamma}(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma)) < \epsilon$, the triangle inequality implies

$$\left| t^{6g-6+2n} \cdot \widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \right) - \widetilde{\mu}_{\mathrm{Thu}}^{\gamma} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \right) \right| < 3\epsilon$$

for every $0 < t < t_0$. As $\epsilon > 0$ is arbitrary, this proves (4.4) and thus concludes the proof of Theorem 4.1. □

The rest of this section is devoted to the proof of Proposition 4.7. To define the compact subsets $\widetilde{K}_{\mathbf{b}}^{\epsilon}(\gamma) \subseteq \widetilde{B}_{\mathbf{b}}(\gamma)$ we approximate the open condition $\lambda \in \mathcal{ML}_{g,n}(\gamma)$ by a sequence of closed conditions.

Filling together with a simple closed multi-curve. Consider the subset

$$\mathcal{Z}_{g,n}(\gamma) := \{\lambda \in \mathcal{ML}_{g,n} \mid i(\lambda, \gamma_i) = 0, \ \forall i = 1, \dots, k\}.$$

This subset is homogeneous and closed. In particular, it is projectively compact. Let $\mathcal{ZS}_{g,n}(\gamma) \subseteq \mathcal{ML}_{g,n}$ be the subset of all simple closed curves on $S_{g,n}$ that belong to $\mathcal{Z}_{g,n}(\gamma)$. This subset is discrete and closed. Notice that every component of γ belongs to $\mathcal{ZS}_{g,n}(\gamma)$. Consider the map $s_\gamma \colon \overline{\mathcal{Y}_{g,n}} \to \mathbf{R}_{\geq 0}$ which to every $\alpha \in \overline{\mathcal{Y}_{g,n}}$ assigns the value

$$s_\gamma(\alpha) := \inf_{\beta \in \mathcal{ZS}_{g,n}(\gamma)} i(\alpha, \beta).$$

We refer to $s_\gamma(\alpha)$ as *the systole of α relative to γ*. As complete, finite area hyperbolic surfaces always have a simple closed geodesic of shortest length, $s_\gamma(\alpha) > 0$ for every $\alpha \in \mathcal{Y}_{g,n}$ and the infimum defining this quantity is attained. The following proposition characterizes the subset $\mathcal{ML}_{g,n}(\gamma) \subseteq \mathcal{ML}_{g,n}$ in terms of this function.

Proposition 4.8. *Given $\lambda \in \mathcal{ML}_{g,n}$,*

$$\lambda \in \mathcal{ML}_{g,n}(\gamma) \ \Leftrightarrow \ s_\gamma(\lambda) > 0.$$

Moreover, if $\lambda \in \mathcal{ML}_{g,n}(\gamma)$, the infimum defining $s_\gamma(\lambda)$ is attained.

Proof. Let us first assume that $\lambda \notin \mathcal{ML}_{g,n}(\gamma)$. By definition, one can find $\eta \in \mathcal{ML}_{g,n}$ such that $i(\gamma, \eta) = i(\lambda, \eta) = 0$. If one of the components of γ is a minimal component of η then $s_\gamma(\lambda) = 0$. Assume then that η has a minimal component η' which is not one of the components of γ. Given $\epsilon > 0$, as η' is minimal and not one of the components of γ, one can follow any half-leaf of η' for long enough so that it comes back near to its starting point in such a way that it can be closed up by adding an arc disjoint from the components of γ and whose tranverse measure with respect to λ is $\leq \epsilon$. This produces a simple closed curve $\beta \in \mathcal{S}_{g,n}^\gamma$ such that $i(\lambda, \beta) \leq \epsilon$. As $\epsilon > 0$ is arbitrary, this shows that $s_\gamma(\lambda) = 0$.

We now assume that $\lambda \in \mathcal{ML}_{g,n}(\gamma)$. Consider the restriction

$$i(\lambda, \cdot)|_{\mathcal{Z}_{g,n}(\gamma)} \colon \mathcal{Z}_{g,n}(\gamma) \to \mathbf{R}_{>0}.$$

This function takes only positive values because of the definitions of $\mathcal{ML}_{g,n}(\gamma)$ and $\mathcal{Z}_{g,n}(\gamma)$. From this fact and the projective compactness of $\mathcal{Z}_{g,n}(\gamma)$ it follows that this function is proper. As $\mathcal{ZS}_{g,n}(\gamma) \subseteq \mathcal{Z}_{g,n}(\gamma)$ is a discrete closed subset, we deduce that $s_\gamma(\lambda) > 0$ and moreover that the infimum defining this quantity is attained. This finishes the proof. □

One can check that the systole relative to γ is continuous as a function on $\overline{\mathcal{Y}_{g,n}}$. We record this and other properties in the following proposition.

Proposition 4.9. *The systole relative to* γ*,*

$$s_\gamma \colon \overline{\mathcal{Y}}_{g,n} \to \mathbf{R}_{\geq 0},$$

is homogeneous, Stab(γ)*-equivariant, and continuous.*

Proof. The homogenity and Stab(γ)-equivariance of s_γ can be checked directly from the definition. We now show that s_γ is continuous. Consider first $\alpha \in \overline{\mathcal{Y}}_{g,n}$ such that $s_\gamma(\alpha) = 0$. Let $\epsilon > 0$ be arbitrary. As $s_\gamma(\alpha) = 0$, one can find $\beta \in \mathcal{ZS}_{g,n}(\gamma)$ such that $i(\alpha, \beta) < \epsilon$. Consider the open neighborhood $U \subseteq \overline{\mathcal{Y}_{g,n}}$ of α given by

$$U := \{\sigma \in \overline{\mathcal{Y}_{g,n}} \mid i(\sigma, \beta) < \epsilon\}.$$

Notice that $s_\gamma(\sigma) < \epsilon$ for every $\sigma \in U$. As $\epsilon > 0$ is arbitrary, this shows that s_γ is continuous at every $\alpha \in \overline{\mathcal{Y}}_{g,n}$ such that $s_\gamma(\alpha) = 0$.

Now consider $\alpha \in \overline{\mathcal{Y}}_{g,n}$ such that $s_\gamma(\alpha) > 0$. Let $1 < \epsilon < 2$ be arbitrary. Let $U' \subseteq \overline{\mathcal{Y}_{g,n}}$ be a compact neighborhood of α. As $\mathcal{Z}_{g,n}(\gamma)$ is projectively compact, one can find a constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{i(\beta, \lambda)}{i(\alpha, \lambda)} \leq C$$

for every $\lambda \in \mathcal{Z}_{g,n}(\gamma)$ and every $\beta \in U'$. In particular, if $\lambda \in \mathcal{Z}_{g,n}(\gamma)$ is such that $i(\alpha, \lambda) > 2Cs_\gamma(\alpha)$, then $i(\beta, \lambda) > 2s_\gamma(\alpha)$ for every $\beta \in U'$. Consider the subset

$$K := \{\lambda \in \mathcal{Z}_{g,n}(\gamma) \mid i(\alpha, \lambda) \leq 2Cs_\gamma(\alpha)\}.$$

As the restriction

$$i(\alpha, \cdot)|_{\mathcal{Z}_{g,n}(\gamma)} \colon \mathcal{Z}_{g,n}(\gamma) \to \mathbf{R}_{>0}$$

is proper (see the proof of Proposition 4.8), the set K is compact. As $\mathcal{ZS}_{g,n}(\gamma) \subseteq \mathcal{Z}_{g,n}(\gamma)$ is a discrete closed subset, $\mathcal{ZS}_{g,n}(\gamma) \cap K$ is finite. Consider the neighborhood $U \subseteq \overline{\mathcal{Y}_{g,n}}$ of α given by

$$U := \left\{\sigma \in U' \mid \tfrac{1}{\epsilon} \cdot i(\alpha, \beta) < i(\sigma, \beta) < \epsilon \cdot i(\alpha, \beta), \ \forall \beta \in \mathcal{S}_{g,n}^\gamma \cap K\right\}.$$

Notice that

$$\tfrac{1}{\epsilon} \cdot s_\gamma(\alpha) \leq s_\gamma(\sigma) \leq \epsilon \cdot s_\gamma(\alpha)$$

for every $\sigma \in U$. As $1 < \epsilon < 2$ is arbitrary, this shows that s_γ is continuous at every $\alpha \in \overline{\mathcal{Y}}_{g,n}$ such that $s_\gamma(\alpha) > 0$. This finishes the proof. □

It follows from Propositions 4.8 and 4.9 that the restriction

$$s_\gamma|_{\overline{\mathcal{Y}_{g,n}}(\gamma)} \colon \overline{\mathcal{Y}_{g,n}}(\gamma) \to \mathbf{R}_{>0}$$

induces a homogeneous, positive, continuous map on the quotient $\overline{\mathcal{Y}_{g,n}}(\gamma)/\mathrm{Stab}(\gamma)$.

No escape of mass. We are now ready to introduce a family of compact subsets satisfying the properties described in Proposition 4.7. For every $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$ and every $\delta > 0$ consider the subset $\widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma) \subseteq \widetilde{B}_{\mathbf{b}}(\gamma)$ given by

$$\widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma) := \left\{ \begin{array}{c|l} \alpha \in \overline{\mathcal{Y}^1_{g,n}}(\gamma)/\mathrm{Stab}(\gamma) & i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1, \dots, k, \\ & s_\gamma(\alpha) \geq \delta. \end{array} \right\}.$$

One should interpret $\widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma)$ as the intersection of $\widetilde{B}_{\mathbf{b}}(\gamma)$ with the *δ-thick part* of $\overline{\mathcal{Y}^1_{g,n}}(\gamma)/\mathrm{Stab}(\gamma)$. Proposition 4.7 is a direct consequence of the following result.

Proposition 4.10. *Let $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$. The subsets $\widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma) \subsetneq \widetilde{B}_{\mathbf{b}}(\gamma)$ are compact and satisfy the following conditions:*

1. $\widetilde{\mu}^\gamma_{Thu}\left(\partial \widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma)\right) = 0$,
2. $\lim_{\delta \to 0} \ \widetilde{\mu}^\gamma_{Thu}\left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma)\right) = 0$,
3. *There exists a constant $C > 0$ such that for every $0 < \delta < 1$,*

$$\limsup_{t \to 0} \ t^{6g-6+2n} \cdot \widetilde{\mu}^{\gamma,t}_{wp}\left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma)\right) \leq C \cdot \delta.$$

For the rest of this section we fix $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$ and show that the subsets $\widetilde{\mathcal{K}}_{\mathbf{b}}^\delta(\gamma) \subseteq \widetilde{B}_{\mathbf{b}}(\gamma)$ satisfy the conditions described in Proposition 4.10.

Bers's Theorem. The following version of Bers's theorem can be proved using arguments similar to those in the proof of [FM12, Theorem 12.8].

Theorem 4.11. *Let $1 \leq k \leq 3g-3+n$ and $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$. There exists a constant $C \geq \max_{i=1,\dots,k} b_i$ such that for any $X \in \mathcal{T}_{g,n}$ and any ordered simple closed multi-curve $\gamma := (\gamma_1, \dots, \gamma_k)$ on $S_{g,n}$ satisfying*

$$\ell_{\gamma_i}(X) \leq b_i, \ \forall i = 1, \dots, k,$$

there exists a completion $\mathcal{P} := (\gamma_1, \dots, \gamma_{3g-3+n})$ of γ to a pair of pants decomposition of $S_{g,n}$ such that

$$\ell_{\gamma_i}(X) \leq C, \ \forall i = 1, \dots, 3g-3+n.$$

Bers's theorem for $\overline{\mathcal{Y}^1_{g,n}}(\gamma)$. Complete γ to a pair of pants decomposition $\mathcal{P}$ of $S_{g,n}$ and further complete $\mathcal{P}$ to a maximal geodesic lamination μ on $S_{g,n}$. Consider the shear coordinates $F_\mu \colon \mathcal{T}_{g,n} \to \mathcal{ML}_{g,n}(\mu)$ of $\mathcal{T}_{g,n}$ with respect to μ. Properties (4.1) and (4.2) allow one to deduce the following analogue of Bers's theorem directly from Theorem 4.11.

Corollary 4.12. *Let $1 \leq k \leq 3g-3+n$ and $\mathbf{b} := (b_1, \dots, b_k) \in (\mathbf{R}_{>0})^k$. There exists a constant $C \geq \max_{i=1,\dots,k} b_i$ such that for any $\alpha \in \overline{\mathcal{Y}^1_{g,n}}(\gamma)$ and any ordered simple closed multi-curve $\gamma := (\gamma_1, \dots, \gamma_k)$ on $S_{g,n}$ satisfying*

$$i(\alpha, \gamma_i) \leq b_i, \ \forall i = 1, \dots, k,$$

there exists a completion $\mathcal{P} := (\gamma_1, \dots, \gamma_{3g-3+n})$ of γ to a pair of pants decomposition of $S_{g,n}$ such that

$$i(\alpha, \gamma_i) \leq C, \ \forall i = 1, \dots, 3g-3+n.$$

Compactness. We now prove that the subsets

$$\widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/\mathrm{Stab}(\gamma)$$

are compact. This result is an analogue of Mumford's compactness criterion; see for instance [FM12, Theorem 12.6]. The proof, although rather technical, hinges on the following basic ideas:

(1) For every $\alpha \in \widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma)$ there exists a pair of pants decomposition of $S_{g,n}$ containing the components of γ that is *short* with respect to α.
(2) There are finitely many *topological types* of pair of pants decompositions of $S_{g,n}$ containing the components of γ.
(3) Thus, when expressed in shear coordinates, the closed set $\widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma)$ is contained in the union of finitely many quotients of compact domains in Dehn-Thurston coordinates times the interval $[0,1]$.

Proposition 4.13. *For every $\delta > 0$ the set $\widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma)$ is compact.*

Proof. Fix $\delta > 0$. Notice that the subset $\mathcal{K}_{\mathbf{b}}^{\delta}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)$ given by

$$\mathcal{K}_{\mathbf{b}}^{\delta}(\gamma) := \left\{ \alpha \in \overline{\mathcal{Y}_{g,n}^1}(\gamma) \;\middle|\; \begin{array}{l} i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1,\dots,k, \\ s_\gamma(\alpha) \geq \delta. \end{array} \right\}$$

is mapped onto the subset $\widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/\mathrm{Stab}(\gamma)$ by the quotient map

$$\overline{\mathcal{Y}_{g,n}}(\gamma) \to \overline{\mathcal{Y}_{g,n}}(\gamma)/\mathrm{Stab}(\gamma).$$

To prove $\widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/\mathrm{Stab}(\gamma)$ is compact, it is enough to show that $\mathcal{K}_{\mathbf{b}}^{\delta}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)$ can written as a finite union of $\mathrm{Stab}(\gamma)$-orbits of compact subsets of $\overline{\mathcal{Y}_{g,n}}(\gamma)$.

Let $C > 0$ be as in Corollary 4.12. Notice that, up to the action of $\mathrm{Stab}(\gamma)$, there are finitely many pair of pants decompositions $\mathcal{P}$ of $S_{g,n}$ containing the components of γ. It follows from Corollary 4.12 that $\mathcal{K}_{\mathbf{b}}^{\delta}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)$ can be written as the union of finitely many $\mathrm{Stab}(\gamma)$-orbits of subsets $\mathcal{C}_{\mathbf{b}}^{\delta}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)$ of the form

$$\mathcal{C}_{\mathbf{b}}^{\delta}(\mathcal{P}) := \left\{ \alpha \in \overline{\mathcal{Y}_{g,n}^1}(\gamma) \;\middle|\; \begin{array}{l} i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1,\dots,k, \\ i(\alpha,\gamma_i) \leq C, \ \forall i = k+1,\dots,3g-3+n, \\ s_\gamma(\alpha) \geq \delta. \end{array} \right\},$$

where $\mathcal{P} := (\gamma_1,\dots,\gamma_{3g-3+n})$ is a pair of pants decomposition of $S_{g,n}$ containing the components of γ. We now show that each one of the $\mathrm{Stab}(\mathcal{P})$-invariant subsets $\mathcal{C}_{\mathbf{b}}^{\delta}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)$ can be written as the $\mathrm{Stab}(\mathcal{P})$-orbit of a compact subset of $\overline{\mathcal{Y}_{g,n}}(\gamma)$. As $\mathrm{Stab}(\mathcal{P}) \subseteq \mathrm{Stab}(\gamma)$, this finishes the proof.

Fix a pair of pants decomposition $\mathcal{P} := (\gamma_1,\dots,\gamma_{3g-3+n})$ of $S_{g,n}$ containing the components of γ. By Proposition 4.8, $\mathcal{C}_{\mathbf{b}}^{\delta}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}$ can be rewritten as

$$\mathcal{C}_{\mathbf{b}}^{\delta}(\mathcal{P}) = \left\{ \alpha \in \overline{\mathcal{Y}_{g,n}^1} \;\middle|\; \begin{array}{l} i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1,\dots,k, \\ i(\alpha,\gamma_i) \leq C, \ \forall i = k+1,\dots,3g-3+n, \\ s_\gamma(\alpha) \geq \delta. \end{array} \right\}.$$

It follows that $\mathcal{C}_{\mathbf{b}}^{\delta}(\mathcal{P})$ is a closed (see Proposition 4.9) subset of the $\mathrm{Stab}(\mathcal{P})$-invariant subset $\mathcal{D}_{\mathbf{b}}^{\delta}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}$ given by

$$\mathcal{D}_{\mathbf{b}}^{\delta}(\mathcal{P}) := \left\{ \alpha \in \overline{\mathcal{Y}_{g,n}^1} \;\middle|\; \begin{array}{l} \delta \leq i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1,\dots,k, \\ \delta \leq i(\alpha,\gamma_i) \leq C, \ \forall i = k+1,\dots,3g-3+n. \end{array} \right\}.$$

If we show that $\mathcal{D}^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}$ is the $\mathrm{Stab}(\mathcal{P})$-orbit of a compact subset $\mathcal{E}^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}$, then $\mathcal{C}^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \overline{\mathcal{Y}_{g,n}}$ will be the $\mathrm{Stab}(\mathcal{P})$-orbit of the compact subset $\mathcal{C}^{\delta}_{\mathbf{b}}(\mathcal{P}) \cap \mathcal{E}^{\delta}_{\mathbf{b}}(\mathcal{P})$, thus finishing the proof.

Complete $\mathcal{P}$ to a maximal geodesic lamination μ of $S_{g,n}$ and consider the shear coordinates of $\overline{\mathcal{Y}_{g,n}}$ with respect to μ,

$$\overline{\Phi_\mu} \colon \overline{\mathcal{Y}_{g,n}} \to ((0,\infty) \times \mathcal{ML}_{g,n}(\mu)) \sqcup (\{0\} \times \mathcal{ML}_{g,n})$$

By (4.2 and (4.3),

$$i(\overline{\Phi_\mu}(\alpha), \gamma_i) = i(\alpha, \gamma_i)$$

for every $\alpha \in \overline{\mathcal{Y}_{g,n}}$ and every $i = 1, \dots, 3g-3+n$. It follows that

$$\overline{\Phi_\mu}(\mathcal{D}^{\delta}_{\mathbf{b}}(\mathcal{P})) = [0,1] \times D^{\delta}_{\mathbf{b}}(\mathcal{P}),$$

where $D^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \mathcal{ML}_{g,n}(\mu)$ is the subset given by

$$D^{\delta}_{\mathbf{b}}(\mathcal{P}) := \left\{ \lambda \in \mathcal{ML}_{g,n}(\mu) \;\middle|\; \begin{array}{l} \delta \leq i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1, \dots, k, \\ \delta \leq i(\alpha,\gamma_i) \leq C, \ \forall i = k+1, \dots, 3g-3+n. \end{array} \right\}.$$

Notice that, as $\mathcal{ML}_{g,n}(\mu) = \mathcal{ML}_{g,n}(\mathcal{P})$ and as $\mathcal{P}$ is a pair of pants decomposition of $S_{g,n}$, $D^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \mathcal{ML}_{g,n}$ can be rewritten as

$$D^{\delta}_{\mathbf{b}}(\mathcal{P}) := \left\{ \lambda \in \mathcal{ML}_{g,n} \;\middle|\; \begin{array}{l} \delta \leq i(\alpha,\gamma_i) \leq b_i, \ \forall i = 1, \dots, k, \\ \delta \leq i(\alpha,\gamma_i) \leq C, \ \forall i = k+1, \dots, 3g-3+n. \end{array} \right\}.$$

As $\overline{\Phi}_\mu$ is $\mathrm{Stab}(\mu)$-equivariant and as the Dehn twists along the components of $\mathcal{P}$ belong to $\mathrm{Stab}(\mu)$, it is enough for our purposes to show that $D^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \mathcal{ML}_{g,n}$ can be written as the orbit of a compact subset of $\mathcal{ML}_{g,n}$ under the action of the group generated by the Dehn twists along the components of $\mathcal{P}$.

Let $(m_i, t_i)_{i=1}^{3g-3+n}$ be a set of Dehn-Thurston coordinates of $\mathcal{ML}_{g,n}$ adapted to $\mathcal{P}$ and denote by $\Theta \subseteq (\mathbf{R}_{\geq 0} \times \mathbf{R})^{3g-3+n}$ its parameter space. Notice that $D^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \mathcal{ML}_{g,n}$ can be described in such coordinates as

$$D^{\delta}_{\mathbf{b}}(\mathcal{P}) = \left\{ (m_i,t_i)_{i=1}^{3g-3+n} \in \Theta \;\middle|\; \begin{array}{l} \delta \leq m_i \leq b_i, \ \forall i = 1, \dots, k, \\ \delta \leq m_i \leq C, \ \forall i = k+1, \dots, 3g-3+n. \end{array} \right\}.$$

Consider the compact subset $E^{\delta}_{\mathbf{b}}(\mathcal{P}) \subseteq \mathcal{ML}_{g,n}$ described in coordinates as

$$E^{\delta}_{\mathbf{b}}(\mathcal{P}) := \left\{ (m_i,t_i)_{i=1}^{3g-3+n} \in \Theta \;\middle|\; \begin{array}{l} \delta \leq m_i \leq b_i, \ \forall i = 1, \dots, k, \\ \delta \leq m_i \leq C, \ \forall i = k+1, \dots, 3g-3+n, \\ 0 \leq t_i \leq m_i, \ \forall i = 1, \dots, 3g-3+n. \end{array} \right\}.$$

Notice that $D^{\delta}_{\mathbf{b}}(\mathcal{P})$ is the orbit of $E^{\delta}_{\mathbf{b}}(\mathcal{P})$ under the action of the group generated by the Dehn twists along the components of $\mathcal{P}$. This finishes the proof. $\square$

Measure estimates. We now show that the subsets $\widetilde{\mathcal{K}}^{\delta}_{\mathbf{b}}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/\mathrm{Stab}(\gamma)$ satisfy the measure estimates described by conditions (1), (2), and (3) in Proposition 4.10. Condition (1) is a direct consequence of Lemma 2.1 and Proposition 4.9. Notice that, as a consequence of Proposition 4.8, $\widetilde{\mathcal{K}}^{\delta}_{\mathbf{b}}(\gamma) \nearrow \widetilde{B}_{\mathbf{b}}(\gamma)$ as $\delta \searrow 0$. Condition (2) then follows from the continuity of the measure $\widetilde{\mu}^{\gamma}_{\mathrm{Thu}}$ on $\overline{\mathcal{Y}_{g,n}}(\gamma)$ and the following result, which can be proved using arguments similar to the ones in the proof of Proposition 4.13.

Lemma 4.14. *The subset $\widetilde{B}_{\mathbf{b}}(\gamma) \subseteq \overline{\mathcal{Y}_{g,n}}(\gamma)/Stab(\gamma)$ has finite $\widetilde{\mu}^{\gamma}_{Thu}$ measure.*

It remains to show that condition (3) of Proposition 4.10 holds.

Proposition 4.15. *There exists $C > 0$ such that for every $0 < \delta < 1$,*

$$\limsup_{t \to 0} \ t^{6g-6+2n} \cdot \widetilde{\mu}_{wp}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \right) \leq C \cdot \delta.$$

Proof. Let $0 < \delta < 1$ be arbitrary. Notice that $\alpha \in \overline{\mathcal{Y}}_{g,n}(\gamma)/\mathrm{Stab}(\gamma)$ belongs to $\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma)$ if and only if

$$i(\alpha, \gamma_i) \leq b_i, \ \forall i = 1, \ldots, k,$$

and at least one of the following conditions holds:

(1) $i(\alpha, \gamma_i) < \delta$ for some $i = 1, \ldots, k$,
(2) $i(\alpha, \beta) < \delta$ for some $\beta \in \mathcal{S}_{g,n}^{\gamma}$ that is not a component of γ.

In particular, for every $t > 0$,

$$\widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \right)$$

is equal to the $\widetilde{\mu}_{\mathrm{wp}}^{\gamma}$ measure of the set of $X \in \mathcal{T}_{g,n}/\mathrm{Stab}(\gamma)$ such that

$$\ell_{\gamma_i}(X) \leq b_i/t, \ \forall i = 1, \ldots, k,$$

and at least one of the following conditions holds:

(1) $\ell_{\gamma_i}(X) < \delta/t$ for some $i = 1, \ldots, k$,
(2) $\ell_{\beta}(X) < \delta/t$ for some $\beta \in \mathcal{S}_{g,n}^{\gamma}$ which is not a component of γ.

This quantity can be estimated using Mirzakhani's integration formulas [Mir07b]. More specifically, following arguments similar to those in the proof of [Ara20b, Proposition 3.9], one can show that, for sufficiently small $t > 0$,

$$\widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \right) \leq \delta \cdot P(1/t^2),$$

where P is a polynomial of degree $3g - 3 + n$ depending only on g, n, γ, and $\mathbf{b}$. It follows that

$$\limsup_{t \to 0} \ t^{6g-6+2n} \cdot \widetilde{\mu}_{\mathrm{wp}}^{\gamma,t} \left(\widetilde{B}_{\mathbf{b}}(\gamma) \backslash \widetilde{\mathcal{K}}_{\mathbf{b}}^{\delta}(\gamma) \right) \leq C \cdot \delta$$

for some constant $C > 0$ depending only on g, n, γ, and $\mathbf{b}$. □

This finishes the proof of Proposition 4.10 and thus of Proposition 4.7. It follows that Theorem 4.1 holds, thus concluding the proof of Theorem 1.16.

5. Counting square-tiled surfaces

Outline of this section. In this section we explain how to combine results and techniques from [Ara20a] with the methods developed in §4 to prove Theorem 1.18. We first give a brief outline of the proofs of the relevant results from [Ara20a] and then explain how Theorem 1.16 can be combined with these results to obtain a proof of Theorem 1.18. We refer the reader to [Ara20a] for a detailed treatment of the arguments discussed here.

Setting. For the rest of this section, let $\alpha := (\alpha_1, \dots, \alpha_k)$ be an ordered simple closed multi-curve on $S_{g,n}$ with $1 \leq k \leq 3g-3+n$ components, $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{Z}_{>0})^k$ be vector of positive integral weights, and $\mathbf{a} \cdot \alpha$ be as in (1.2). Recall that for every $L > 0$ we consider the counting function

$$s(\mathbf{a} \cdot \alpha, L) := \# \left\{ \begin{array}{c} \text{square-tiled surfaces with horizontal core multi-curve} \\ \text{of the same topological type as } \mathbf{a} \cdot \alpha \text{ and } \leq L \text{ squares} \end{array} \right\} / \sim,$$

where $\sim$ denotes the equivalence relation induced by cut and paste operations. We are interested in the asymptotic behavior of $s(\mathbf{a} \cdot \alpha, L)$ as $L \to \infty$.

Notation. To discuss the relevant results and techniques from [Ara20a], we first introduce some notation. A marked square-tiled surface (S, φ) of genus g with n punctures is a square-tiled surface S together with a homeomorphism $\varphi \colon S_{g,n} \to S$ called a *marking*. Denote by $\mathcal{QT}_{g,n}(\mathbf{Z})$ the set of all marked square-tiled surfaces of genus g with n punctures up to cut and paste operations and isotopy of markings. The group $\mathrm{Mod}_{g,n}$ acts on $\mathcal{QT}_{g,n}(\mathbf{Z})$ by changing the markings. The quotient $\mathcal{QT}_{g,n}(\mathbf{Z}) := \mathcal{QT}_{g,n}(\mathbf{Z})/\mathrm{Mod}_{g,n}$ is the set of square-tiled surfaces of genus g with n punctures up to cut and paste operations.

Denote by $\mathcal{ML}_{g,n}(\mathbf{Z}) \subseteq \mathcal{ML}_{g,n}$ the set of all integrally weighted simple closed multi-curves on $S_{g,n}$ up to isotopy. Consider the map $\Im \colon \mathcal{QT}_{g,n}(\mathbf{Z}) \to \mathcal{ML}_{g,n}(\mathbf{Z})$ which to every marked square-tiled surface (S, φ) assigns the pullback through φ of the horizontal core multi-curve of S. Let $[\Im] \colon \mathcal{QM}_{g,n}(\mathbf{Z}) \to \mathcal{ML}_{g,n}(\mathbf{Z})/\mathrm{Mod}_{g,n}$ denote the map induced on quotients. Consider the map $\mathrm{Area} \colon \mathcal{QM}_{g,n}(\mathbf{Z}) \to \mathbf{Z}_{>0}$ which to every square-tiled surface S assigns the area of S, or, equivalently, the number of squares of S. In terms of this notation,

$$s(\mathbf{a} \cdot \alpha, L) = \{S \in \mathcal{QM}_{g,n}(\mathbf{Z}) \mid [\Im](S) \in \mathrm{Mod}_{g,n} \cdot (\mathbf{a} \cdot \alpha), \ \mathrm{Area}(S) \leq L\}.$$

Counting square tiled-surfaces and Thurston volumes. In [Ara20a] we describe the function $s(\mathbf{a} \cdot \alpha, L)$ in terms of countings of integrally weighted simple closed multi-curves in the following way. Consider the map $\Re \colon \mathcal{QT}_{g,n}(\mathbf{Z}) \to \mathcal{ML}_{g,n}(\mathbf{Z})$ which to every marked square-tiled surface (S, φ) assigns the pullback through φ of the vertical core multi-curve of S. Denote the induced map on quotients by $[\Re] \colon \mathcal{QM}_{g,n}(\mathbf{Z}) \to \mathcal{ML}_{g,n}(\mathbf{Z})/\mathrm{Mod}_{g,n}$. Let $\Delta \subseteq \mathcal{ML}_{g,n}(\mathbf{Z}) \times \mathcal{ML}_{g,n}(\mathbf{Z})$ be the set of pairs of integrally weighted simple closed multi-curves that do not fill $S_{g,n}$. These definitions give rise to a bijection

$$\begin{array}{cccc} \Psi \colon & \mathcal{QT}_{g,n}(\mathbf{Z}) & \to & \mathcal{ML}_{g,n}(\mathbf{Z}) \times \mathcal{ML}_{g,n}(\mathbf{Z}) - \Delta \\ & (S, \varphi) & \mapsto & (\Re(S, \varphi), \Im(S, \varphi)) \end{array}.$$

This bijection is $\mathrm{Mod}_{g,n}$-equivariant and maps area of square tiled surfaces to geometric intersection number of integrally weighted simple closed multi-curves. In [Ara20a, §3] we use this bijection to show that

$$s(\mathbf{a} \cdot \alpha, L) = \{\mathbf{b} \cdot \beta \in \mathcal{ML}_{g,n}(\mathbf{Z}) \cap \mathcal{ML}_{g,n}(\mathbf{a} \cdot \alpha)/\mathrm{Stab}(\mathbf{a} \cdot \alpha) \mid i(\mathbf{a} \cdot \alpha, \mathbf{b} \cdot \beta) \leq L\}.$$

To study the asymptotics of these counting functions we consider the measures

$$\mathbf{m}_L := \frac{1}{L^{6g-6+2n}} \sum_{\mathbf{b} \cdot \beta \in \mathcal{ML}_{g,n}(\mathbf{Z})} \delta_{\frac{1}{L} \cdot \mathbf{b} \cdot \beta}.$$

By work of Masur [Mas85, Theorem 2],

$$\mu_{\mathrm{Thu}} = 2^{2g-3+n} \cdot \lim_{L \to \infty} \mathbf{m}_L. \tag{5.1}$$

For a computation of the explicit scaling factor in (5.1) see [MT19]. Equation (5.1) is an analogue of the definition of the Lebesgue measure as a limit of rescaled integer point counting measures. Consider the measures $\mu_{\text{Thu}}^{\mathbf{a}\cdot\alpha} := \mu_{\text{Thu}}|_{\mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)}$ and $\mathbf{m}_L^{\mathbf{a}\cdot\alpha} := \mathbf{m}_L|_{\mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)}$ on $\mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)$ and denote by $\widetilde{\mu}_{\text{Thu}}^{\mathbf{a}\cdot\alpha}$ and $\widetilde{\mathbf{m}}_L^{\mathbf{a}\cdot\alpha}$ their local pushforwards to $\mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)/\text{Stab}(\mathbf{a}\cdot\alpha)$. Directly from (5.1) we deduce

$$\widetilde{\mu}_{\text{Thu}}^{\mathbf{a}\cdot\alpha} = 2^{2g-3+n} \cdot \lim_{L\to\infty} \widetilde{\mathbf{m}}_L^{\mathbf{a}\cdot\alpha}. \tag{5.2}$$

Consider the subset

$$A(\mathbf{a}\cdot\gamma) := \{\lambda \in \mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)/\text{Stab}(\mathbf{a}\cdot\alpha) \mid i(\mathbf{a}\cdot\alpha,\lambda) \leq 1\}.$$

Recall the definition of $\epsilon_{g,n} \in \mathbf{Z}_{>0}$ in (1.10). Notice that

$$s(\mathbf{a}\cdot\alpha, L) = \epsilon_{g,n} \cdot \widetilde{\mathbf{m}}_L^{\mathbf{a}\cdot\alpha}(A(\mathbf{a}\cdot\gamma)), \tag{5.3}$$

where $\epsilon_{g,n}$ accounts for orbifold considerations. We would like to combine (5.3) with (5.2) and apply Portmanteau's theorem to obtain asymptotics for the counting functions $s(\mathbf{a}\cdot\alpha, L)$ as $L \to \infty$. As in §4, we run into the issue that the subset $A(\mathbf{a}\cdot\gamma) \subseteq \mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)/\text{Stab}(\mathbf{a}\cdot\alpha)$ does not have compact closure. The same methods used to deal with this issue in §4 also work in this case. See [Ara20a, Proposition 3.5] for an alternative argument using period coordinates of strata of quadratic differentials. Overall, we deduce the following.

Theorem 5.1. [Ara20a, Proposition 3.4] *Let $\mathbf{a}\cdot\alpha \in \mathcal{ML}_{g,n}(\mathbf{Z})$ be an integrally weighted simple closed multi-curve on $S_{g,n}$. Then,*

$$\lim_{L\to\infty} \frac{s(\mathbf{a}\cdot\alpha, L)}{L^{6g-6+2n}} = \frac{\epsilon_{g,n} \cdot \widetilde{\mu}_{Thu}^{\mathbf{a}\cdot\alpha}\left(\{\lambda \in \mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)/\text{Stab}(\mathbf{a}\cdot\alpha) \mid i(\mathbf{a}\cdot\alpha,\lambda) \leq 1\}\right)}{2^{2g-3+n}}.$$

Thurston volumes and Weil-Petersson volumes. Recall that the frequency $r(\alpha,\mathbf{a}) \in \mathbf{Q}_{>0}$ of the integrally weighted simple closed multi-curve $\mathbf{a}\cdot\alpha \in \mathcal{ML}_{g,n}(\mathbf{Z})$ is defined as the limit of Weil-Petersson volumes of expanding subsets of quotients of $\mathcal{T}_{g,n}$ in (1.4). Directly from Theorem 1.16 we deduce the following.

Corollary 5.2. *Let $\mathbf{a}\cdot\alpha \in \mathcal{ML}_{g,n}(\mathbf{Z})$ be an integrally weighted simple closed multi-curve on $S_{g,n}$. Then,*

$$r(\alpha,\mathbf{a}) = \widetilde{\mu}_{Thu}^{\mathbf{a}\cdot\alpha}\left(\{\lambda \in \mathcal{ML}_{g,n}(\mathbf{a}\cdot\alpha)/\text{Stab}(\mathbf{a}\cdot\alpha) \mid i(\mathbf{a}\cdot\alpha,\lambda) \leq 1\}\right).$$

Remark 5.3. In [Ara20a] we give a different proof of Corollary 5.2 using an indirect argument which relies on a result of Mirzakhani [Mir08b, Theorem 1.3] and the involution on strata of quadratic differentials given by rotation by 90° [Ara20a, §4].

Theorem 1.18 now follows directly from Theorem 5.1 and Corollary 5.2.

6. Counting filling closed hyperbolic multi-geodesics

Outline of this section. In this section we explain how to combine results from §4 with techniques of Mirzakhani in [Mir16] to prove Theorem 1.20. We give a brief overview of Mirzakhani's work in [Mir16] and introduce a general asymptotic formula that can be proved directly using her methods; see Theorem 6.5. We then use this formula in combination with Proposition 4.4 to prove Theorem 1.20.

Setting. For the rest of this section, let $\gamma := (\gamma_1,\ldots,\gamma_k)$ be an ordered filling closed multi-curve on $S_{g,n}$ with $k \geq 1$ components and $X \in \mathcal{T}_{g,n}$ be a marked, oriented, complete, finite area hyperbolic structure on $S_{g,n}$.

Outline of the proof of Theorem 1.20. We begin with an elementary reduction of the proof of Theorem 1.20 to a more concrete statement, which we introduce below as Theorem 6.1. The proof of Theorem 6.1 reduces to a simple application of the general asymptotic formula alluded to above and Proposition 4.4.

A first reduction of the proof of Theorem 1.20. We refer the reader back to the statement of Theorem 1.20 for the notation that will be used in the following discussion. As γ is filling, its stabilizer $\mathrm{Stab}(\gamma) \subseteq \mathrm{Mod}_{g,n}$ is finite. Consider the family of rescaled counting measures $\{\overline{\mu}_{\gamma,X}^L\}_{L>0}$ on $(\mathbf{R}_{\geq 0})^k$ given by

$$\overline{\mu}_{\gamma,X}^L := \sum_{\phi \in \mathrm{Mod}_{g,n}} \delta_{\frac{1}{L}\cdot \vec{\ell}_{\phi\cdot\gamma}(X)}.$$

Notice that, for every $L > 0$,

$$\overline{\mu}_{\gamma,X}^L = |\mathrm{Stab}(\gamma)| \cdot \mu_{\gamma,X}^L,$$
$$(I_\gamma)_*(\mu_{\mathrm{Thu}}^\gamma) = |\mathrm{Stab}(\gamma)| \cdot (\widetilde{I}_\gamma)_*(\widetilde{\mu}_{\mathrm{Thu}}^\gamma).$$

It follows that, to prove Theorem 1.20, it is equivalent to show

$$\lim_{L\to\infty} \frac{\overline{\mu}_{\gamma,X}^L}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot (I_\gamma)_*(\mu_{\mathrm{Thu}}^\gamma) \tag{6.1}$$

in the weak-$\star$ topology for measures on $(\mathbf{R}_{\geq 0})^k$.

By standard approximation arguments, to prove (6.1), it is equivalent to show

$$\lim_{L\to\infty} \frac{\mu_{\gamma,X}^L(A)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot (I_\gamma)_*(\mu_{\mathrm{Thu}}^\gamma)(A)$$

for boxes $A := \prod_{i=1}^k [0, b_i) \subseteq (\mathbf{R}_{\geq 0})^k$ with $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$ arbitrary. By Lemma 2.1, we can instead consider closed boxes $A := \prod_{i=1}^k [0, b_i] \subseteq (\mathbf{R}_{\geq 0})^k$ with $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$ arbitrary.

Fix $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$. For every $L > 0$ consider the counting function

$$f(X, \gamma, \mathbf{b}, L) := \#\{\phi \in \mathrm{Mod}_{g,n} \mid \ell_{\phi\cdot\gamma_i}(X) \leq b_i L, \ \forall i = 1, \ldots, k\}.$$

In terms of the counting functions $c(X, \gamma, \mathbf{b}, L)$ introduced in (1.7),

$$f(X, \gamma, \mathbf{b}, L) = |\mathrm{Stab}(\gamma)| \cdot c(X, \gamma, \mathbf{b}, L).$$

Directly from the definitions we see that

$$f(X, \gamma, \mathbf{b}, L) = \overline{\mu}_{\gamma,X}^L \left(\prod_{i=1}^k [0, b_i] \right).$$

Thus, the proof of (6.1), and so of Theorem 1.20, reduces to the following result.

Theorem 6.1. *For every* $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$,

$$\lim_{L\to\infty} \frac{f(X, \gamma, \mathbf{b}, L)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot \mu_{Thu}(\{\lambda \in \mathcal{ML}_{g,n} \mid i(\gamma_i, \lambda) \leq b_i, \ \forall i = 1, \ldots, k\}).$$

The rest of this section is devoted to the proof of Theorem 6.1. The proof combines Proposition 4.4 with techniques of Mirzakhani in [Mir16]. More precisely, we use the general asymptotic formula introduced in Theorem 6.5 below.

Overview of Mirzakhani's work. In [Mir16], Mirzakhani proved asymptotic formulas analogous to the one in Theorem 1.4 for counting functions of filling closed hyperbolic multi-geodesics with respect to total hyperbolic length. As highlighted in [Mir16, §1.2], such formulas hold for length functions more general than total hyperbolic length. We now introduce one such class of length functions and give a precise statement of the corresponding asymptotic formula; see Theorem 6.5. In the course of the following discussion we give a brief overview of Mirzakhani's work.

Let $m \in \mathbf{N}$ be arbitrary. Every linear function $\mathcal{L}\colon \mathbf{R}^m \to \mathbf{R}$ cuts out a *positive open half-space* and a *positive closed half-space* in $\mathbf{R}^m$ corresponding to the sets

$$H_{>0}(\mathcal{L}) := \{x \in \mathbf{R}^m \mid \mathcal{L}(x) > 0\},$$
$$H_{\geq 0}(\mathcal{L}) := \{x \in \mathbf{R}^m \mid \mathcal{L}(x) \geq 0\}.$$

A *convex polytope* $P \subseteq \mathbf{R}^m$ is an intersection of finitely many positive open/closed half-spaces of $\mathbf{R}^m$. The boundary $\partial P \subseteq \mathbf{R}^m$ of a convex polytope $P \subseteq \mathbf{R}^m$ is its topological boundary when considered as a subset of $\mathbf{R}^m$.

Let $P \subseteq \mathbf{R}^m$ be a convex polytope. We say that a function $\mathcal{F}\colon P \to \mathbf{R}$ is *asymptotically linear* if there exists a linear function $\mathcal{L}\colon P \to \mathbf{R}$ and a constant $c \in \mathbf{R}$ such that

$$\lim_{x \in P\colon d(x,\partial P) \to \infty} \mathcal{F}(x) - \mathcal{L}(x) = c,$$

where d denotes the standard Euclidean distance on $\mathbf{R}^m$. We say that a function $\mathcal{F}\colon P \to \mathbf{R}$ is *asymptotically piecewise linear* if P can be partitioned into finitely many convex polytopes on which $\mathcal{F}$ restricts to an asymptotically linear function.

The basic example of an asymptotically piecewise linear function in the setting of this paper is the hyperbolic length of a closed curve on $S_{g,n}$ interpreted as a function on $\mathcal{T}_{g,n}$. More precisely, the following holds.

Theorem 6.2. [Mir16, Theorem 4.1] *Let γ be a closed curve on $S_{g,n}$. The hyperbolic length function*

$$\ell_\gamma\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$$

is asymptotically piecewise linear with respect to any set of Fenchel-Nielsen coordinates $(\ell_i,\tau_i)_{i=1}^{3g-3+n}$. More precisely, after identifying

$$\mathcal{T}_{g,n} = (\mathbf{R}_{>0} \times \mathbf{R})^{3g-3+n}$$

using the Fenchel-Nielsen coordinates $(\ell_i,\tau_i)_{i=1}^{3g-3+n}$, the length function

$$\ell_\gamma\colon \mathcal{T}_{g,n} = (\mathbf{R}_{>0} \times \mathbf{R})^{3g-3+n} \to \mathbf{R}_{>0}$$

is asymptotically piecewise linear.

Let $\mathcal{P} := (\gamma_1,\dots,\gamma_{3g-3+n})$ be a pair of pants decomposition of $S_{g,n}$. Fix a set of Fenchel-Nielsen coordinates $(\ell_i,\tau_i)_{i=1}^{3g-3+n}$ of $\mathcal{T}_{g,n}$ adapted to $\mathcal{P}$, i.e., whose length parameters correspond to the lengths of the components of $\mathcal{P}$. After identifying

$$\mathcal{T}_{g,n} = (\mathbf{R}_{>0} \times \mathbf{R})^{3g-3+n}$$

using these coordinates, we can partition $\mathcal{T}_{g,n}$ into a countable union of convex polytopes of the form

$$\mathcal{C}_\mathcal{P}^{\mathbf{m}} := \{Y \in \mathcal{T}_{g,n} \mid m_i \cdot \ell_i(Y) \leq \tau_i(Y) < m_{i+1} \cdot \ell_{i+1}(Y)\}$$

with $\mathbf{m} := (m_1,\dots,m_{3g-3+n}) \in \mathbf{Z}^{3g-3+n}$ arbitrary. These polytopes are, up to finite index, fundamental domains for the action of $\mathrm{Stab}(\mathcal{P})$ on $\mathcal{T}_{g,n}$. We say that a function $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ is *bounding* with respect to the set of Fenchel-Nielsen coordinates $(\ell_i, \tau_i)_{i=1}^{3g-3+n}$ if for every $Y \in \mathcal{T}_{g,n}$ there exist constants $L_0 > 0$ and $C > 0$ such that for every $\mathbf{m} := (m_1, \dots, m_{3g-3+n}) \in \mathbf{Z}^{3g-3+n}$, every $L > L_0$, and every $i \in \{1, \dots, 3g-3+n\}$,

$$Z \in \mathrm{Mod}_{g,n} \cdot Y \cap \mathcal{C}_{\mathcal{P}}^{\mathbf{m}} \cap \mathcal{F}^{-1}([0,L]) \Rightarrow \ell_i(Z) \leq C \cdot \frac{L}{\max\{|m_i|, |m_i+1|\}}.$$

The basic example of a bounding function is the total hyperbolic length of a weighted filling closed multi-curve on $S_{g,n}$. More precisely, the following holds.

Proposition 6.3. [Mir16, §9.4] *Let $\gamma := (\gamma_1, \dots, \gamma_k)$ with $k \geq 1$ be an ordered filling closed multi-curve on $S_{g,n}$ and $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{R}_{>0})^k$ be a vector of positive weights on the components of γ. The total hyperbolic length function*

$$\ell_{\mathbf{a}\cdot\gamma}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$$

is bounding with respect to any set of Fenchel-Nielsen coordinates on $\mathcal{T}_{g,n}$.

Another important property of the total hyperbolic length of a weighted filling closed multi-curve on $S_{g,n}$ is its properness when interpreted as a function on $\mathcal{T}_{g,n}$. We record this fact in the following lemma.

Lemma 6.4. [Ker83, Lemma 3.1] *Let $\gamma := (\gamma_1, \dots, \gamma_k)$ with $k \geq 1$ be an ordered filling closed multi-curve on $S_{g,n}$ and $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{R}_{>0})^k$ be a vector of positive weights. The total hyperbolic length function $\ell_{\mathbf{a}\cdot\gamma}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ is proper.*

Let $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ be a positive, continuous, proper function that is asymptotically piecewise linear and bounding with respect to some set of Fenchel-Nielsen coordinates. For every $L > 0$ consider the counting function

$$f(X, \mathcal{F}, L) := \#\{\phi \in \mathrm{Mod}_{g,n} \mid \mathcal{F}(\phi \cdot X) \leq L\}.$$

Using Wolpert's magic formula [Wol85, Theorem 1.3] and the properties of $\mathcal{F}$, one can check that the following limit, which is an analogue of (1.4), exists,

$$r(\mathcal{F}) := \lim_{L\to\infty} \frac{\mu_{\mathrm{wp}}(\{Y \in \mathcal{T}_{g,n} \mid \mathcal{F}(Y) \leq L\})}{L^{6g-6+2n}}. \tag{6.2}$$

We are now ready to introduce the general asymptotic formula that will be used in the proof of Theorem 6.1. The arguments in the proof of [Mir16, Theorem 1.1] directly yield the following result.

Theorem 6.5. *Let $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ be a positive, continuous, proper function that is asymptotically piecewise linear and bounding with respect to some set of Fenchel-Nielsen coordinates. Then*

$$\lim_{L\to\infty} \frac{f(X, \mathcal{F}, L)}{L^{6g-6+2n}} = \frac{B(X) \cdot r(\mathcal{F})}{b_{g,n}}.$$

Remark 6.6. According to Theorem 6.2, Proposition 6.3, and Lemma 6.4, given any set of positive weights $\mathbf{a} := (a_1, \dots, a_k) \in (\mathbf{R}_{>0})^k$ on the components of γ, the total hyperbolic length function $\ell_{\mathbf{a}\cdot\gamma}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ satisfies the hypothesis of Theorem 6.5. It follows that we can recover [Mir16, Theorem 1.1] from Theorem 6.5 by setting $\mathcal{F} := \ell_{\mathbf{a}\cdot\gamma}$.

Topological interpretation of $r(\mathcal{F})$. The arguments introduced in the proof of Theorem 4.1 can also be used to give a topological interpretation of the limit $r(\mathcal{F})$ in (6.2) for a particular class of maps $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ which we now describe.

Let $F\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ be a continuous, homogeneous, and proper function. In particular, F is positive away from the origin. Consider the map $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ which to every $Y \in \mathcal{T}_{g,n}$ assigns the positive value

$$\mathcal{F}(Y) := F(\ell_{\gamma_1}(Y), \dots, \ell_{\gamma_k}(Y)). \tag{6.3}$$

More generally, consider the map $\overline{\mathcal{F}}\colon \overline{\mathcal{Y}_{g,n}} \to \mathbf{R}_{>0}$ which to every $\alpha \in \overline{\mathcal{Y}_{g,n}}$ assigns the positive value

$$\overline{\mathcal{F}}(\alpha) := F(i(\gamma_1, \alpha), \dots, i(\gamma_k, \alpha)). \tag{6.4}$$

For any map $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ as in (6.3), the following holds.

Proposition 6.7. *Let $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ be as in (6.3). Then,*

$$r(\mathcal{F}) = \mu_{Thu}\left(\{\lambda \in \mathcal{ML}_{g,n} \mid \overline{\mathcal{F}}(\lambda) \leq 1\}\right),$$

where $\overline{\mathcal{F}}\colon \overline{\mathcal{Y}_{g,n}} \to \mathbf{R}_{\geq 0}$ is as in (6.4).

Proof. By Proposition 4.4,

$$\lim_{t \to 0} t^{6g-6+2n} \cdot \mu_{\mathrm{wp}}^t = \mu_{\mathrm{Thu}}$$

in the weak-$\star$ topology for measures on $\overline{\mathcal{Y}_{g,n}}$. Consider the subset of $\overline{\mathcal{Y}_{g,n}}$ given by

$$D(\overline{\mathcal{F}}) := \{\alpha \in \overline{\mathcal{Y}_{g,n}} \mid \overline{\mathcal{F}}(\alpha) \leq 1\}.$$

Using the properness of the function $F\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ and the fact that γ is filling, one can check that $D(\overline{\mathcal{F}}) \subseteq \overline{\mathcal{Y}_{g,n}}$ is compact. By Lemma 2.1,

$$\mu_{\mathrm{Thu}}(\partial D(\overline{\mathcal{F}})) = 0.$$

It follows from Portmanteau's theorem that

$$\lim_{t \to 0} t^{6g-6+2n} \cdot \mu_{\mathrm{wp}}^t(D(\overline{\mathcal{F}})) = \mu_{\mathrm{Thu}}(D(\overline{\mathcal{F}})).$$

Letting $t := 1/L$ for $L > 0$ and taking $L \to \infty$ we deduce

$$\lim_{L \to \infty} \frac{\mu_{\mathrm{wp}}^{1/L}(D(\overline{\mathcal{F}}))}{L^{6g-6+2n}} = \mu_{\mathrm{Thu}}(D(\overline{\mathcal{F}})). \tag{6.5}$$

As the function $F\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ is homogeneous,

$$\mu_{\mathrm{wp}}^{1/L}(D(\overline{\mathcal{F}})) = \mu_{\mathrm{wp}}(\{Y \in \mathcal{T}_{g,n} \mid \mathcal{F}(Y) \leq L\}) \tag{6.6}$$

for every $L > 0$. Notice also that

$$\mu_{\mathrm{Thu}}(D(\overline{\mathcal{F}})) = \mu_{\mathrm{Thu}}\left(\{\lambda \in \mathcal{ML}_{g,n} \mid \overline{\mathcal{F}}(\lambda) \leq 1\}\right). \tag{6.7}$$

Putting together (6.5), (6.6), and (6.7), we conclude

$$r(\mathcal{F}) := \lim_{L \to \infty} \frac{\mu_{\mathrm{wp}}(\{Y \in \mathcal{T}_{g,n} \mid \mathcal{F}(Y) \leq L\})}{L^{6g-6+2n}} = \mu_{\mathrm{Thu}}\left(\{\lambda \in \mathcal{ML}_{g,n} \mid \overline{\mathcal{F}}(\lambda) \leq 1\}\right).$$

□

Proof of Theorem 6.1. We are now ready to prove Theorem 6.1 and thus finish the proof of Theorem 1.20.

Proof of Theorem 6.1. Let $\mathbf{b} := (b_1, \ldots, b_k) \in (\mathbf{R}_{>0})^k$ be arbitrary. Consider the function $F\colon (\mathbf{R}_{\geq 0})^k \to \mathbf{R}_{\geq 0}$ which to every $(x_1, \ldots, x_k) \in (\mathbf{R}_{\geq 0})^k$ assigns the value

$$F(x_1, \ldots, x_k) := \max\{x_1/b_1, \ldots, x_k/b_k\}.$$

Let $\mathcal{F}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$ be the map induced by F as defined in (6.3). Notice that, as a consequence of Theorem 6.2, $\mathcal{F}$ is asymptotically piecewise linear with respect to any set of Fenchel-Nielsen coordinates. Let $\mathbf{1} := (1, \ldots, 1) \in (\mathbf{Q}_{>0})^k$. Consider the total hyperbolic length functions $\ell_{\mathbf{1}\cdot\gamma}\colon \mathcal{T}_{g,n} \to \mathbf{R}_{>0}$. The bound

$$\ell_{\mathbf{1}\cdot\gamma} \leq k \cdot \max\{b_1, \ldots, b_k\} \cdot \mathcal{F}$$

ensures that, for every $L > 0$,

$$\mathcal{F}^{-1}([0, L]) \subseteq \ell_{\mathbf{1}\cdot\gamma}^{-1}([0, k \cdot \max\{b_1, \ldots, b_k\} \cdot L]).$$

This fact together with Proposition 6.3 implies $\mathcal{F}$ is bounding with respect to any set of Fenchel-Nielsen coordinates. The same fact together with Lemma 6.4 implies $\mathcal{F}$ is proper. By Theorem 6.5, it follows that

$$\lim_{L\to\infty} \frac{f(X, \mathcal{F}, L)}{L^{6g-6+2n}} = \frac{B(X) \cdot r(\mathcal{F})}{b_{g,n}}. \tag{6.8}$$

Notice that

$$f(X, \mathcal{F}, L) = f(X, \gamma, \mathbf{b}, L) \tag{6.9}$$

for every $L > 0$. As a consequence of Proposition 6.7,

$$r(\mathcal{F}) = \mu_{\mathrm{Thu}}(\{\lambda \in \mathcal{ML}_{g,n} \mid i(\gamma_i, \lambda) \leq b_i,\ \forall i = 1, \ldots, k\}). \tag{6.10}$$

Putting together (6.8), (6.9), and (6.10), we conclude

$$\lim_{L\to\infty} \frac{f(X, \gamma, \mathbf{b}, L)}{L^{6g-6+2n}} = \frac{B(X)}{b_{g,n}} \cdot \mu_{\mathrm{Thu}}(\{\lambda \in \mathcal{ML}_{g,n} \mid i(\gamma_i, \lambda) \leq b_i,\ \forall i = 1, \ldots, k\}). \qquad \square$$

References

[AA20] Francisco Arana-Herrera and Jayadev S. Athreya, *Square-integrability of the Mirzakhani function and statistics of simple closed geodesics on hyperbolic surfaces*, Forum Math. Sigma **8** (2020), e9, 33. MR 4061979

[Ara20a] Francisco Arana-Herrera, *Counting square-tiled surfaces with prescribed real and imaginary foliations and connections to Mirzakhani's asymptotics for simple closed hyperbolic geodesics*, Journal of Modern Dynamics **16** (2020), 81–107.

[Ara20b] ———, *Equidistribution of families of expanding horospheres on moduli spaces of hyperbolic surfaces*, Geometriae Dedicata (2020).

[DGZZ19] Vincent Delecroix, Elise Goujard, Peter Zograf, and Anton Zorich, *Masur-Veech volumes, frequencies of simple closed geodesics and intersection numbers of moduli spaces of curves*, arXiv e-prints (2019), arXiv:1908.08611.

[Do13] Norman Do, *Moduli spaces of hyperbolic surfaces and their Weil-Petersson volumes*, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, pp. 217–258. MR 3184165

[EM18] Viveka Erlandsson and Gabriele Mondello, *Ergodic invariant measures on the space of geodesic currents*, arXiv e-prints (2018), arXiv:1807.02144.

[EMM19] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi, *Effective counting of simple closed geodesics on hyperbolic surfaces*, arXiv e-prints (2019), arXiv:1905.04435.

[EPS20] Viveka Erlandsson, Hugo Parlier, and Juan Souto, *Counting curves, and the stable length of currents*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 6, 1675–1702. MR 4092896

[ES16] Viveka Erlandsson and Juan Souto, *Counting curves in hyperbolic surfaces*, Geom. Funct. Anal. **26** (2016), no. 3, 729–777. MR 3540452

[ES20] ———, *Geodesic currents and Mirzakhani's curve counting*, In preparation, 2020.

[EU18] V. Erlandsson and C. Uyanik, *Length functions on currents and applications to dynamics and counting*, ArXiv e-prints (2018).

[FLP12] Albert Fathi, François Laudenbach, and Valentin Poénaru, *Thurston's work on surfaces*, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012, Translated from the 1979 French original by Djun M. Kim and Dan Margalit. MR 3053012

[FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR 2850125

[Ker83] Steven P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), no. 2, 235–265. MR 690845

[KM96] D. Y. Kleinbock and G. A. Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Sinaĭ's Moscow Seminar on Dynamical Systems, Amer. Math. Soc. Transl. Ser. 2, vol. 171, Amer. Math. Soc., Providence, RI, 1996, pp. 141–172. MR 1359098

[Liu19] Mingkun Liu, *Length statistics of random multicurves on closed hyperbolic surfaces*, arXiv e-prints (2019), arXiv:1912.11155.

[Mar70] Grigoriy A. Margulis, *On some aspects of the theory of Anosov systems*, Ph.D. Thesis, 1970, Springer-Verlag, Berlin, 2003. MR 2035655

[Mar16] Bruno Martelli, *An Introduction to Geometric Topology*, ArXiv e-prints (2016).

[Mas85] Howard Masur, *Ergodic actions of the mapping class group*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 455–459. MR 787893

[Mir04] Maryam Mirzakhani, *Simple geodesics on hyperbolic surfaces and the volume of the moduli space of curves*, ProQuest LLC, Ann Arbor, MI, 2004, Thesis (Ph.D.)–Harvard University. MR 2705986

[Mir07a] ———, *Random hyperbolic surfaces and measured laminations*, In the tradition of Ahlfors-Bers. IV, Contemp. Math., vol. 432, Amer. Math. Soc., Providence, RI, 2007, pp. 179–198. MR 2342816

[Mir07b] ———, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222. MR 2264808

[Mir07c] ———, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. **20** (2007), no. 1, 1–23. MR 2257394

[Mir08a] ———, *Ergodic theory of the earthquake flow*, Int. Math. Res. Not. IMRN (2008), no. 3, Art. ID rnm116, 39. MR 2416997

[Mir08b] ———, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Ann. of Math. (2) **168** (2008), no. 1, 97–125. MR 2415399

[Mir16] ———, *Counting Mapping Class group orbits on hyperbolic surfaces*, arXiv e-prints (2016), arXiv:1601.03342.

[MT19] Leonid Monin and Vanya Telpukhovskiy, *On normalizations of Thurston measure on the space of measured laminations*, Topology Appl. **267** (2019), 106878, 12. MR 4008568

[Pap88] Athanase Papadopoulos, *Sur le bord de Thurston de l'espace de Teichmüller d'une surface non compacte*, Math. Ann. **282** (1988), no. 3, 353–359. MR 967017

[Pap91] ———, *On Thurston's boundary of Teichmüller space and the extension of earthquakes*, Topology Appl. **41** (1991), no. 3, 147–177. MR 1135095

[PH92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992. MR 1144770

[PP93] A. Papadopoulos and R. C. Penner, *The Weil-Petersson symplectic structure at Thurston's boundary*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 891–904. MR 1089420

[PS15] A. Papadopoulos and W. Su, *On the Finsler structure of Teichmüller's metric and Thurston's metric*, Expo. Math. **33** (2015), no. 1, 30–47. MR 3310926

[RS19] Kasra Rafi and Juan Souto, *Geodesic currents and counting problems*, Geom. Funct. Anal. **29** (2019), no. 3, 871–889. MR 3962881

[SB01] Yaşar Sözen and Francis Bonahon, *The Weil-Petersson and Thurston symplectic forms*, Duke Math. J. **108** (2001), no. 3, 581–597. MR 1838662

[Thu86] William P. Thurston, *Minimal stretch maps between hyperbolic surfaces*, arXiv Mathematics e-prints (1986), math/9801039.

[Wol85] Scott Wolpert, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), no. 4, 969–997. MR 796909

Email address: `farana@stanford.edu`

Department of Mathematics, Stanford University, 450 Jane Stanford Way, Building 380, Stanford, CA 94305-2125, USA