Probability I: Sample spaces and counting

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Biol 799 - Lecture 4

January 4, 2017

Topics

- Random processes
- Sample spaces
- Basic probability rules
  - Complementarity
  - Addition
  - Multiplication
- Disjoint and independent sets
- The binomial distribution

Random

Can we predict a coin flip *mechanistically?*

No

Can we predict a coin flip *probabilistically?*

Yes!

Coin flips

50% chance Heads

50% chance Tails

This "random" result tells us everything we need to know about the very complex problem of the coin-flip.

In short...

$$1 + 1 = 2$$

but what is ...

 +  ?

What does **random** mean?

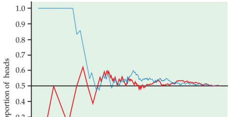
- A **random event** X can take some values in $k = (x_1, x_2, x_3, ...)$ ... but we can not predict X exactly.
- BUT, if X were repeated many times, a *fixed pattern* would emerge. This pattern is the **probability distribution**

$$f(k) = P(X = k)$$

Note: the values k is called the **sample space**.

What does **random** mean?

We can not describe it well exactly ONCE, but we can describe what will happen if it is repeated many times.
This is the *frequentist* interpretation of probability.



Definitions

- The **sample space** is a **set** (or **list**) of *all*, possible, non-overlapping outcomes of a random process.
- An **event** is a subset of the sample space.
- A **probability model** (or *measure*) is the probability $(0 < P < 1)$ for a given **event** in the **sample space**

Types of sample spaces

- Discrete, finite
  - All outcomes can be enumerated (even if it is a lot of outcomes)
    - Examples: coin tosses, rolls of the dice, card picks
- Continuous, infinite
  - Like a continuous variable, there are an uncountable number of outcomes in a continuous sample space
    - Examples: time to your next text message, length of pups, colors in the visible spectrum
- Goal: to estimate the probability of an event $P(A)$ in sample space S

Enumerating discrete sample spaces

- For discrete sample spaces, you can **count** or **enumerate** all possibilities.
- Under certain assumptions, you can build the **probability model** of an event.

Example: A single coin flip

- The sample space of X = **a single coin flip** is:
  - **H:**  and **T:** 
- We denote this: $S = \{H, T\}$- possible events are just H or T.
- The probability model is written:
  $P(X = H) = 0.5$ and $P(X = T) = 0.5$

Example:Two coin flips

- The sample space of X = **two coin flips** is:
  S = **HH:** , **HT:** , **TH:**  and **TT:** 
- Is $HT = TH$? It depends on your question!
- If NO, the probability model is:

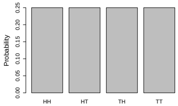
$$P(X = HH) = 0.25, P(X = HT) = 0.25$$
$$P(X = TH) = 0.25, P(X = TT) = 0.25$$

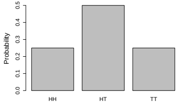
- If YES, the probability model is:

$$P(X = HH) = 0.25$$
$$P(X = HT) = 0.50$$
$$P(X = TT) = 0.25$$

Example:Two coin flips

S = **HH:** , **HT:** , **TH:** and **TT:** 



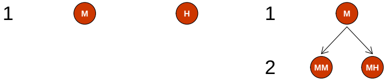


- If $HT \neq TH$:
  $P(X = HH) = 0.25$,
  $P(X = HT) = 0.25$,
  $P(X = TH) = 0.25$,
  $P(X = TT) = 0.25$
- If $HT = TH$:
  $P(X = HH) = 0.25$
  $P(X = HT) = 0.50$
  $P(X = TT) = 0.25$

The sample space depends on the question!

A basketball player shoots three free throws.

- Question I: What are the possible sequences of hits and misses?



- S = {MMM, MMH, MHM, MHH, HMM, HMH, HHM, HHH}
- Note: $k = 2^3 = 8$

The sample space depends on the question!

A basketball player shoots three free throws.

- Question II: How many baskets will the basketball player make total?

try 0:	0
try 1:	0 1
try 2:	0 1 2
try 3:	0 1 2 3

- S = {0, 1, 2, 3}

Continuous spaces are different

- The sample space can not be enumerated.
- When we work with these, we need to describe them with a mathematical function that takes values on the continuous real numbers.
- For now, we'll stick to discrete spaces.

Goals and Rules of Probability

- Rules about sample spaces:
  - $0 \leq P(A) \leq 1$ for any event A
  - $P(S) = 1$
- Rules about combining probabilities
  - **Complement rule**: For any event A, where A^c is the event "not A":
    $P(A^c) = 1 - P(A)$
  - **Addition rule**: If A and B are **disjoint** events, then:
    $P(A \text{ or } B) = P(A) + P(B)$
  - **Multiplication rule**: If A and B are **independent** events, then:
    $P(A \text{ and } B) = P(A) \times P(B)$

Another example system

In the 2006 NBA playoffs, Shaq shot 37% from free throw line.

In the 2011 playoffs, Ray Allen shot 96% from free throw line.

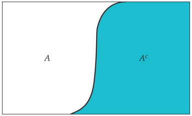
Sample space rules

- $0 \leq P(A) \leq 1$
  - P(heads) = 0.5
  - P(Shaq makes a FT) = 0.37
  - P(Allen makes a FT) = 0.96
- $P(S) = 1$
  - P(heads) + P(tails) = 1
  - P(Shaq makes a FT) + P(Shaq misses a FT) = 1
  - P(Allen makes a FT) + P(Allen misses a FT) = 1
  - P(Shaq makes either 0,1,2,3 FT in 3 attempts) = 1
- $P(A^c) = 1 - P(A)$
  - P(heads) = 1 - P(tails) = 0.5
  - P(Shaq misses a FT) = 1-P(Shaq makes a FT) = 0.63
  - P(Shaq makes 0/3) = 1-P(Shaq makes 1,2 or 3/3) = ?

Complements

Event A divide sample space into two pieces:

- Event happened: A
- Event did not happen: A^c
  - $A^c = A$ "complement"



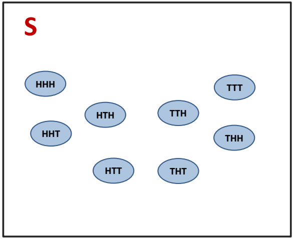
Rule of complements: $P(A^c) = 1 - P(A)$

Combining events: UNION

UNION: A or B - $A \cup B$

- Example: Three coin tosses with exactly one head OR first flip is a tail
- S = {{HHH}, {HHT}, {HTH}, {HTT}
  {THH}, {THT}, {TTH}, {TTT}}
- A = {{HTT}, {THT}, {TTH}}
- B = {{THH}, {THT}, {TTH}, {TTT}}
- $A \cup B$ = {{HTT}, {THH}, {THT}, {TTH}, {TTT}}
- $(A \cup B)^C$ ={HHH}, {HHT} {HTH}

In a Venn diagram

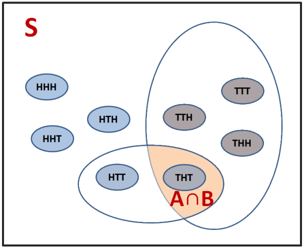


Combining events: INTERSECTION

INTERSECTION: A and B - $A \cap B$

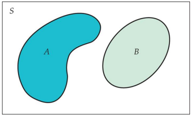
- Example: Three coin tosses with exactly one head AND first flip is a tail
- S = {{HHH}, {HHT}, {HTH}, {HTT}
  {THH}, {THT}, {TTH}, {TTT}}
- A = {{HTT}, {THT}, {TTH}}
- B = {{THH}, {THT}, {TTH}, {TTT}}
- $A \cap B$ ={{HTT}}
- $(A \cup B)^C$ =...

In a Venn diagram



Addition rule for disjoint events

Two events A and B are disjoint if they have no outcomes in common and can never happen together. The probability that A OR B occurs is the sum of their individual probabilities



Addition rule for disjoint events:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

Independence

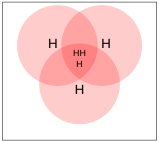
- If events A and B are **independent**, then $P(A)$ has no impact on $P(B)$.
  - *Example*:
    - You flip a coin twice,
    - $P(\text{Heads first})$ has no effect on $P(\text{Tails second})$
  - *Counterexample*:
    - You draw a card from a deck of 52 once: $P(\text{black card on first draw}) = 0.5$.
    - You draw a second card from a deck without replacing the first: $P(\text{black card on second draw}) = 25/51 < 0.5$.
  - *Possible counterexample*:
    - You shoot a basketball once.
    - Is $P(\text{You make the second}|\text{You missed the first}) = P(\text{You make a second}|\text{You made the first})$?

Multiplication rule for **Independent Events**

- If A and B are independent: $P(A \cap B) = P(A) \times P(B)$
- Note: $P(B|A) = P(B)$

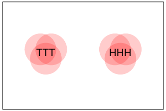
Example 1: Three Heads

- What is the probability of flipping three heads in three tosses?
- Note: $P(H) = 0.5$;
- Coin flips are independent;
- So $P(HHH) = P(H) \times P(H) \times P(H)$



Example 2: A run of three

- What is the probability of getting three in a row?
- Now we combine "AND" and "OR":



$$
\begin{aligned}
P(HHH \cup TTT) &= P(HHH) + P(TTT) \\
&= P(H \cap H \cap H) + P(T \cap T \cap T) \\
&= P(H)P(H)P(H) + P(T)P(T)P(T) \\
&= (0.5)^3 + (0.5)^3 = \mathbf{0.25}
\end{aligned}
$$

- So what is the probability of a 2/1 split?
- $P(2/1 \text{ split}) = P((HHH \cup TTT)^c) = 1 - P(HHH \cup TTT) = 0.75$

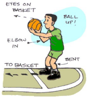
Example

- Note that every outcome has the same probability,
- But that is only because $P(H) = P(T) = P(H^c)$

	Toss:		
	First	Second	Third
1	**H**	**H**	**H**
2	H	H	T
3	H	T	H
4	H	T	T
5	T	H	H
6	T	H	T
7	T	T	H
8	T	T	T

Uniform probability spaces

- There is a class of random processes for which each outcome has equal probability, for example:
  - Coin flips
  - Dice rolls
  - Cards from a shuffled deck
- But not:'
  - Free throws



Part II: Permutations and Combinations

A surprising fact

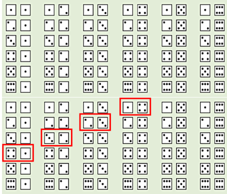
A lot of the theory underlying classical statistical inference can be derived from considering (in great detail) *independent* events from *equal probability* sample spaces!

Consider rolling 2 dice

Question: What is the probability that the sum is 5?

What is the probability that the sum is 5?

The sample space consists of 36 equally probable events:



$S = \{(1,1),(1,2),(1,3),....\}$

- How do we know? We counted: $N_S = 6 \times 6$
  - Note: A and B are independent, so $P(A \cap B) = P(A)P(B)$.
- How many sum to 5? We counted: (1,4), (2,3), (3,2), (4,1)
  - $N_A = 4$
- $P(D_1 + D_2 = 5) = N_A/N_S = 4/36 = 0.111$

Lots of probability problems are just counting problems!

- What's the probability of 1 die giving an odd number?
  - S has 6 outcomes, A (Odds) had 3 outcomes, $N_A/N_S = 3/6 = 0.5$
- What's the probability of 2 dice giving a sum > 9?
  - S has 36 outcomes, A (> 9) has six outcomes, $N_A/N_S = 6/36 = 0.166$
- What's the probability that at least 2 people in a class of 23 people have the same birthday?
  - Yikes!
- What's the probability that after 20 coin flips, you'll get exactly 10 heads?
  - Yikes!

Counting is not always easy!

Counting Rules

What's for lunch?

- Food: Sushi, Teriyaki, Udon noodle
- Drink: Fanta, Green Tea, H_20

How many different meals can I make?

Counting Rules

Fundamental counting rule

Let A_1 be a set with n_1 elements and A_2 be a set with n_2 elements. If one element is taken from A_1 and one element is taken from A_2, there are:

$$n_1 \times n_2$$

possible unique outcomes.

Answer

$3 \times 3 = 9$

Counting Rules

What's for dinner?

- Food: Escargots, Fondue, Grenouilles
- Drink: Bordeaux, Burgundy, Beaujolais
- Dessert: Crème fraîche, Tarte aux pommes, Sorbet aux pêches

How many people at a table can have a unique meal?

Counting Rules

Multiplicative rule

Let A_1, A_2, A_3 be k sets with n_1, n_2, ... n_k elements (respectively) in each set. If one element is taken from each set, then there are

$$n_1 \times n_2 \times ... \times n_k$$

possible unique outcomes.

Answer

$3 \times 3 \times 3 = 27$

Counting Rules

How do I rank my favorite animals?

Some animals:

- Aardvark, Baboon, Cheetah, Dolphin

How many different ways can I rank them according to how cool I think they are?

Counting Rules

Factorial rule for permutations

A set of n elements can be ordered $n!$ different ways

Definition of factorial

$$n! = n(n-1)(n-2)(n-3)...1$$

And: $1! = 0! = 1$

Answer

$4! = 4 \times 3 \times 2 \times 1 = 24$

Counting Rules

How do I rank my favorite four out of eleven animals?

Some animals:

- Aardvark, Baboon, Cheetah, Dolphin, Egret, Flamingo, Giraffe, Hippo, Iguana, Jackal, Kangaroo

How many different ways can I rank the 4 coolest ones?

Counting Rules

Permutations (order matters)

A selection of r elements from a set of n total elements can be **rank ordered** in

$$\frac{n!}{(n-r)!}$$

different ways.

Answer

$\frac{11!}{(11-4)!} = \frac{11!}{7!} = 11 \times 10 \times 9 \times 8 = 7920$

Counting Rules

How do I pick four animals I want to study?

Some animals:

- Aardvark, Baboon, Cheetah, Dolphin, Egret, Flamingo, Giraffe, Hippo, Iguana, Jackal, Kangaroo

How many different ways can I separate this group into 4 that I want to study and 7 that I don't?

Counting Rules

Combinations (order doesn't matters)

A selection of r elements from a set of n total elements can be **chosen** in

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

different ways.

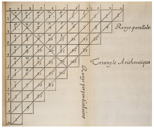
"Choose" function

- We call this creature: $\binom{n}{r}$ "*n choose r*"
- It is the number of ways we can pick r unique cases from a set of n
- It is also written: $_rC_n$, and called: "the binomial coefficient".

Answer

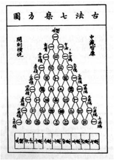
$\frac{11!}{4!(11-4)!} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$

Blaise Pascal (1623 - 1662)



French mathematician - described "Pacal's triangle" in *Treatise on the Arithmetical Triangle* (1653).

Zhu Shijie 朱世杰 (1270 - 1330)



- Great Chinese mathematician - described the triangle in *The Precious Mirror of the Four Elements* (1303).
- Attributes it to **Jia Xian** (1050).
- Also attributed to **Omar Khayyam** (Persia: 1048-1131) - *Khayyam's Triangle*
- Who attributes it to **Al-Karaji** (Persia: 953-1029)
- Though it was known by **Pingala** (India: 2nd century)

Back to the Birthday Problem

What is the probability that in a class of 23 students, at least 2 will have a matching birthday?

- What is S? All possible sequences of birthdays (**multiplicative rule**):
  - $N_S = 365^{23}$
- What is A? All possible sequences where at least 2 people have the same birthday.
  - That's a bit tricky.
- What is A^c? All possible sequences where NO ONE shares a birthday (**permutations rule**).
  - $N_A = 365 \times 364 \times 363 \times \ldots \times 343 = \frac{365!}{(365-23)!}$
  - $P(A^c) = \frac{N_{A^c}}{N_S} = \frac{365!/342!}{365^{23}}$
  - $P(A) = 1 - P(A^c) = 1 - \frac{365!/342!}{365^{23}} \approx 0.507$

Back to the Coin Problem

What is the probability that after flipping 10 coins, you'll get exactly 5 heads?

- Sample size (**multiplicative rule**)
  - $N_S = 2^{10}$
- What is the event size: N_A?
- We can define a sequence of events by 5 numbers chosen from 1 to 10. This is the same as choosing a combination of 5 unique numbers from 10 total, and we don't care about the order (**combinations rule**):

$$N_A = \binom{10}{5} = \frac{10!}{5!(10-5)!}$$

- $P(\text{5 heads in 10 tosses}) = \frac{10!/(5!(10-5)!)}{2^{10}} \approx 0.246$

A different way to look at the Coin Problem

What is the probability that after flipping 10 coins, you'll get exactly 5 heads?

- First: We need 5 heads (H) and 5 tails (T) to happen:
  - $P(T) = P(H) = 1/2$
  - $P(HHHHH) = (1/2)^5$, $P(TTTTT) = (1/2)^5$
- But there are many ways in which these sequences can happen!
  - $P(\text{5 heads in 10 tosses}) = K(1/2)^5(1/2)^5$
  - What is K?
- **Combinations Rule!**

$$K = \binom{10}{5} = \frac{10!}{5!(10-5)!}$$

- $P(\text{5 heads in 10 tosses}) = \frac{10!}{5!(10-5)!}(1/2)^5(1/2)^5 \approx 0.246$

More flexible way of thinking about the problem...

Example: what is the probability that Shaq will make 8 free throws out of 10?

- We need 8 successes (H) and 2 failures (M) to happen:
  - $P(H) = p = 0.374$ and $P(M) = 1 - p = 0.626$
  - $P(HHHHHHHH) = p^8$, $P(MM) = (1-p)^2$
- How many ways can the sequence of 8 hits happen? **Combinations Rule!**

$$K = \binom{10}{8} = \frac{10!}{8!2!}$$

- $P(\text{8 hits in 10 FT's}) = \frac{10!}{8!2!}p^8(1-p)^2 \approx 0.67\%$

(Note: that number is written in PERCENT!)

Binomial Distribution

The Binomial Distribution...

... is a **discrete probability distribution** that tells you the exact probability of k successes out of n tries, if each try is an independent event with probability p:

$$f(k|n,p) = \Pr(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

where $k = \{0, 1, 2...n\}$. Note the following properties:

$$\sum_{k=0}^{n} f(k|n,p) = 1$$

Note that we derived this distribution from **probability rules** and **counting rules**.