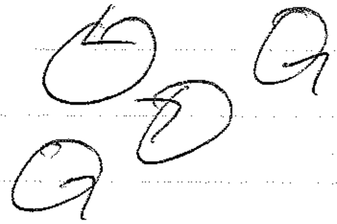
Navier Stokes Turbulence and
4 Dissipation Processes

Many plasma systems are characterized as being turbulent. This means for example that transport of various quantities results from turbulent convection rather than classical collisional diffusion.



In a turbulent medium a fluid element can be convected, producing an effective transport of energy momentum etc. Often it is the gradient of the quantity being transported, e.g. flow velocity in the case of momentum, which is the source of free energy. Examples of systems in which turbulence is believed to dominate transport — "anomalous transport" — are many.

In fluid systems.

① pipe flow – weakly dissipative limit
⇒ anomalous drag

② Raleigh–Benard convection — liquid heated from below in a gravitational system.
"anomalous thermal transport"

③ Atmospheric dynamics – storm fronts, etc.

In plasma systems

① solar convection zone
⇒ like Raleigh–Benard convection

(2) Earth's magnetotail — convection of magnetic flux

(3) Accretion ~~discs~~ discs — transport of angular momentum

(4) Magnetized ~~plasma~~ confinement systems for fusion — energy, particle and momentum transport $\perp$ to $\underset{\sim}{B}$

Turbulence in plasma and neutral fluid systems have features in common — (1) often nearly incompressible
(2) cascade processes are believed to be important.
$\Rightarrow$ energy is transferred to shorter and shorter scale lengths as a result of nonlinear interactions $\Rightarrow$ why?
$\Rightarrow$ energy transfer is most effective between comparable scales
$\Rightarrow$ energy transfer is local in k space
$\Rightarrow$ cascade
$\Rightarrow$ not always true for plasma systems.

$\Rightarrow$ start with fluid turbulence
$\Rightarrow$ same as unmagnetized plasma in incompressible limit

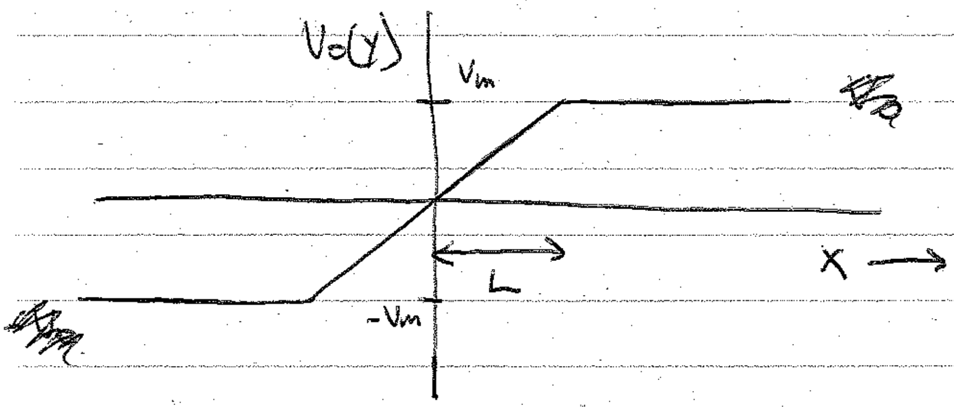
Kelvin Helmholtz Instability

Fundamental to understanding cascades in plasmas and fluids are instabilities driven by sheared flows

$\Rightarrow$ produce the turbulence driving cascade of energy

Ramped shear flow system

Equil. $V = V_0(x)\,\hat{y}$



Momentum eqn $\longrightarrow$ 2-D in x-y plane $\longrightarrow$ taps energy

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\nabla\right)\underset{\sim}{V} = -\frac{1}{\rho}\nabla P$$

For velocities small compared to the sonic speed of the medium the flow is nearly incompressible

$\Rightarrow$ eliminate compressible motion by taking $\hat{z}\cdot\nabla\times$

$\Rightarrow$ write $\underset{\sim}{V} = \hat{z}\times\nabla\varphi$

$$\frac{\partial}{\partial t}\hat{z}\cdot\nabla\times\underset{\sim}{V} + \hat{z}\cdot\nabla\times\ \underset{\sim}{V}\cdot\nabla\underset{\sim}{V} = 0$$

$$\hat{z}\cdot\nabla\times\underset{\sim}{v} = \hat{z}\cdot\nabla\times(\hat{z}\times\nabla\varphi)$$

$$= \hat{z}\cdot\left[\hat{z}\nabla^2\varphi - \cancel{\hat{z}\cdot\nabla}^{\,0}\,\nabla\varphi\right] = \nabla^2\varphi$$

$$\frac{\partial}{\partial t}\nabla^2\varphi + \nabla\cdot\ \hat{z}\times\nabla\varphi\cdot\nabla\ \underbrace{\hat{z}\times(\hat{z}\times\nabla\varphi)}_{-\nabla\varphi} = 0$$

$$\frac{\partial}{\partial t}\nabla^2\varphi + \nabla\cdot\ \hat{z}\times\nabla\varphi\cdot\nabla\ \nabla\varphi = 0$$

$$\frac{\partial}{\partial t}\nabla^2\varphi + \hat{z}\times\nabla\varphi\cdot\nabla\ \nabla^2\varphi + \sum_j \cancel{\hat{z}\times\nabla(\nabla_j\varphi)\cdot\nabla(\nabla_j\varphi)}^{\,0} = 0$$

$$\boxed{\left(\frac{\partial}{\partial t} + \hat{z}\times\nabla\varphi\cdot\nabla\right)\nabla^2\varphi = 0}$$ $\nabla^2\varphi$ = vorticity

⇒ justify incompressibility

⇒ consider source $\underset{\sim}{S}$ of flow

⇒ linear response

$$\frac{\partial}{\partial t}\underset{\sim}{V} = -\frac{1}{\rho}\nabla P + \underset{\sim}{S}$$

$$\frac{\partial P}{\partial t} + P_0\Gamma\nabla\cdot\underset{\sim}{V} = 0$$

$$\frac{\partial}{\partial t}\nabla\cdot \underset{\sim}{V} = -\frac{1}{\rho}\nabla^2 p + \nabla\cdot \underset{\sim}{S}'$$

$$\frac{\partial^2}{\partial t^2}\nabla\cdot \underset{\sim}{V} = +\frac{\Gamma p_0}{\rho}\nabla^2\nabla\cdot \underset{\sim}{V} + \nabla\cdot \dot{\underset{\sim}{S}}'$$

$\sim \omega^2$ $\quad\quad \sim c_s^2 k^2$

$\Rightarrow$ assume $\omega^2 \ll k^2 c_s^2$

$\tilde{V} = \hat{z}\times\nabla\tilde{\varphi}$

$\tilde{V}_x = -\frac{\partial}{\partial y}\tilde{\varphi}$

$$\nabla\cdot \underset{\sim}{V} \sim \frac{1}{\omega}\nabla\cdot \underset{\sim}{S}' \;\frac{\omega^2}{k^2 c_s^2}$$

$$\frac{\partial}{\partial t}\nabla\times \underset{\sim}{V} = \nabla\times \underset{\sim}{S}'$$

$$\frac{\nabla\cdot \underset{\sim}{V}}{\hat{z}\cdot\nabla\times \underset{\sim}{V}} \sim \frac{\omega^2}{k^2 c_s^2} \ll 1$$

$$\hat{z}\cdot\nabla\times \underset{\sim}{V} \sim \frac{\hat{z}\cdot}{\omega}\nabla\times \underset{\sim}{S}'$$

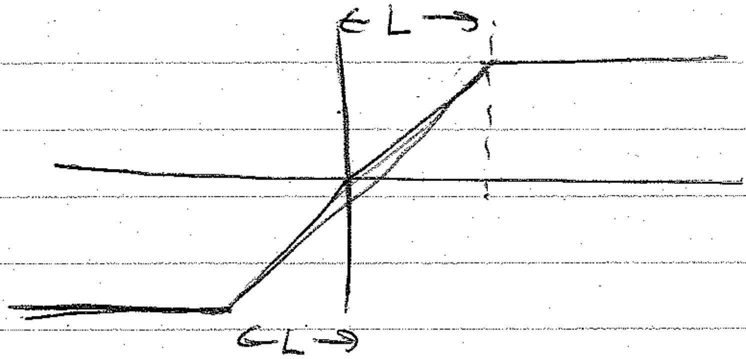
$$\varphi = \varphi_0(x) + \tilde{\varphi}(x)e^{iky}$$

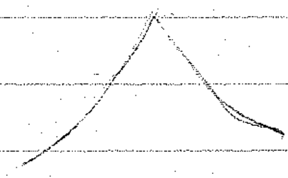
<u>Linearization</u> $\quad \underset{\sim}{V_0} = \hat{z}\times\hat{i}\frac{\partial}{\partial x}\varphi_0 = \frac{\partial\varphi_0}{\partial x}\hat{y}$

$$\left(\frac{\partial}{\partial t} + V_0 \overset{iky}{\cancel{\frac{\partial}{\partial x}}}\right)\nabla^2\tilde{\varphi} + \left(\overset{-iky}{\cancel{-\frac{\partial}{\partial y}}}\tilde{\varphi}\right)\frac{\partial^2}{\partial x^2}V_0 = 0$$

$$\left(\overset{-i\omega}{\cancel{\frac{\partial}{\partial t}}} + iky V_0\right)\left(\frac{\partial^2}{\partial x^2} - k_\perp^2\right)\tilde{\varphi} - iky\,\tilde{\varphi}\,V_0'' = 0$$

$$\left(\frac{\partial^2}{\partial x^2} - k^2\right)\tilde{\varphi} = -\frac{k_y\tilde{\varphi}\,V_0''}{\omega - k_y V_0}$$





⇒ solve for $\tilde{\psi}$ in regions $x > L$, $|x| < L$ and $x < -L$

⇒ jump conditions at $x = \pm L$

⇒ at marginal stability

$$\left(\frac{\partial^2}{\partial x^2} - k^2\right)\tilde{\psi} = +\tilde{\psi}\,\frac{V_0''}{V_0}$$

⇒ ~~exact solution~~ note $\frac{V_0''}{V}$ is even

⇒ $\tilde{\psi}$ has even and odd solutions

⇒ even

$|x| < L$

$$\tilde{\psi} = \tilde{\psi}_0 \frac{\cosh(kx)}{\cosh(kL)}$$

$\tilde{\psi}$ continuous at $x = L$

$x > L$

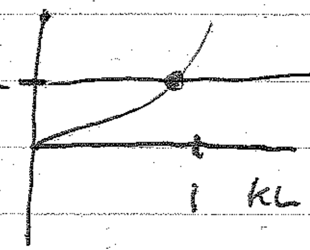
$$\tilde{\psi} = \tilde{\psi}_0\, e^{-k(x-L)}$$

jump in slope $\frac{\partial^2}{\partial x^2}\tilde{\psi} = +\tilde{\psi}_0 \frac{V_0''}{V_m}$

$$\left.\frac{\partial \psi}{\partial x}\right|_{L-\epsilon}^{L+\epsilon} = +\frac{\tilde{\psi}_0}{V_m}\left(-\frac{V_m}{L}\right)$$

$$\frac{e^{x}+e^{-x}}{2}$$

$$-k\cancel{\tilde{\phi}_0} - k\frac{\sin kL}{\cos kL}\cancel{\tilde{\phi}_0} = -\frac{\cancel{\tilde{\phi}_0}}{L}\cancel{k}$$



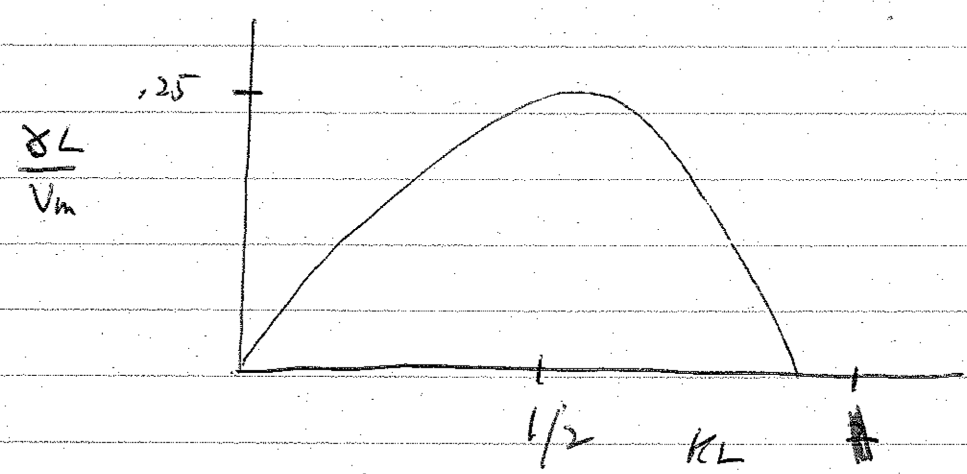
$$\boxed{KL(1+\tanh KL) = +1}$$

marginal stability

$\Rightarrow$ for finite growth rate can't assume symmetry

$\Rightarrow$ solutions in three regions <u>Hwk</u>

$$\left(\frac{\gamma L}{V_m}\right)^2 + \left[\left(kL-\frac{1}{2}\right)^2 - \frac{1}{4}e^{-4kL}\right] = 0$$



$\Rightarrow$ generic

$\Rightarrow \gamma \to 0$ at $k = 0$

$\gamma \to 0$ for $KL \sim 1$

General Stability

Quadratic form

$$\int dx\, \tilde{\varphi}^* \left(\frac{\partial^2}{\partial x^2} - k^2\right)\tilde{\varphi} + k_y \int dx \frac{|\tilde{\varphi}|^2 V_0''}{\omega - k_y V_0} = 0$$

$$-\int dx \left[\left|\frac{\partial\tilde{\varphi}}{\partial x}\right|^2 + k^2|\tilde{\varphi}|^2\right] + k_y \int dx \frac{|\tilde{\varphi}|^2 V_0''(\omega_0 - i\gamma - k_y V_0)}{\underbrace{(\omega_0+i\gamma-k_yV_0)(\omega_0-i\gamma-k_yV_0)}_{(\omega_0-k_yV_0)^2+\gamma^2}} = 0$$

separate into real and imaginary

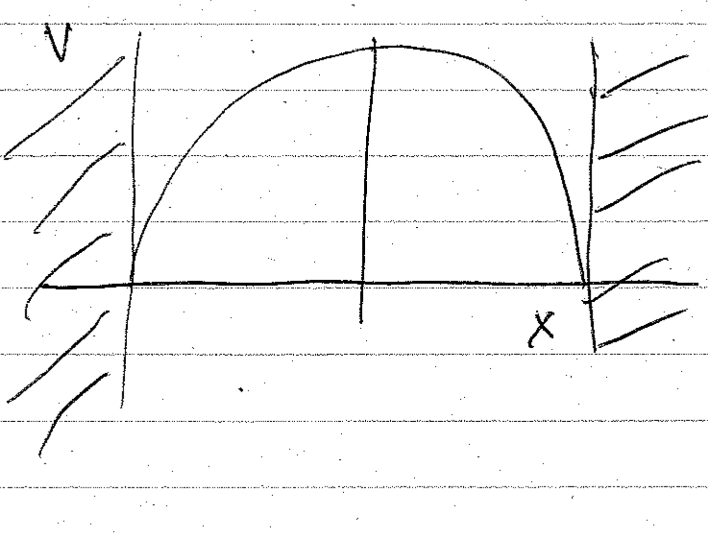
Imag.

$$\gamma \int dx \frac{|\tilde{\varphi}|^2 V_0''}{\gamma^2 + (\omega_0 - k_y V_0)^2} = 0 \Rightarrow$$

V_0'' must have $\pm$ values for instability

$\Rightarrow$ inflection pt.

$$\int dx \left[\left|\frac{\partial\tilde{\varphi}}{\partial x}\right|^2 + k^2|\tilde{\varphi}|^2\right] = -k_y^2 \int dx \frac{|\tilde{\varphi}|^2 V_0'' V_0}{\gamma^2 + (\omega_0 - k_y V_0)^2}$$



always stable

3-D Navier-Stokes Eqns

$$\frac{\partial}{\partial t}\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{V} = -\frac{1}{\rho_0}\nabla P + \nu\nabla^2\mathbf{V}$$

$$\nabla\cdot\mathbf{V} = 0$$

$\Rightarrow$ valid for subsonic motion

$\Rightarrow$ dissipation is an important control parameter

$\Rightarrow$ compare size of dissipative term with convective nonlinearity

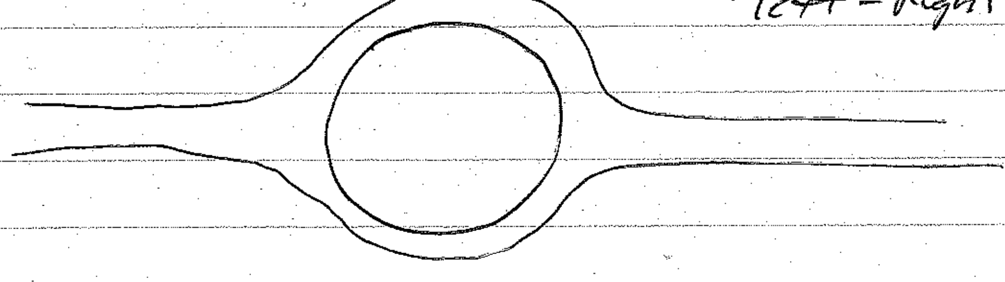
$$\frac{V\cdot\nabla}{\nu\nabla^2} \sim \frac{VL}{\nu} \equiv R = \text{Reynold's number}$$

$\Rightarrow$ for a specified system geometry this is the only parameter

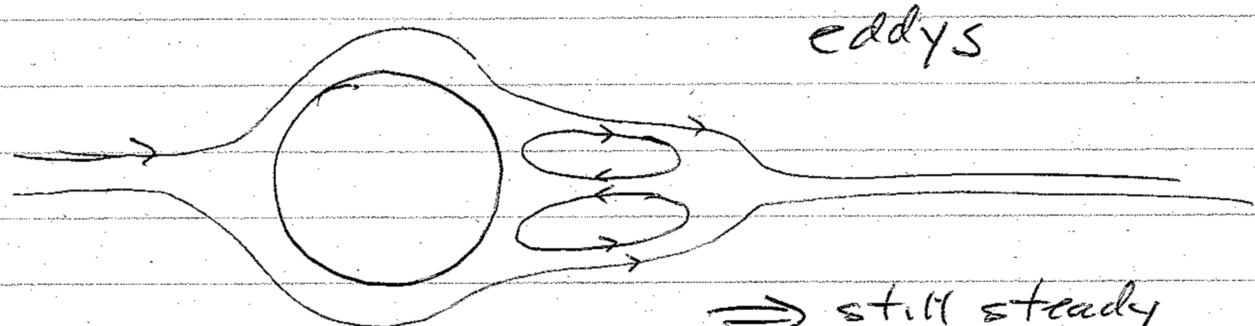
Flow past a cylinder

$R \sim 1$

left-right symmetry



$R \gtrsim 5 \Rightarrow$ eddys downstream



$\Rightarrow$ still steady

$R \gtrsim 40 \Rightarrow$ periodic in time

$\Rightarrow$ Karman Street.

$R \gtrsim 40 - 75$ z invariance broken

chaotic in time

Energy Spectrum

It is useful to describe the turbulence in terms the energy spectrum.

$$E^V = \frac{1}{2} \frac{\int_V d\underline{x}\ \underline{v} \cdot \underline{v}}{V}$$

$\Rightarrow$ where take $\rho = \rho_0 = $ const

$\Rightarrow$ for homogeneous system the local energy density is independent of location

$\Rightarrow$ for isotropic system indep of direction.

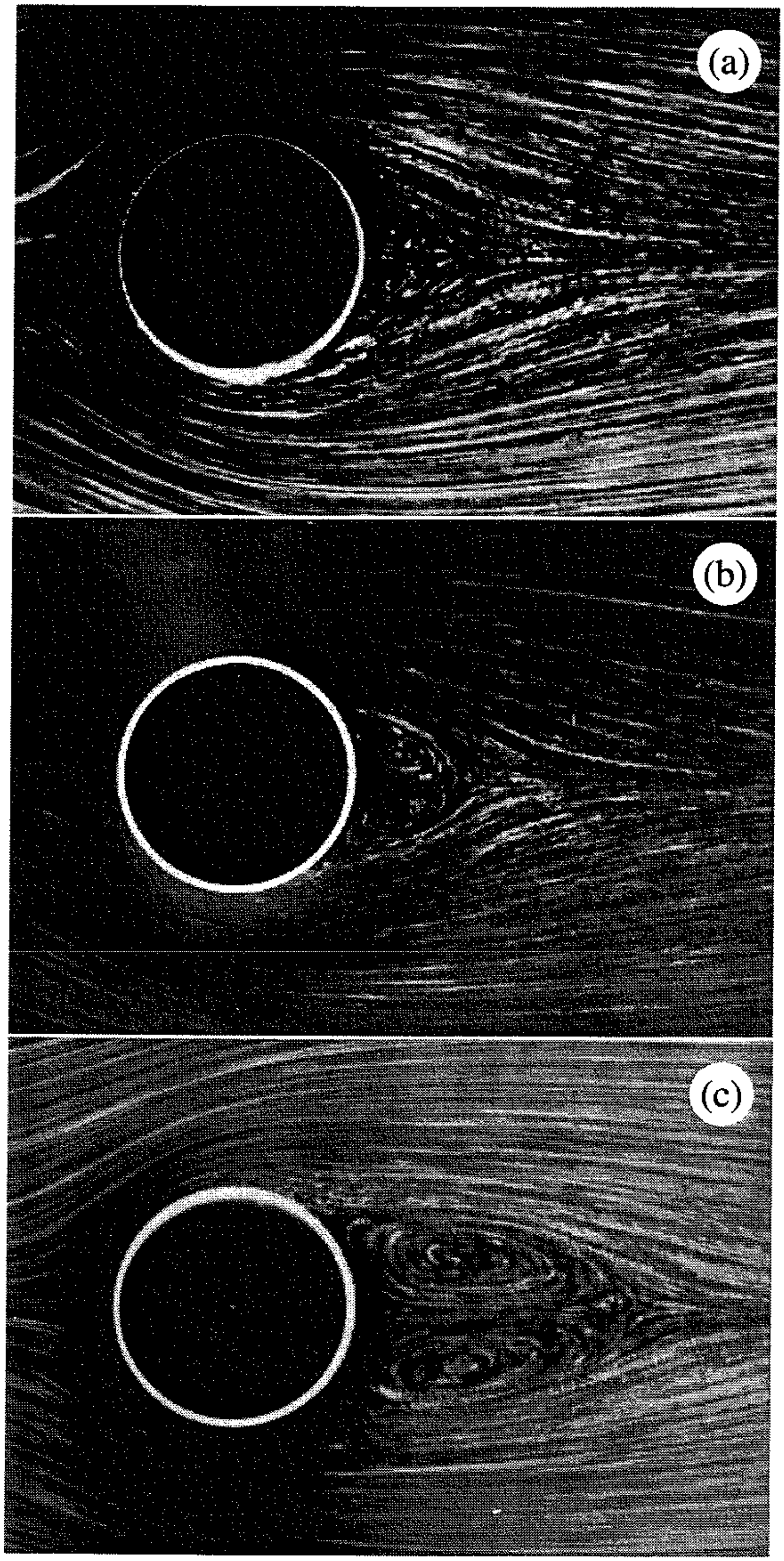


Fig. 1.4. Circular cylinder at $R = 9.6$ (a), $R = 13.1$ (b) and $R = 26$ (c) (Van Dyke 1982). Photograph S. Taneda.

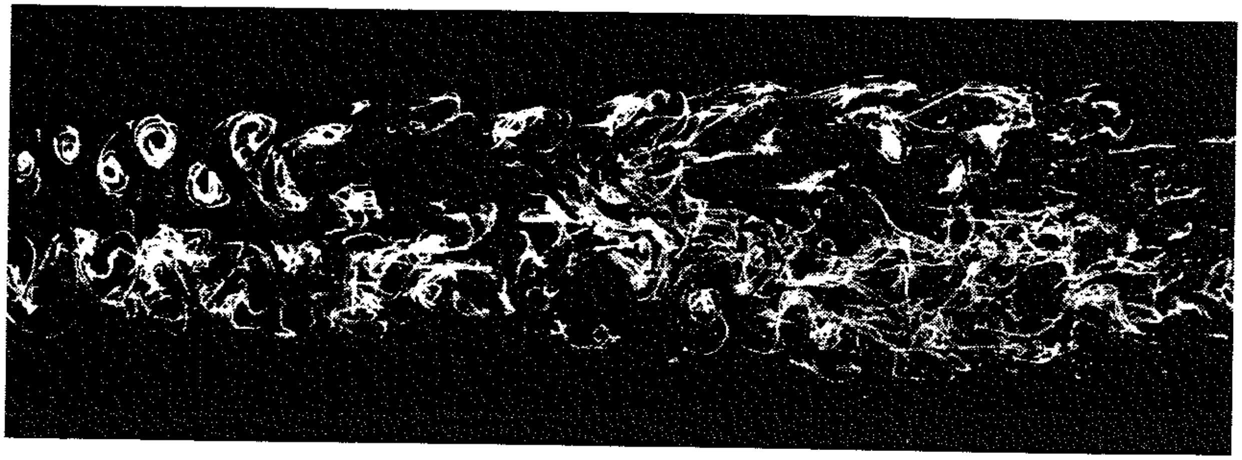


Fig. 1.9. Wake behind two identical cylinders at $R = 240$. Courtesy R. Dumas.

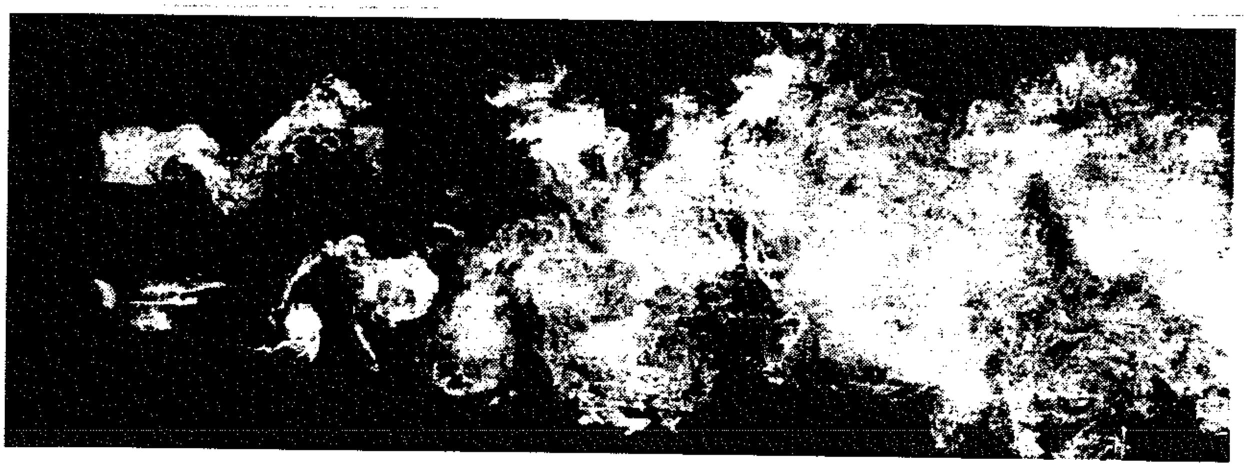


Fig. 1.10. Wake behind two identical cylinders at $R = 1800$. Courtesy R. Dumas.

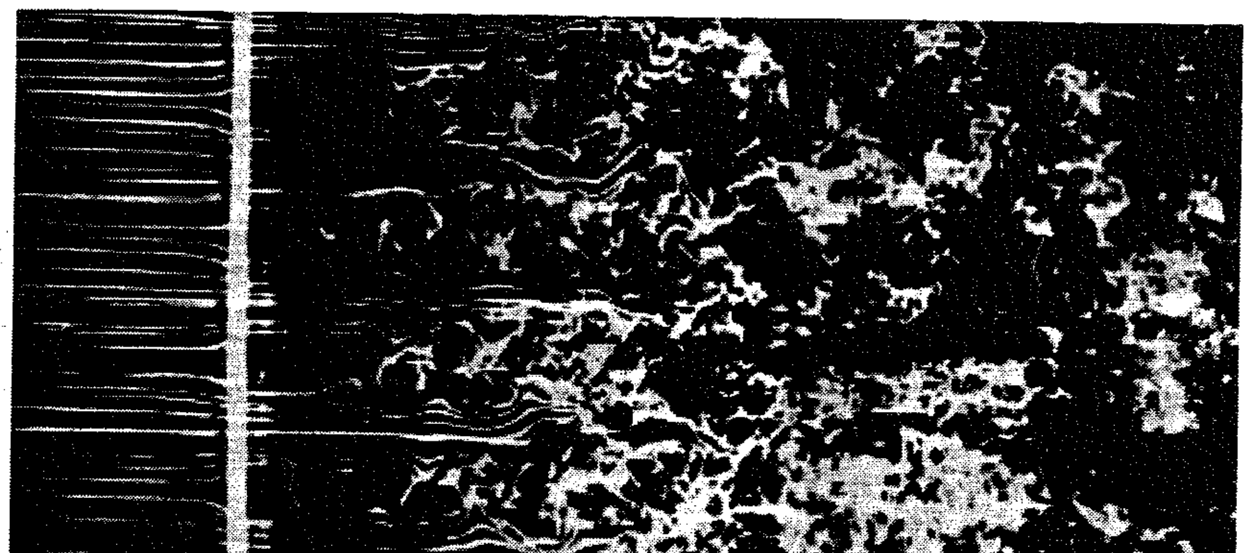


Fig. 1.11. Homogeneous turbulence behind a grid. Photograph T. Corke and H. Nagib.

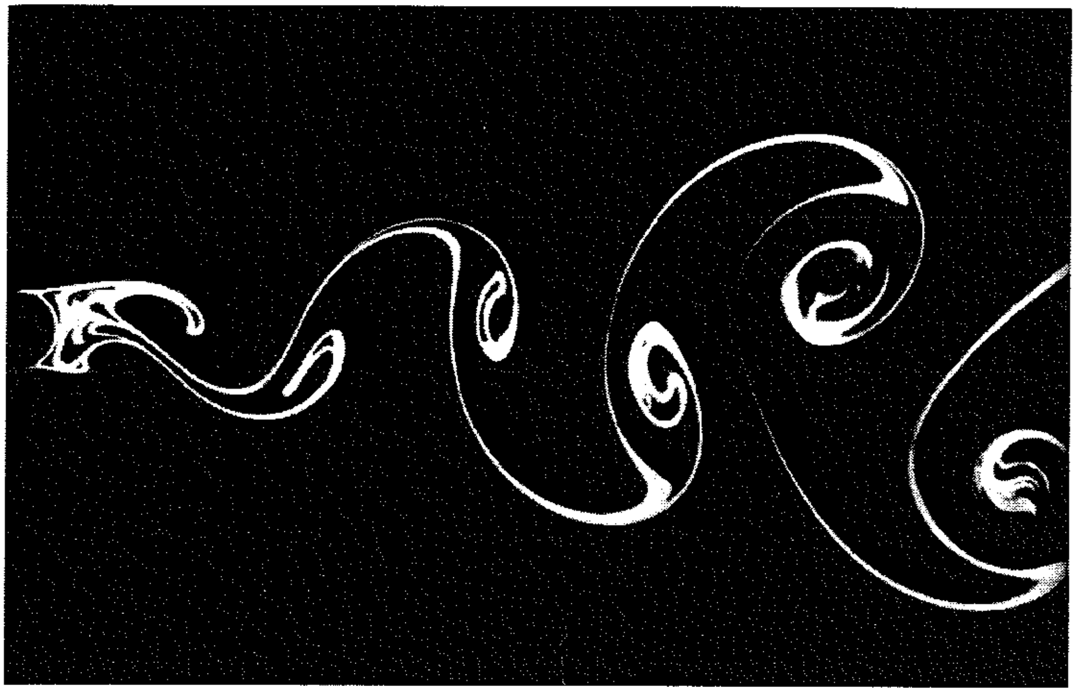


Fig. 1.6. Kármán vortex street behind a circular cylinder at $R = 140$ (Van Dyke 1982). Photograph S. Taneda.

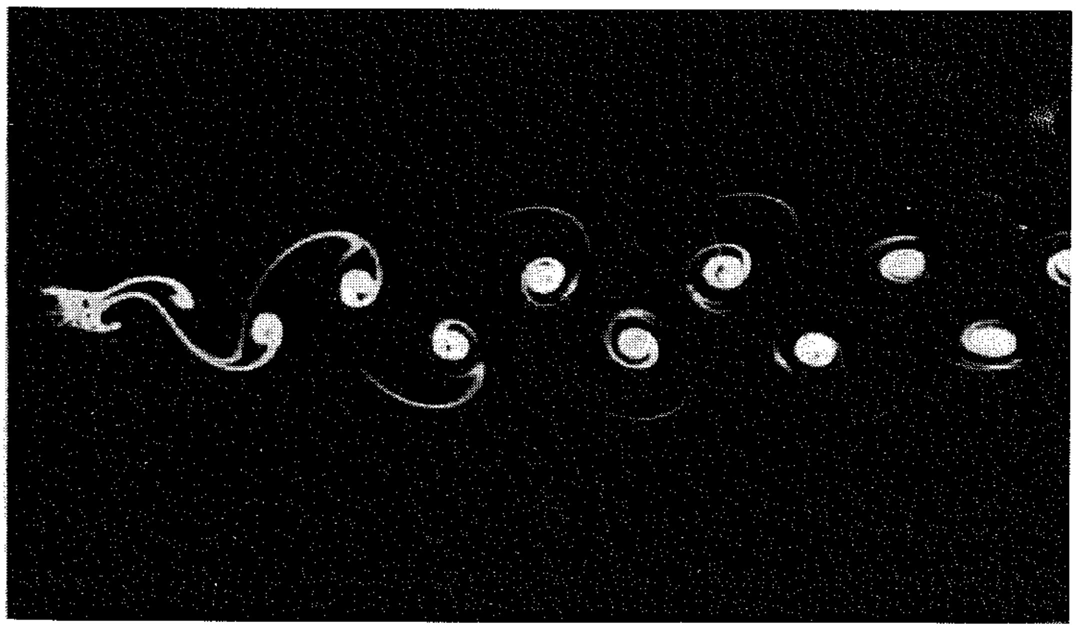


Fig. 1.7. Kármán vortex street behind a circular cylinder at $R = 105$ (Van Dyke 1982). Photograph S. Taneda.

$$E^V = \frac{1}{2}\frac{1}{(2\pi)^3}\int d\underset{\sim}{k}\,|v_k|^2$$

$$= \frac{1}{2}\frac{1}{(2\pi)^3}\int dk\, 4\pi k^2 |v_k|^2$$

$$\equiv \int dk\, \mathcal{E}_k$$

$\mathcal{E}_k$ = energy spectrum

This spectrum has been measured in a wide variety of physical systems.

$\Rightarrow$ show wind tunnel results.

multiply by ω

$$\int_V d\underset{\sim}{x}\, \omega\left(\frac{\partial}{\partial t} + v\cdot\nabla\right)\omega = 0$$

$$\boxed{\frac{\partial}{\partial t}\int d\underset{\sim}{x}\, \omega^2 = 0}$$

$\Omega = \int d\underset{\sim}{x}\, \omega^2$ = mean square vorticity
= const.
= enstrophy

3-D

$$\frac{\partial}{\partial t}\underset{\sim}{V} + \underset{\sim}{V}\cdot\nabla\underset{\sim}{V} = -\frac{1}{\rho_0}\nabla P + \nu\nabla^2\underset{\sim}{V}$$

$$\nabla\cdot\underset{\sim}{V} = 0$$

$\Rightarrow$ must annihilate the ∇P by taking $\nabla\times$

$$\underset{\sim}{\omega} = \nabla\times\underset{\sim}{V}$$

$$\underset{\sim}{V}\times(\nabla\times\underset{\sim}{V}) = \nabla\frac{V^2}{2} - \underset{\sim}{V}\cdot\nabla\underset{\sim}{V}$$

$$\frac{\partial}{\partial t}\underset{\sim}{\omega} - \nabla\times(\underset{\sim}{V}\times\underset{\sim}{\omega}) = \nu\nabla^2\underset{\sim}{\omega}$$

Ideal Invariants

$\Rightarrow$ give information about cascade directions.

2-D $\quad \omega = \nabla^2 \varphi \qquad \underline{v} = \hat{z} \times \nabla \varphi$

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) \omega = 0$$

multiply by φ

$$\varphi \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) \nabla^2 \varphi = 0$$

$$\underbrace{\int_V d\underline{x} \; \varphi \frac{\partial}{\partial t} \nabla^2 \varphi}_{} + \int_V d\underline{x} \; \underbrace{\varphi \, v \cdot \nabla \, \nabla^2 \varphi}_{} = 0$$

$$\underbrace{- \int d\underline{x} \; \nabla \varphi \cdot \frac{\partial}{\partial t} \nabla \varphi}_{} \qquad \underbrace{\int_V d\underline{x} \; (\underline{v} \cdot \nabla \varphi) \nabla^2 \varphi}_{0}$$

$$\underbrace{\int d\underline{x} \; \frac{\partial}{\partial t} \frac{1}{2} |\nabla \varphi|^2}_{}$$

$$\int d\underline{x} \; \frac{\partial}{\partial t} \frac{1}{2} |\underline{v}|^2 \qquad \bar{w} = \int d\underline{x} \; \frac{1}{2} |\underline{v}|^2$$

$$\boxed{\frac{\partial}{\partial t} \int_V d\underline{x} \; \frac{1}{2} |\underline{v}|^2 = 0}$$

energy conservation

<u>energy conservation</u>

$\Rightarrow$ need to introduce vector potential for $\underset{\sim}{v}$

$$\underset{\sim}{v} = \nabla \times \underset{\sim}{A}^v \quad \Rightarrow \quad \nabla \cdot \underset{\sim}{v} = 0$$

choose $\nabla \cdot \underset{\sim}{A}^v = 0$

always do this since $A \to A + \nabla \varphi$ does not affect $\underset{\sim}{v}$

~~Take~~ Operate on $\underset{\sim}{\omega}$ eqn with $\underset{\sim}{A}^v \cdot$

$$\int d\underset{\sim}{x} \left[\underset{\sim}{A}^v \cdot \frac{\partial}{\partial t} \underset{\sim}{\omega} + \underset{\sim}{A}^v \cdot \nabla \times (\underset{\sim}{v} \times \underset{\sim}{\omega}) = \nu \underset{\sim}{A}^v \cdot \nabla^2 \underset{\sim}{\omega} \right]$$

$$\nabla \cdot (A^v \times \dot{v}) = \dot{v} \cdot \nabla \times \underset{\sim}{A}^v - \underset{\sim}{A}^v \cdot \nabla \times \dot{\underset{\sim}{v}}$$

$$\int d\underset{\sim}{x} \, \underset{\sim}{A}^v \cdot \nabla \times \dot{\underset{\sim}{v}} = \int dx \left(\dot{\underset{\sim}{v}} \cdot \nabla \times \underset{\sim}{A}^v - \cancel{\nabla \cdot \underset{\sim}{A}^v \times \dot{\underset{\sim}{v}}} \right)$$

$$= \frac{1}{2} \frac{\partial}{\partial t} \int d\underset{\sim}{x} |\underset{\sim}{v}|^2$$

$$\nabla \cdot \underset{\sim}{A}^v \times \underset{\sim}{v} = \underset{\sim}{v} \cdot \nabla \times \underset{\sim}{A}^v - \underset{\sim}{A}^v \cdot \nabla \times \underset{\sim}{v}$$

$$\int d\underset{\sim}{x} \, \underset{\sim}{A}^v \cdot \nabla \times (\underset{\sim}{v} \times \underset{\sim}{\omega}) = \int d\underset{\sim}{x} \, (\underset{\sim}{v} \times \underset{\sim}{\omega}) \cdot \nabla \times \underset{\sim}{A}^v$$

$$= \int dx \, (\underset{\sim}{v} \times \underset{\sim}{\omega}) \cdot \underset{\sim}{v} = 0$$

$$\underset{\sim}{A}^v \cdot \nabla^2 \underset{\sim}{\omega} = \underset{\sim}{A}^v \cdot \nabla \times (\nabla^2 \underset{\sim}{v})$$

$$\Rightarrow \int d\underset{\sim}{x} \, \underset{\sim}{A}^v \cdot \nabla^2 \underset{\sim}{\omega} = \int d\underset{\sim}{x} \, \underset{\sim}{v} \cdot \nabla^2 \underset{\sim}{v}$$

$$\nabla \times (\nabla \times \underset{\sim}{v}) = \nabla \cancel{\nabla \cdot \underset{\sim}{v}} - \nabla^2 \underset{\sim}{v}$$

$$- \int d\underset{\sim}{x} \, \underset{\sim}{v} \cdot (\nabla \times \underset{\sim}{\omega}) = - \int d\underset{\sim}{x} \, |\underset{\sim}{\omega}|^2$$

$$\boxed{\frac{\partial}{\partial t}\int_V d\underset{\sim}{x}\,|\underset{\sim}{v}|^2 = -\nu\int d\underset{\sim}{x}\,|\underset{\sim}{\omega}|^2}$$

$$\frac{\partial E^V}{\partial t} = 0 \quad \text{for } \nu = 0$$

$$\boxed{E_V = \int_V d\underset{\sim}{x}\,|\underset{\sim}{v}|^2}$$

$$\frac{\partial}{\partial t}\int d\underset{\sim}{x}\;\underset{\sim}{v}\cdot\underset{\sim}{\omega} = \int d\underset{\sim}{x}\left(\underbrace{\dot{\underset{\sim}{v}}\cdot\underset{\sim}{\omega}}_{\underset{\sim}{v}\cdot\dot{\underset{\sim}{\omega}}} + \underset{\sim}{v}\cdot\dot{\underset{\sim}{\omega}}\right)$$

$$= 2\int d\underset{\sim}{x}\;\underset{\sim}{v}\cdot\dot{\underset{\sim}{\omega}}$$

$$= 2\int d\underset{\sim}{x}\;\underset{\sim}{v}\cdot\left[-\cancelto{0}{\nabla\times(\underset{\sim}{v}\times\underset{\sim}{\omega})} + \nu\nabla^2\underset{\sim}{\omega}\right]$$

$$= 2\nu\int d\underset{\sim}{x}\;\underset{\sim}{v}\cdot\nabla^2\underset{\sim}{\omega}$$

$$= -2\nu\int d\underset{\sim}{x}\;\underset{\sim}{\omega}\cdot\nabla\times\underset{\sim}{\omega}$$

$$\boxed{\frac{\partial}{\partial t}\frac{1}{2}\int d\underset{\sim}{x}\;\underset{\sim}{v}\cdot\underset{\sim}{\omega} = -\nu\int d\underset{\sim}{x}\;\underset{\sim}{\omega}\cdot\nabla\times\underset{\sim}{\omega}}$$

$$H^V = \frac{1}{2}\int d\underset{\sim}{x}\;\underset{\sim}{v}\cdot\underset{\sim}{\omega} = \text{helicity}$$

Vortex Stretching ~~and Vorticity~~

In 3-D Navier Stokes ~~[illegible]~~ there is a direct cascade of both energy and helicity to small spatial scales. This cascade leads to the Kolmogorov $E \sim k^{-5/3}$ spectrum.

It has been suggested that this cascade is driven by vortex stretching:

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} - \underbrace{\nabla \times (\underset{\sim}{v} \times \underset{\sim}{\omega})} = \nu \nabla^2 \underset{\sim}{\omega}$$

$$\underset{\sim}{\omega} \cdot \nabla \cancel{\ } \underset{\sim}{v} \cancel{\ } - \underset{\sim}{v} \cdot \nabla \underset{\sim}{\omega}$$

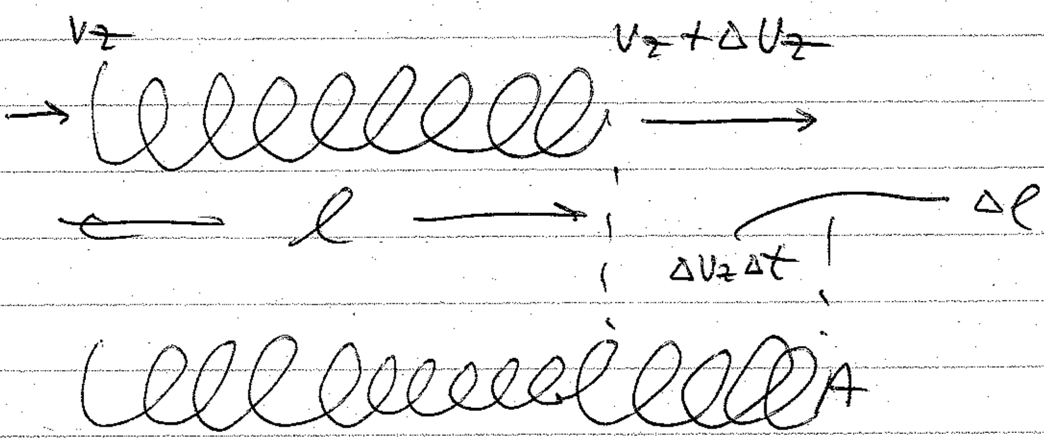
$$\frac{d}{dt} \underset{\sim}{\omega} = + \underbrace{\underset{\sim}{\omega} \cdot \nabla \underset{\sim}{v}}_{\text{vortex stretching term}} + \nu \nabla^2 \underset{\sim}{\omega}$$

Note that the vortex stretching term is absent in 2-D case. Consider a vortex with $\underset{\sim}{\omega}$ in z direction and z dependent flow in z.

$$\frac{d}{dt} \omega_z = + \omega_z \frac{\partial}{\partial z} v_z + \nu \nabla^2 \omega_z$$

It v_z increases with z, this produces increase in ω_z. Why?

$\Rightarrow$ vortex is stretched



$$\Delta\omega_z = \omega_z \frac{\Delta V_z \Delta t}{l} = \omega_z \frac{\Delta l}{l}$$

Al = volume = const.

$$\Rightarrow \Delta A l + \Delta l A = 0$$

$$\frac{\Delta l}{l} = -\frac{\Delta A}{A}$$

$$\Delta\omega_z = -\omega_z \frac{\Delta A}{A}$$

$$\Rightarrow \omega_z A = \text{const}$$

$\omega_z r^2 = \text{const.}$ = angular momentum

$\int_A \underset{\sim}{\omega} \cdot dA = \oint \underset{\sim}{v} \cdot d\underset{\sim}{l}$

$r \downarrow \Rightarrow \omega \uparrow$

Vortex stretching drives energy cascade in 3-D NS turbulence

$r \downarrow \Rightarrow k \uparrow$

Kolmogorov spectrum in 3-D NS turbulence

Suppose that energy is entering a fluid system at a scale length $L \sim K_{in}^{-1}$.

If the viscosity is small then ~~the~~ energy must cascade to a very large k before being dissipated. ~~In a steady system the rate at which energy~~ The range of k between the injection scale and the dissipation scale k_d is called the inertial range.

If the system is in a steady state such that the injection of energy balances dissipation then the rate of energy transfer ϵ must be the same at any local value of k in the inertial range.

Break up k space ~~into~~ into a discrete set of scales $l_n \sim k_n^{-1}$. Define V_l to be the velocity difference between locations separated by l_n.

$$V_l \sim V(x+l) - V(x)$$

Eddy turnover time

$$\frac{V_n}{l_n} \sim \frac{1}{\tau_n}$$

This is the time required by energy to transfer to the adjacent scale n+1 so

$$\mathcal{E} = \frac{E_n}{\tau_n} \sim V_n^2 \frac{V_n}{l_n} \sim \frac{V_n^3}{l_n} \Rightarrow V_n \sim (\mathcal{E} l_n)^{1/3}$$

$$E_n \sim V_n^2 \sim \mathcal{E}^{2/3} l_n^{2/3} \sim E(k_n)\,\Delta k_n$$

$$\sim E(k_n)\frac{1}{l_n}$$

$$\boxed{E(k_n) \sim \mathcal{E}^{2/3} \frac{1}{K_n^{5/3}}}$$

$\Rightarrow$ this law agrees remarkably well with observation data in a variety of systems

$\Rightarrow$ wind, water

Dissipation Scales

At small enough spatial scales the dissipation rate becomes comparable to the energy transfer rate $\Rightarrow l_d$

$$\frac{V_d}{l_d} \sim \frac{\nu}{l_d^2} \qquad \frac{V_d}{\cancel{l_d}} \sim \nu \frac{1}{l_d^{\cancel{2}}}$$

$$\mathcal{E}^{1/3} l_d^{1/3} \sim \frac{\nu}{l_d}$$

$$l_d \sim \frac{\nu^{3/4}}{\epsilon^{1/4}} = \text{Kolmogorov micro scale}$$

By definition the local Reynold's number is unity at the dissipation ~~[illegible]~~ scale.

~~Note that the~~

Energy transfer ~~rate~~

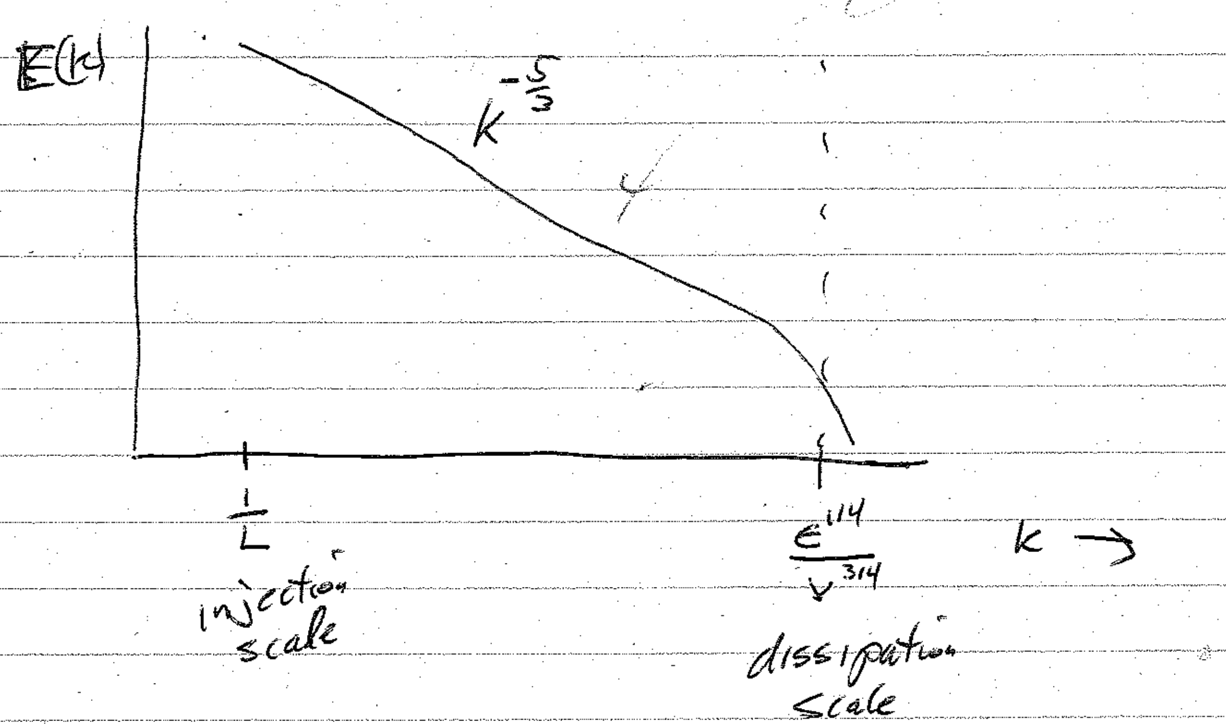
Note that ~~[illegible]~~ the energy transfer rate ϵ is governed by the energy ~~injected into the~~ and characteristic scale length of the injection scale.

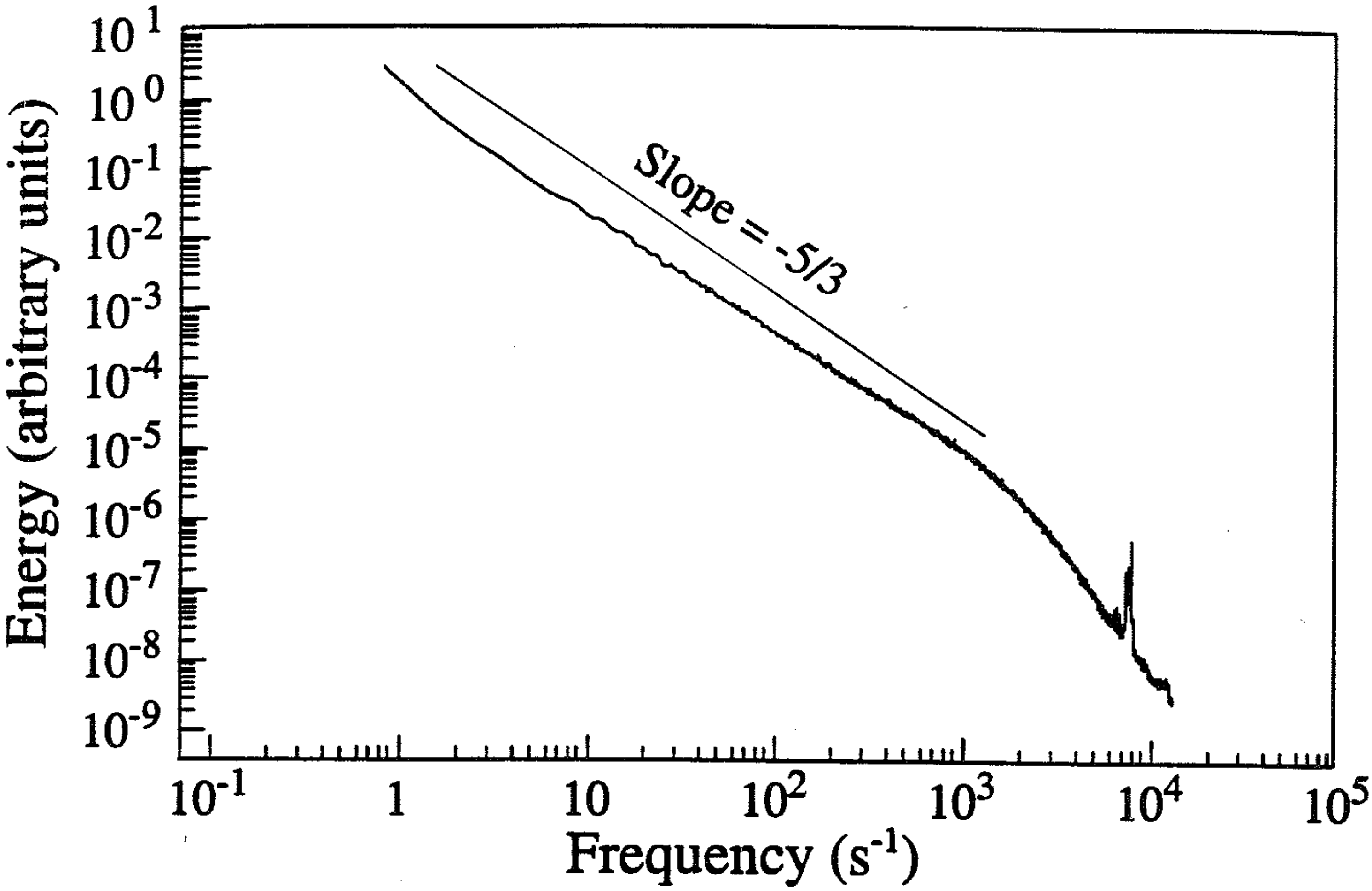
$$E_{in} \sim \epsilon^{2/3} L^{2/3}$$

so

$$\boxed{\epsilon = \frac{E_{in}^{3/2}}{L}}$$

~~[illegible] Turbulence~~





5.4. Energy spectrum in the time domain for data from S1. Reynolds number = 2720. Courtesy Y. Gagne and M. Marchand.

over a suitable range. The larger the Reynolds number, the wider range.

ig. 5.4 shows the energy spectrum for the best data obtained so far n the S1 wind tunnel. This is again a log–log plot. The horizontal axis frequency which can be reinterpreted as a wavenumber by use of the lor hypothesis. A power-law scaling k^{-n} with an exponent n close to is observed over a very substantial range of about three decades of enumber. This range is called the *inertial range*, a name which will

Invariants:

$$W = \int_V dx \tfrac{1}{2}|v|^2$$

$$\Omega \equiv \int_V dx\, \omega^2$$

2-D NS Turbulence

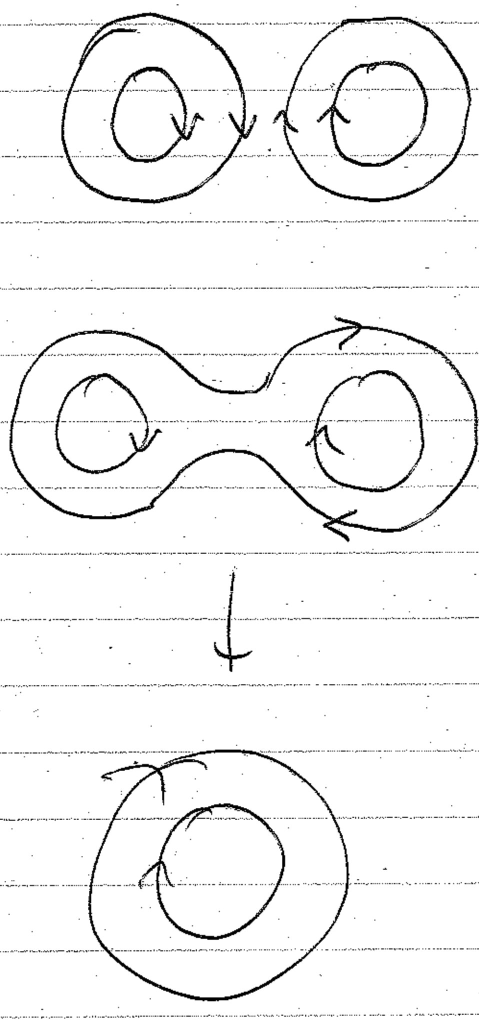
The nonlinearities in the 2-D NS equations differ greatly from 3-D

$\Rightarrow$ no vortex stretching.

$\Rightarrow$ convection of vorticity.

$\Rightarrow$ nonlinear behavior is dominated by the merger of like sign vortices

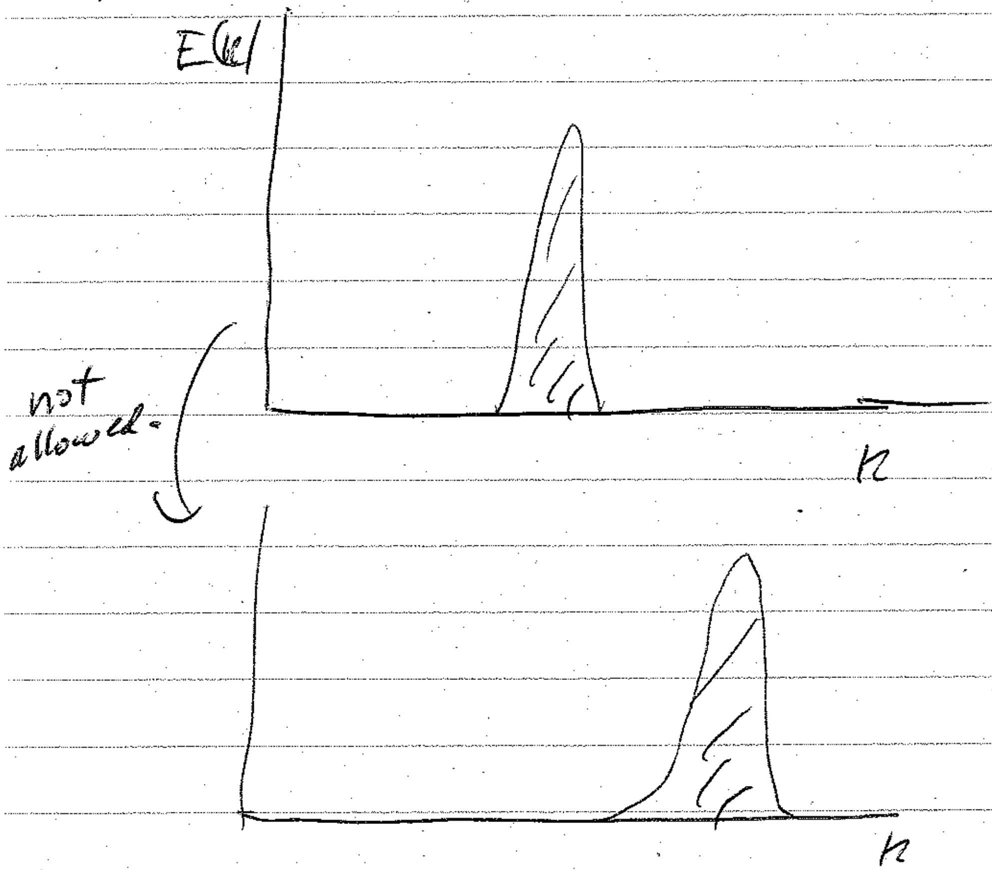
$\Rightarrow$ smaller and smaller number of vortices which interact only occasionally.



$\Rightarrow$ implication that transfer of energy to short scales is not operational

$\Rightarrow$ inverse cascade in 2-D NS.

The presence of two quadratic nonlinearities
$\Rightarrow$ energy and enstrophy forces a dual cascade
$\Rightarrow$ both invariants can't be satisfied in a simple cascade



$\sum_k k^2 E_k^2$ not preserved if $\sum_k E_k^2$ is preserved.

$\Rightarrow$ direct cascade of enstrophy

$\Rightarrow$ inverse cascade of energy

Scaling law for Enstrophy cascade

$\Rightarrow$ cascade rate of enstrophy ϵ_Ω

$$\epsilon_\Omega \approx \left(\frac{V_n^2}{l_n^2}\right)\frac{1}{\tau_n} \sim \frac{V_n^3}{l_n^3}$$

$$V_n \sim \epsilon_\Omega^{1/3}\, l_n$$

$$E_n \sim V_n^2 \sim \epsilon_\Omega^{2/3}\, l_n^2 \sim E(k)\frac{1}{l_n}$$

$$\boxed{E(k) \sim \epsilon_\Omega^{2/3}\frac{1}{k^3}}$$

Consider a source at an intermediate scale

$\Rightarrow$ cascade of energy to long scales

$\Rightarrow$ cascade of enstrophy to short scales

$\Rightarrow$