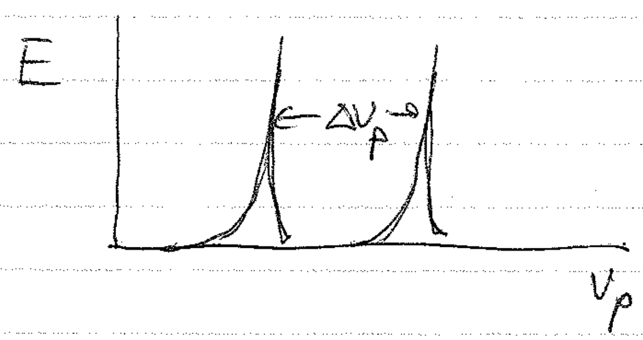
Irreversible processes and chaotic motion

To have an irreversible process must have ~~some~~ a resonant interaction between a particle and a wave. In quasilinear theory the resonance must be exact. $\Longrightarrow$ $V_p = V$

Particle trapping broadens the resonance.

A second form of broadening can also take place if the kick which a particle gets from a wave is ~~suff~~ sufficiently large. If have spectrum with discrete phase speeds



Quasilinear theory says no diffusion but if kick from one wave ΔV is sufficient for particle to resonate with other wave ΔV_p ie $\Delta V > \Delta V_p$, diffusion can take place.

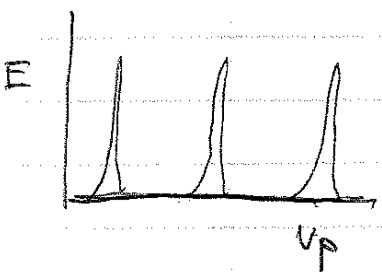
$\Longrightarrow$ particle motion becomes chaotic even in a discrete spectrum.

Chaotic Motion in a Periodic Spectrum: the Standard Map

Consider a system with a periodic distribution of phase velocities

$k =$ fixed

$\omega = n\,\Delta\omega \qquad n =$ integer



$$v_p = \frac{\omega}{k} = n\,\frac{\Delta\omega}{k}$$

$$E = \sum_{n=-\infty}^{\infty} E_n \sin kx \, e^{in\Delta\omega t}$$

take $E_n = E_0 = \text{const}$

$$E = E_0 \sin kx \sum_{n=-\infty}^{\infty} e^{in\Delta\omega t}$$

$\Rightarrow$ note periodic over $\tau = \frac{2\pi}{\Delta\omega}$

Over interval τ can represent

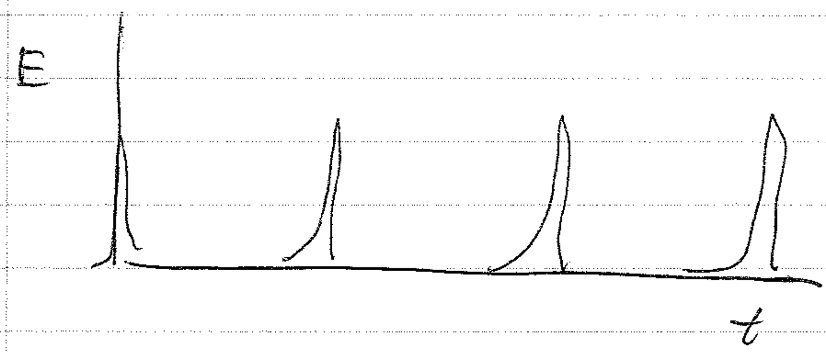
$$\delta(t) = \sum_{-\infty}^{\infty} a_n e^{+in\Delta\omega t}$$

$$\int_0^{\tau} dt\, \delta(t)\, e^{-in'\Delta\omega t} = a_{n'}\,\tau$$

$$a_n = \frac{1}{\tau} = \frac{\Delta\omega}{2\pi}$$

$$\delta(t) = \sum_{-\infty}^{\infty} \frac{\Delta\omega}{2\pi} e^{in\Delta\omega t}$$

$$E = E_0 \sin kx \, \frac{2\pi}{\Delta\omega} \sum_{m=-\infty}^{\infty} \delta\left(t - \frac{2\pi}{\Delta\omega} m\right)$$



$\Longrightarrow$ particle sees a series of kicks

<u>particle motion in this spectrum:</u>

$$\frac{dv}{dt} = \frac{q}{m_p} \frac{2\pi}{\Delta\omega} E_0 \sin kx \sum_m \delta\left(t - \frac{2\pi}{\Delta\omega} m\right)$$

$$\frac{dx}{dt} = v$$

<u>Map</u>

Let x_m, v_m be position and velocity of particle just before mth kick

$$v_{m+1} = \frac{q}{m_p} \frac{2\pi}{\Delta\omega} E_0 \sin(k x_m) + v_m$$

$$x_{m+1} = x_m + v_{m+1} \frac{2\pi}{\Delta\omega}$$

$\Longrightarrow$ since v_{m+1} is a constant over $\tau = \frac{2\pi}{\ldots}$

Define $\theta_m = k x_m$

$$p_m = k \frac{2\pi}{\Delta\omega} v_m$$

$$\boxed{\begin{aligned} p_{m+1} &= p_m + K \sin\theta_m \\ \theta_{m+1} &= \theta_m + p_{m+1} \end{aligned}}$$

Standard Map.

$$K = k \frac{q}{m_p} \left(\frac{2\pi}{\Delta\omega}\right)^2 E_0$$

$$= k \Delta x \qquad \Delta x = \frac{q}{m_p} E_0 \tau^2$$

effective displacement due to wave kick.

Surface of Section

plot points (θ_m, p_m) as map is iterated to study motion of particles

θ_m is periodic over 2π

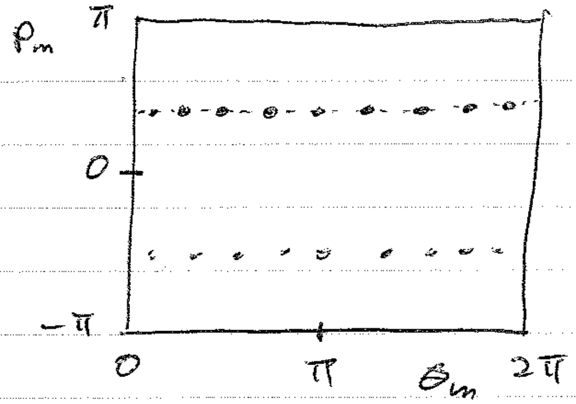
p_m?

Let $g_m = p_m + 2\pi$

$$g_{m+1} = g_m + K \sin\theta_m$$

$$\theta_{m+1} = \theta_m + g_{m+1} - \cancel{2\pi}$$

take (θ_m, p_m) mod 2π



For $K=0$

$$P_{m+1} = P_m = \text{const} = P_0$$
$$\theta_{m+1} = \theta_m + P_0$$

For $\frac{P_0}{2\pi}$ irrational dots fill out a line

For $\frac{P_0}{2\pi}$ rational

$$P_0 = 2\pi\left(\frac{s}{\omega}\right)$$

After ω iterations point returns to initial position θ_0

Fixed Points

For $K \neq 0$ there are points for which positions map to themselves

$$\theta_m = 0, \pi \quad (\text{or } 2\pi)$$
$$P_m = 0$$

$$P_{m+1} = 0 + K \sin\theta_m = 0$$
$$\theta_{m+1} = \theta_m + P_{m+1} = \theta_m$$

<u>stability of fixed pts:</u>

Let $\theta_m = \underbrace{0, \pi}_{\text{either}} + \delta\theta_m$

$P_m = \delta P_m$ where $\delta\theta_m$, δP_m are small

$$\delta P_{m+1} = \delta P_m + K\cos(0,\pi)\,\delta\theta_m$$

$$\delta\theta_{m+1} = \delta\theta_m + \delta P_m + K\cos(0,\pi)\,\delta\theta_m$$

$$\begin{pmatrix} \delta P_{m+1} \\ \delta\theta_{m+1} \end{pmatrix} = \begin{bmatrix} 1 & K\cos(0,\pi) \\ 1 & 1 + \underbrace{K\cos(0,\pi)}_{\bar{K}} \end{bmatrix} \begin{pmatrix} \delta P_m \\ \delta\theta_m \end{pmatrix}$$

equations homogeneous in m

$\Rightarrow$ solution of form

$$\delta P_{m+1} = \lambda\, \delta P_m$$

$\Rightarrow$ similar to exponential behavior for continuous system

$\Rightarrow$ dispersion equation for λ

$$\begin{vmatrix} 1-\lambda & \bar{K} \\ 1 & 1-\lambda+\bar{K} \end{vmatrix} = 0$$

$$(1-\lambda)^2 + (1-\lambda)\bar{K} - \bar{K} = 0$$

$$\lambda^2 - 2\lambda - \bar{K}\lambda + 1 + \cancel{\bar{K}} - \cancel{\bar{K}} = 0$$

$$\lambda^2 - (2+\bar{K})\lambda + 1 = 0$$

$$\lambda = \frac{2+\bar{K} \pm \sqrt{(2+\bar{K})^2 - 4}}{2}$$

$|\lambda| > 1 \Rightarrow$ unstable

$\bar{K} > 0 \Rightarrow$ unstable $\theta = 0$ fixed pt is unstable

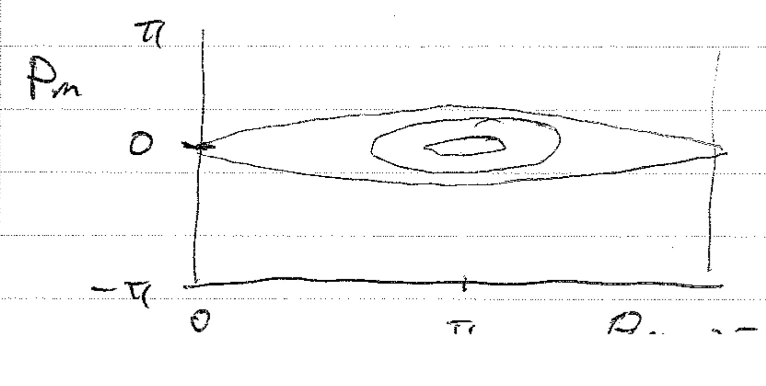
$\bar{K} < -4$ unstable

$-4 < \bar{K} < 0 \Rightarrow$ stable

$\theta = 0, 2\pi$ unstable for any K.

$\theta = \pi$ is a stable fixed pt for some range of K.

$\Rightarrow$ trajectories map out an island structure in the surface of section



island width for small K:

take increment in P_m, θ_m for a given iteration small

$\Longrightarrow$ back to continuous representation

$$\dot{P}_{\cancel{m}} = \frac{P_{m+1} - P_m}{1} = K \sin\theta$$

$$\dot{\theta} = \frac{\theta_{m+1} - \theta_m}{1} = P$$

$$\left.\begin{array}{l} \dot{P} = K\sin\theta \\ \dot{\theta} = P \end{array}\right\} \qquad \left(\ddot{\theta} = K\sin\theta\right)\dot{\theta}$$

$$\frac{\dot{\theta}^2}{2} + K\cos\theta = \text{const}$$

$\cancel{\ddot{\theta} = K\cos\theta}$

$\Longrightarrow$ separatrix $\quad \dot{\theta} = 0$ at $\theta = 0$

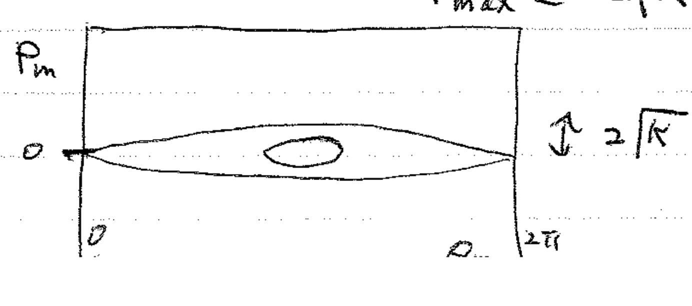
$$\frac{\dot{\theta}^2}{2} + K\cos\theta = K$$

$$\frac{\dot{\theta}^2_{max}}{2} = K(1-\cos\theta)_{max}$$

$$= 2K$$

$$\dot{\theta}_{max} = 2\sqrt{K}$$

$$P_{max} = 2\sqrt{K}$$



For $K \gtrsim 0.7$ chaotic orbits around separatrix,

http://order.ph.utexas.edu/standardmap/standardmap.html

KAM surface
⇒ ~~…~~ line bounding chaotic orbits
⇒ last surface is broken for $K \gtrsim 0.97$ John Greene
⇒ corresponds to single kick exceeding spacing of phase speeds of waves.
⇒ particles can reach arbitrary energies above this amplitude

Quasilinear diffusion for large K

Well above chaotic threshold want to calculate velocity diffusion rate

$$P_{m+1} = \sum_{\ell=0}^{m} K \sin\theta_\ell + \cancel{P_0}^{\,0}$$

$$P_{m+1}^2 = K^2 \sum_{\ell=0}^{m} \sum_{\ell'=0}^{m} \sin\theta_\ell \sin\theta_{\ell'}$$

$$\theta_1 = \theta_0 + K\sin\theta_0$$

$$\theta_2 = \cdots\cdots$$

average over θ_0, $\theta_1, \theta_2 \cdots$ will change a lot for small change in θ_0

$\theta_{\ell+1}$ is nearly decorrelated from θ_ℓ

$$\langle p_{m+1}^2 \rangle = K^2 \sum_{\ell=0}^{m} \sum_{\ell'=0}^{m} \underbrace{\langle \sin\theta_\ell \sin\theta_{\ell'} \rangle}_{\approx \frac{1}{2}\delta_{\ell\ell'}}$$

$$= \frac{1}{2} K^2 m$$

$$D \equiv \text{diffusion rate} \equiv \lim_{m\to\infty} \frac{\langle p_m^2 \rangle}{2m}$$

$$\boxed{D = \frac{1}{4} K^2}$$

$\Rightarrow$ dimensional units

$$D \sim V^2 \tau \sim \frac{1}{K^2}\frac{1}{\tau}$$

$$V \sim \frac{\Delta\omega}{2\pi}\frac{1}{K} \sim \frac{1}{K\tau}$$

$$D = \cancel{\frac{1}{K^2}} \cancel{\frac{\Delta\omega}{2\pi}} \frac{1}{4} \cancel{K^2} \frac{g^2}{mp^2} \left(\frac{2\pi}{\Delta\omega}\right)^{\cancel{4}\,3} E_0^2$$

$$\boxed{D = \frac{1}{4} \frac{g^2}{mp^2} \left(\frac{2\pi}{\Delta\omega}\right)^3 E_0^2}$$

Rechester - White PRL 44, 1586, 1980

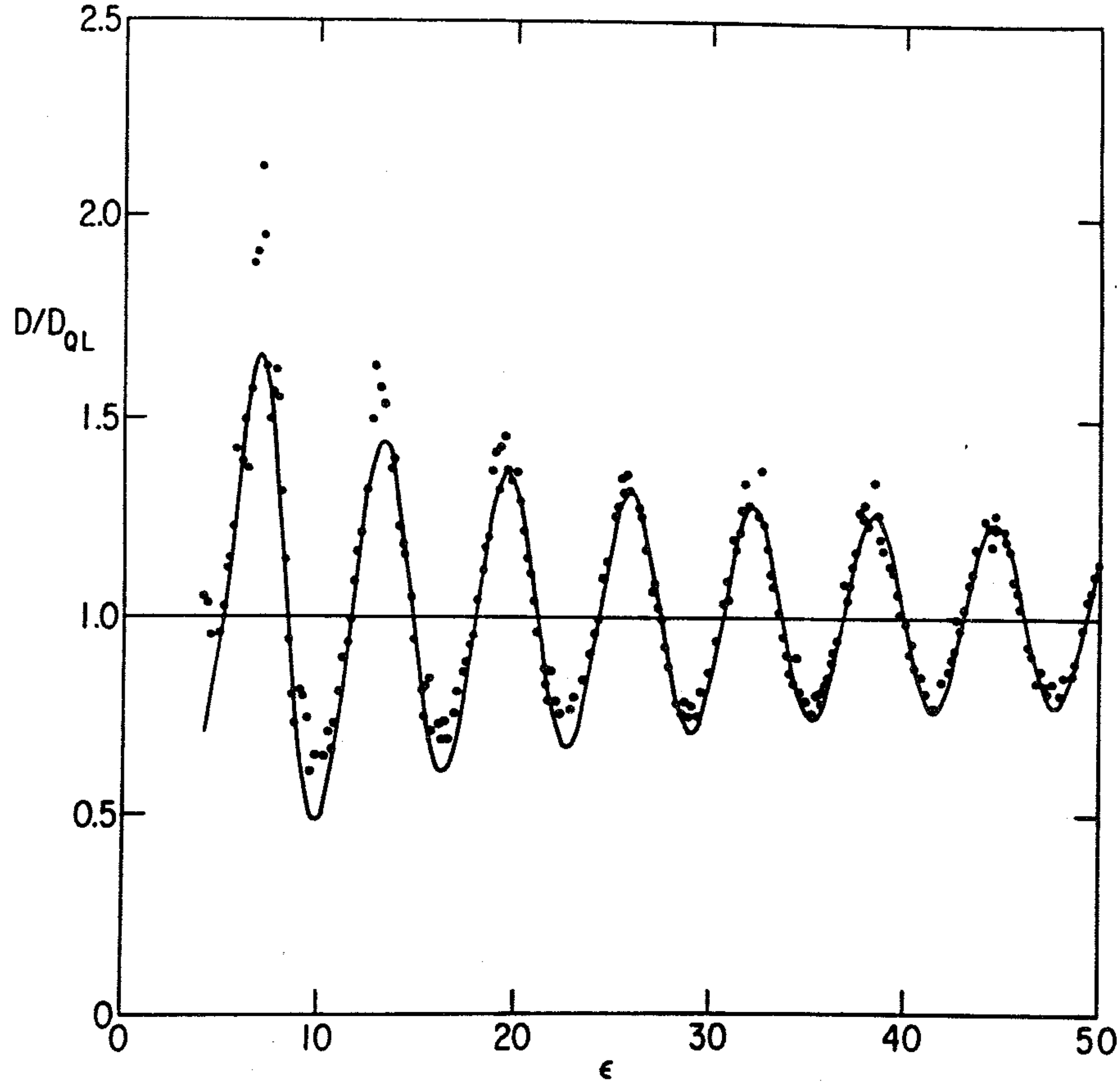


FIG. 1. The ratio of the numerically obtained diffusion to the quasilinear value as a function of ϵ. Here $\sigma = 10^{-5}$. Also plotted is the analytic expression given by Eq. (20).

We have made extensive numerical study of Eqs. (4) and (5), introducing a small random step in x with a normal distribution of mean σ. Some

Chaotic Behavior in Coherent Waves

Normally a coherent wave can produce trapping but the dynamic of the particles are not chaotic but deterministic. A magnetic field changes the behavior dramatically in some cases.

Consider an ~~tran~~ electric field transverse to a magnetic field.

$\Rightarrow$ linear theory says no damping unless $\omega = n\Omega$

$\Rightarrow$ no diffusion.

$\Rightarrow$ If $B \to 0$ might recover resonance with

$$\omega = \underset{\sim}{k} \cdot \underset{\sim}{v}$$

$\Rightarrow$ possibly have trapping

$\Rightarrow$ linear theory must break down. How?

equations of motion:

$$\frac{dv_x}{dt} = \frac{q}{m} E \cos(kx - \omega t) + v_y \Omega$$

$$\frac{dv_y}{dt} = -v_x \Omega = -\frac{dx}{dt}\Omega$$

$$v_y + x\Omega = \text{const} = v_{y0}$$

$$\ddot{x} = \frac{q}{m} E \cos(kx - \omega t) + (v_{y0} - x\Omega)\Omega$$

eliminate v_{y0} by shifting x and t

$$\xi = kx$$

$$\Omega t \to \tau$$

$$\ddot{\xi} + \xi = \alpha \cos(\xi - \nu\tau)$$

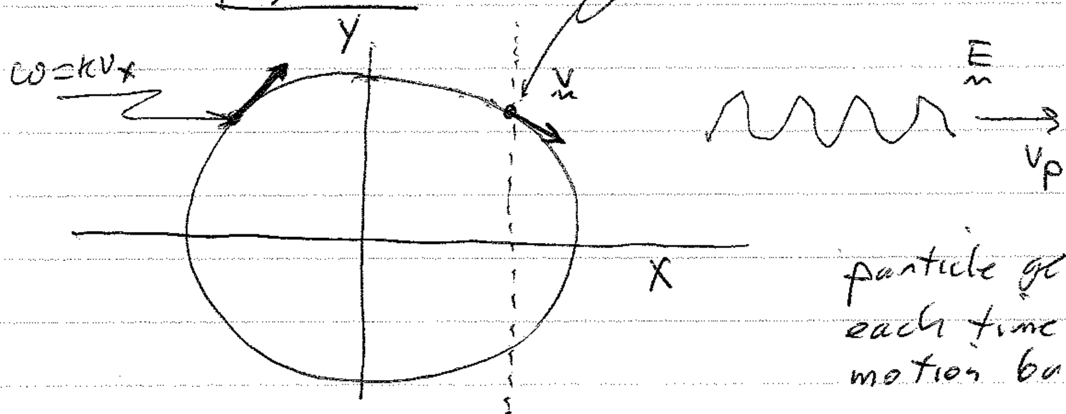
$$\alpha = \frac{kq}{\Omega^2 m} E, \quad \nu = \frac{\omega}{\Omega}$$

Have stochastic behavior for both $\nu < 1$ and $\nu > 1$

<u>$\nu > 1$</u> Karney, 1978

plasma heating with lower hybrid wave. $\nu \sim \sqrt{\frac{m_i}{m_e}}$

<u>physics</u>



particle gets kick each time cyclotron motion brings it in resonance with wave.

Motion stochastic when wave phase shifts significantly when comes around again.

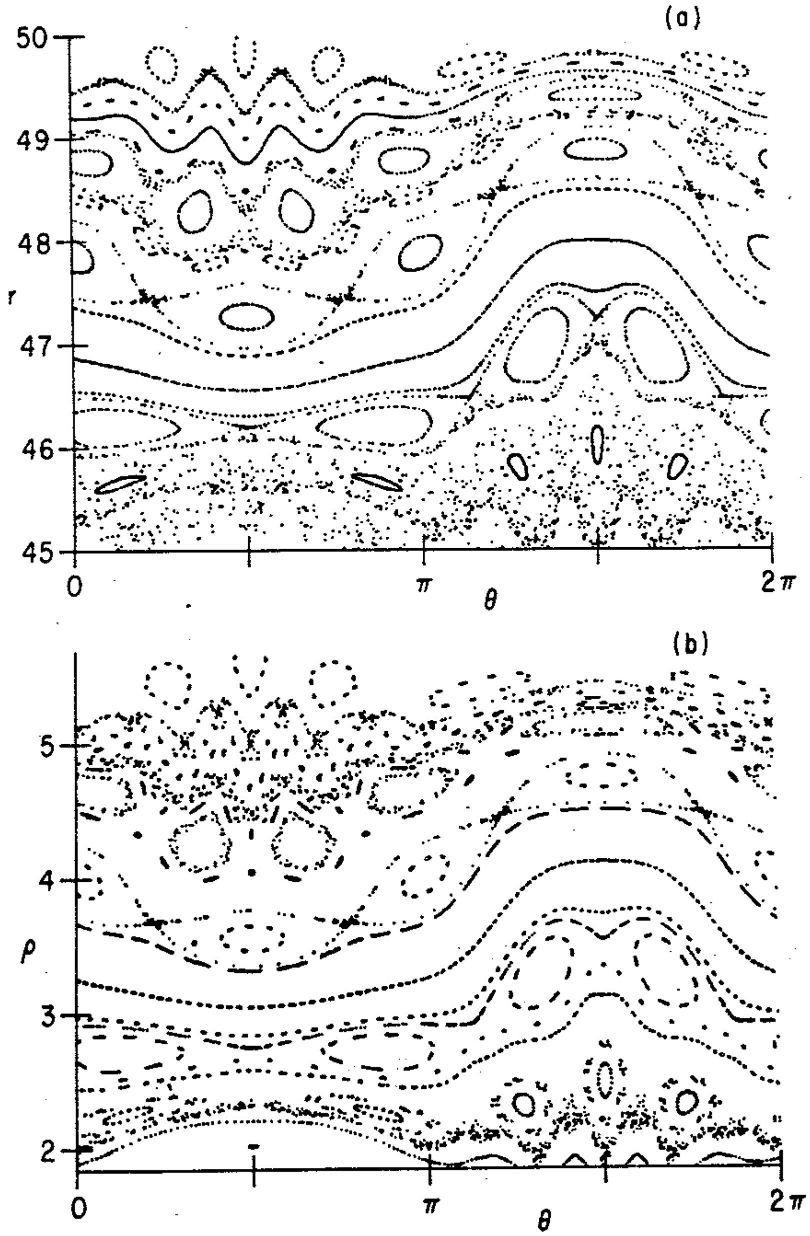


FIG. 2. Comparison of the difference equations, Eq. (16), with the Lorentz force law, Eq. (2). (a) The mapping of the (θ, r) plane using Eq. (2) with $\nu = 30.23$ and $\alpha = 2.2$. [This is taken from Fig. 3(b) of (I).] (b) The mapping of the (θ, ρ) plane under T using Eq. (16), with $\delta = 0.23$ and $A = 0.1424$, which is given by Eq. (14) with $\nu = 30.23$, $\alpha = 2.2$, and $r = 47.5$. In each case the trajectories of 24 particles are followed for 300 orbits. Thus, in (b) the points, $T^j(\theta_0, \rho_0)$ for $0 \leq j \leq 300$, are plotted for 24 different initial conditions, (θ_0, ρ_0).

$$F(u_{j+1}, v_j) = u_{j+1} v_j + 2\pi\delta(u_{j+1} - v_j) + 2\pi A(\sin u_{j+1} + \sin v_j)\,. \tag{21}$$