Wave Particle Interactions

Plasma particles can exchange energy with waves even in the absense of classical dissipative processes through resonance interactions with waves

$\Longrightarrow$ particles moving close to the phase speed of a wave effectively see a DC field and can therefore give or take energy from wave

$\Longrightarrow$ particles moving with very different velocities see an ocillatory field and there is typically no "irreversible" energy exchange.

The type of interaction takes on several forms. depending on the nature of the wave spectrum

~~(1) For a broad wave spectrum the~~

(1) For a narrow wave spectrum particles interact with a wave for a long time and can have particle trapping.

Consider a particle (electron) moving in the frame of the wave (wave potential is stationary) $\varphi(x)$

$-e \longrightarrow v$

$\varphi(x) = \varphi_0 \cos kx$

$\longrightarrow x$

In this frame the energy of the particle is conserved.

$$\tfrac{1}{2} m v^2 - e\varphi = \text{const} = E$$

$$\tfrac{1}{2} m v^2 = E + e\varphi = E + e\varphi_0 \cos kx$$

for $E < e\varphi_0$ particles are trapped. since $v^2 \Rightarrow 0$

~~Consider~~

What is the bounce time of deeply trapped particles? $E = -e\varphi_0 + \epsilon^2$

$$\tfrac{1}{2} m v^2 = E + e\,\varphi_0\left(1 - \tfrac{1}{2}k^2 x^2\right)$$

$$= \underbrace{(E + e\varphi_0)}_{\epsilon^2} - \tfrac{1}{2} e\varphi_0 k^2 x^2$$

$$m \cancel{v}\, \dot{v} = -\cancel{\tfrac{1}{2}}\, e\varphi_0 k^2 x \cancel{\dot{x}}$$

$$\ddot{x} + \frac{e\varphi_0 k^2}{m} x = 0 \qquad \boxed{\omega_B^2 = k^2 \frac{e\varphi_0}{m}}$$

What do the constant energy contours look like in the v, x phase space?

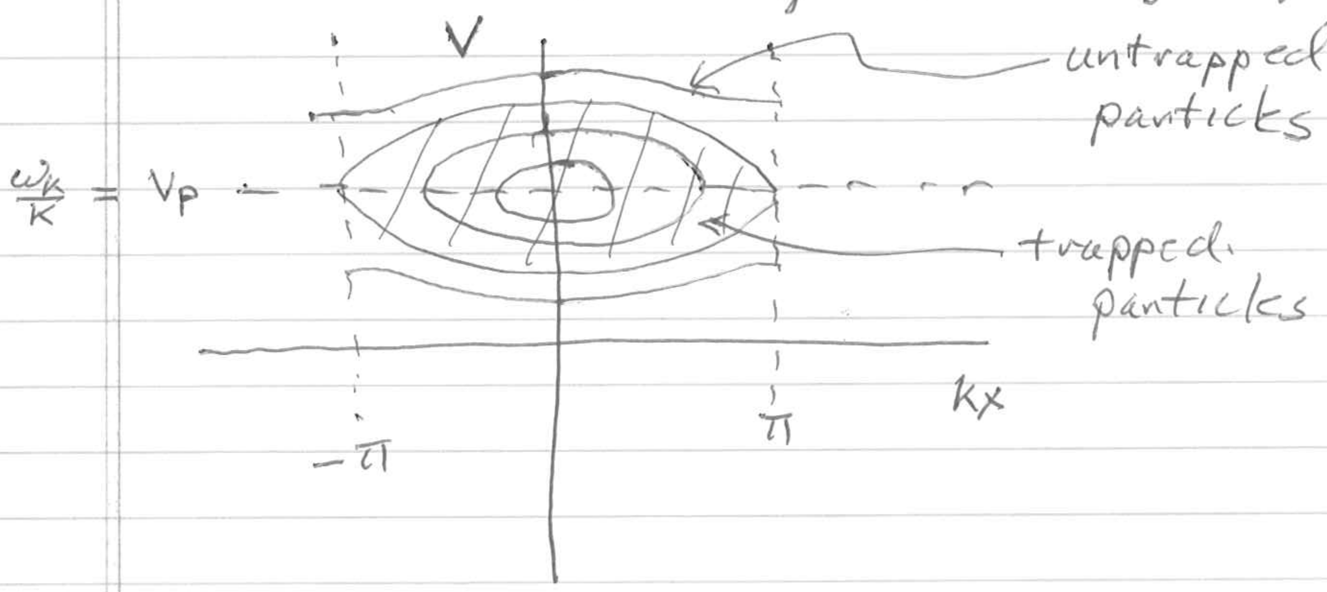
Near $x=0$:

$$E = \frac{1}{2}mv^2 - e\varphi_0\left(1 - \frac{1}{2}k^2x^2\right)$$

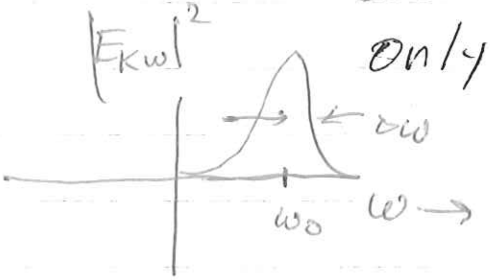
$$E + e\varphi_0 = \frac{1}{2}\left(mv^2 + k^2 e\varphi_0 x^2\right)$$

$$2\left(\frac{E + e\varphi_0}{m}\right) = v^2 + \frac{k^2 e\varphi_0}{m} x^2$$

$\Rightarrow$ ellipses in v, x space



(2) For a broader wave spectrum the particle sees many waves, and see's a coherent acceleration only for a time



$$\tau_c \sim \frac{2\pi}{\Delta\omega}$$

where $\Delta\omega$ is the spectral width.
If

$$\omega_B \tau_c \ll 1$$

then there will be no particle trapping and the energy exchange with the waves can be treated in using a statistical approach. Often the nonlinearities of the system can then be considered weak and one can expand the equations of motion in powers of the wave amplitude

$\Longrightarrow$ quasilinear theory

When there is particle trapping this does not work. Motion is then fully nonlinear.

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~~Quasi~~

Quasilinear Treatment of wave-particle interactions: bump-on-tail system

Consider a plasma with a weak electron beam with $n_b \ll n_0$, where n_b is the beam density and n_0 the background ion density.

⟹ ~~the~~ beam will drive ~~waves with~~ plasma waves

⟹ take ions to be stationary

Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \underbrace{\frac{e}{m}\frac{\partial \varphi}{\partial x}}\frac{\partial f}{\partial v} = 0$$

$$\dot{v} = -\frac{e}{m}E = \frac{e}{m}\frac{\partial \varphi}{\partial x}$$

$$\frac{\partial}{\partial x}E = 4\pi e\, n_0\left(1 - \frac{1}{n_0}\underbrace{\int dv\, f}_{\text{electrons}}\right)$$

$$\| $$

$$-\frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = -4\pi e\, n_0\left(1 - \frac{1}{n_0}\int dv\, f\right)$$

Linear theory

$$\varphi \sim \mathrm{Re}\left(\varphi_k \, e^{ikx} e^{-i\omega t}\right)$$

$$f = f_0(v,t) + \mathrm{Re}\left[f_k \, e^{ikx} e^{-i\omega t}\right]$$

$\Rightarrow$ allow $f_0(v,t)$ to vary in time slowly

$\Rightarrow$ linearizing in φ_k, f_k

$$(-\cancel{i}\omega + \cancel{i}kv) f_k + \frac{e}{m} \cancel{i} k \varphi_k \frac{\partial f_0}{\partial v} = 0$$

$$k^2 \varphi_k = 4\pi e \left(-\int dv \, f_k\right)$$

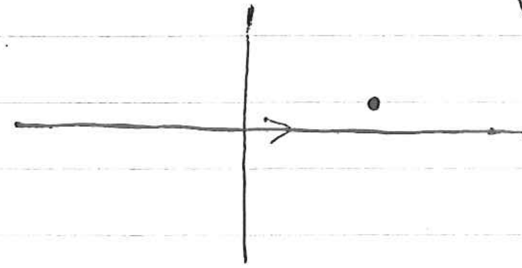
$$k^2 \varphi_k = 4\pi e \left(+\int dv \frac{\frac{e}{m} k \varphi_k \frac{\partial f_0}{\partial v}}{-\omega + kv}\right)$$

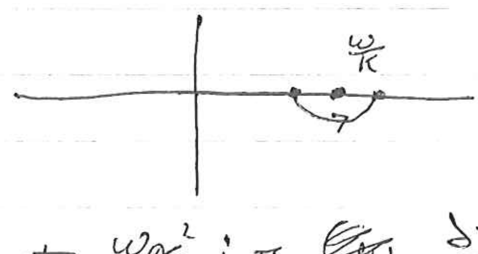
$$1 + \frac{\omega_{pe}^2}{k n_0} \int dv \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} = 0$$

growing modes

$\mathrm{Im}\,\omega > 0$

v plane





$$1 - \frac{\omega_{pe}^2}{k^2} \frac{P}{n_0} \int dv \frac{\frac{\partial f_0}{\partial v}}{v - \frac{\omega}{k}} - \frac{\omega_{pe}^2}{k^2} \frac{i\pi}{n_0} \left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega}{k}} = 0$$

take $\frac{\omega}{k} \gg v_{te}$ for background

$$1 - \frac{\omega_{pe}^2}{k^2} \frac{P}{n_0} \underbrace{\int dv \frac{f_0}{\left(v - \frac{\omega}{k}\right)^2}}_{\frac{k^2}{\omega^2} + v_{te} \text{ corrections}} - i\pi \frac{\omega_{pe}^2}{k^2} \frac{1}{n_0} \frac{\partial f_0}{\partial v}\bigg|_{\frac{\omega}{k}} = 0$$

$$1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + (\,)\right) - i\pi \frac{\omega_{pe}^2}{k^2} \frac{1}{n_0} \frac{\partial f_0}{\partial v}\bigg|_{\frac{\omega}{k}} = 0$$

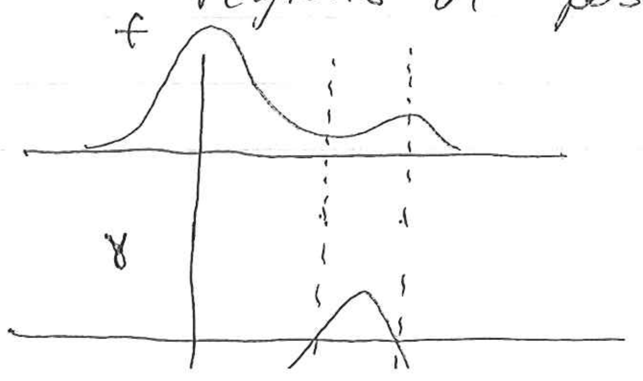
$$\omega_k^2 = \omega_{pe}^2 + 3 \frac{T_e k^2}{m_e}$$

growth rate

$$+ 2 \frac{\cancel{\omega_{pe}^2}}{\omega^3} \cancel{i} \gamma - \cancel{i} \pi \frac{\cancel{\omega_{pe}^2}}{k^2} \frac{\partial f_0}{\partial v}\bigg|_{\frac{\omega_k}{k}} = 0$$

$$\boxed{\gamma_k = \frac{\pi}{2} \frac{\omega_{pe}^3}{k^2} \frac{1}{n_0} \frac{\partial f_0}{\partial V}\bigg|_{\frac{\omega_k}{k}}}$$

$\Longrightarrow$ growth rate positive in regions of positive slope



quasilinear theory

Assume that the dominant nonlinear process occurs due to the resonant interaction of the waves with the particles

$\Rightarrow$ neglect wave-wave interactions

$\Rightarrow$ find equation for the uniform part of f

generally

$$\frac{\partial f_p}{\partial t} + i p v f_p + \frac{e}{m} i p \phi_p \frac{\partial f_0}{\partial v} + \frac{e}{m} i k \phi_k \frac{\partial}{\partial v} f_{p-k} = 0$$

$$\cancel{+ \frac{e}{m} i p \phi_p \frac{\partial}{\partial v} f_{k-p} = 0}$$

take $p = 0$

$$\frac{\partial f_0}{\partial t} + \frac{e}{m} i k \phi_k \frac{\partial}{\partial v} f_{-k} = 0$$

$$f_K = \frac{e}{m} k \varphi_K \frac{\partial f_0}{\partial v} \frac{1}{\omega_k - kv}$$

$$\frac{\partial f_0}{\partial t} + \frac{e}{m} \sum_k ik \varphi_k \frac{\partial}{\partial v} \left(\frac{e}{m} \varphi_k \frac{-k}{\omega_{-k} + kv} \right) \frac{\partial f_0}{\partial v}$$

From reality condition on φ

$$\omega_{-k} = -\omega_k^*$$

$$\varphi_{-k} = \varphi_k^* \qquad \omega_k \Rightarrow \omega_k^R + i\gamma_k$$

$$\underbrace{\varphi}_{\text{real function}} = \varphi_K e^{ikx} e^{-i\omega_K t} + \varphi_{-K} e^{-ikx} e^{-i\omega_{-k} t}$$

$$\boxed{\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0 = 0}$$

$$D = + \frac{e^2}{m^2} \sum_K \frac{|\varphi_K|^2 \, i k^2 \overbrace{(-\omega_k^R + kv - i\gamma_k)}^{\text{cancels during sum } k, -k}}{(-\omega_k^R + i\gamma_k + kv)(-\omega_k^R + kv - i\gamma_k)}$$

$$\boxed{D(v) = \frac{e^2}{m^2} \sum_k \frac{|\varphi_K|^2 k^2 \gamma_k}{\gamma_k^2 + (\omega_k^R - kv)^2}}$$

$$\frac{\partial}{\partial t} |\varphi_k|^2 = 2\gamma_k |\varphi_k|^2 \qquad \gamma_k = \frac{\pi}{2} \frac{\omega_{pe}^3}{k^2} \frac{1}{n_0} \frac{\partial f_0}{\partial v} \Big|_{\frac{\omega_k}{k}}$$

Conservation Laws

number density $n = \int dv\, f_0$

$$\frac{\partial}{\partial t}\int dv\, f_0 - \cancel{\int dv \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0}^{\,0} = 0$$

$$\frac{\partial}{\partial t}\int dv\, f_0 = 0 \Rightarrow n \text{ is constant}$$

momentum density

$$\frac{\partial}{\partial t}\int dv\, v f_0 - \underbrace{\int dv\, v \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0}_{} = 0$$

$$= \int dv\, D(v) \frac{\partial f_0}{\partial v}$$

$$\int dv\, D \frac{\partial f_0}{\partial v} = \frac{e^2}{m^2}\int dv \frac{|\phi_k|^2 k^2 \gamma_k}{\gamma_k^2 + (\omega_k - kv)^2} \frac{\partial f_0}{\partial v}$$

$$\frac{\gamma_k}{\gamma_k^2 + (\omega_k - kv)^2} = -\frac{(\operatorname{Im} \omega^*)}{(\omega - kv)(\omega^* - kv)}$$

$$= -\operatorname{Im} \frac{1}{\omega - kv}$$

$$\int dv \, \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} = -\frac{k}{\omega_{pe}^2} n_0$$ from dispersion relation.

$$\int dv \, D \frac{\partial f_0}{\partial v} = + \sum_k \mathrm{Im}\, k^2 |\mathcal{E}_k|^2 \left(+ \frac{k}{\omega_{pe}^2} \right) \frac{e^2}{m^2}$$

$= 0$ since is odd in k and is real.

Energy Density

$$\frac{\partial}{\partial t} \int dv \, v^2 f_0 - \underbrace{\int dv \, v^2 \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0}_{-2 \int dv \, v D \frac{\partial f_0}{\partial v}} = 0$$

$$\int dv \, v D \frac{\partial f_0}{\partial v} = \frac{e^2}{m^2} \int dv \sum_k \frac{|\mathcal{E}_k|^2 k^2 \gamma_k v}{\gamma_k^2 + (\omega_k - kv)^2} \frac{\partial f_0}{\partial v}$$

$$= -\frac{e^2}{m^2} \mathrm{Im} \int dv \sum_k |\mathcal{E}_k|^2 k \underbrace{\frac{kv}{\omega_k - kv}}_{k \left(\frac{kv - \omega + \omega}{\omega - kv} \right)} \frac{\partial f_0}{\partial v}$$

odd in k

$$= -\frac{e^2}{m^2} \mathrm{Im} \int dv \sum_k |\mathcal{E}_k|^2 \frac{k\omega}{\omega - kv} \frac{\partial f_0}{\partial v}$$

$$= + \frac{e^2}{m^2} \sum_k |\mathcal{E}_k|^2 \, \mathrm{Im}\, k\omega \frac{k}{\omega_{pe}^2} n_0$$

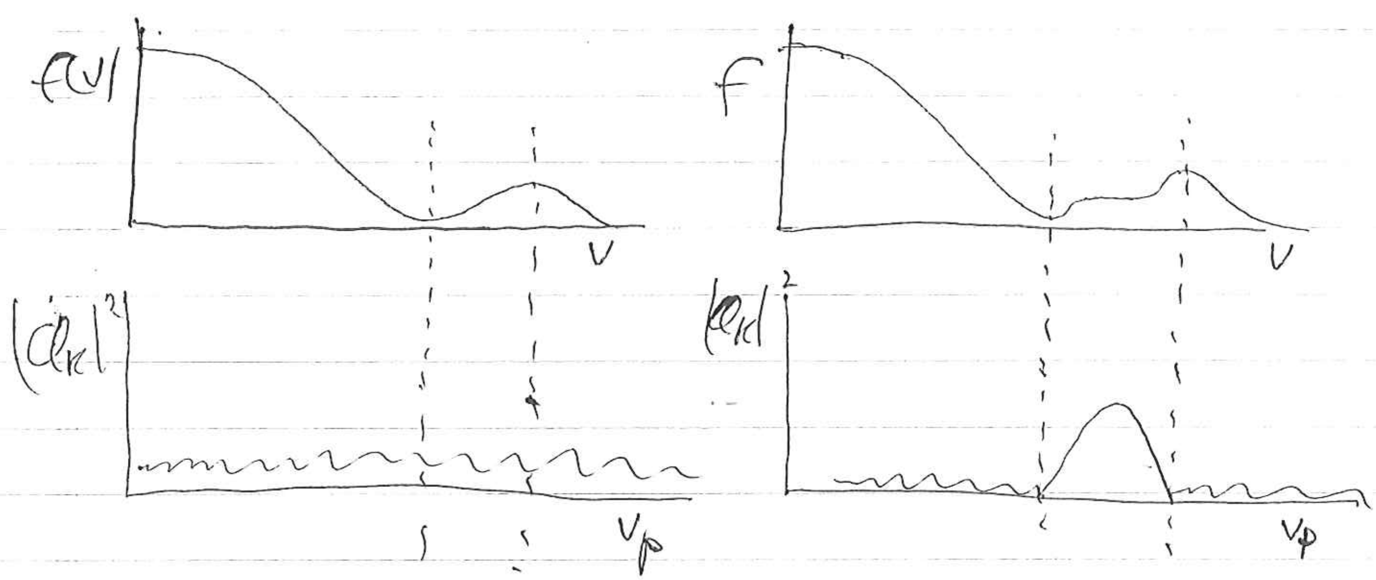
$$= \frac{e^2}{m^2} \frac{1}{\omega_{pe}^2} \sum k^2 |\mathcal{E}_k|^2 \gamma_k n_0$$

$$\frac{\partial}{\partial t}\int dv\, v^2 f_0 + \frac{e^2}{m^2}\frac{n_0}{\omega_{pe}^2}\frac{\partial}{\partial t}\sum_K k^2|\varphi_k|^2 = 0$$

$$\frac{\partial}{\partial t}\left(n_0\int dv\, \tfrac{1}{2}mv^2 f_0 + \frac{1}{8\pi}\sum_K |E_K|^2\right) = 0$$

field energy + particle energy
= const.

Evolution of system



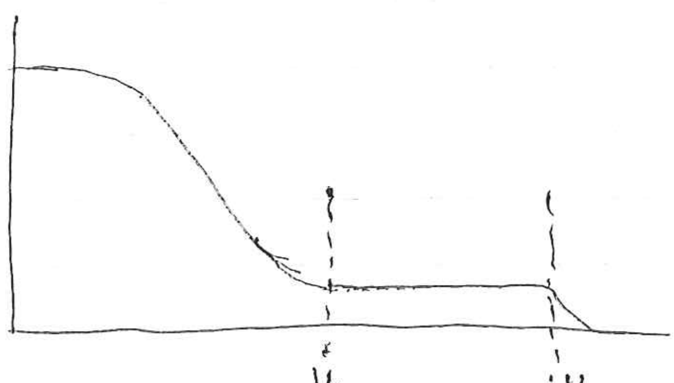
Resonant particles are dragged to smaller velocities, giving up energy to the waves

Steady state

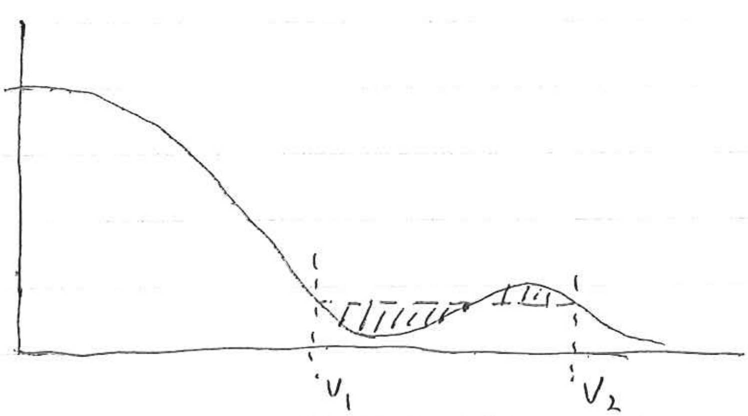
$$\frac{\partial f_0}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0 = 0$$

$$D \frac{\partial}{\partial v} f_0 = \text{const} = 0$$

$$\Rightarrow \frac{\partial f_0}{\partial v} = 0 \text{ in regions where } D \neq 0$$



Distribution flattened between V_1, V_2



Draw horizontal line between V_1 and V_2 on initial distribution. Must have equal areas above and below.

$\Rightarrow$ number conservation

$$f_0(v_1, t=0) \simeq f_0(v_2, t=0) = f_0(v_1 < v < v_2, t=\infty)$$
$$\equiv f_p$$

$$\int_{v_1}^{v_2} dv \, f(v, t=0) = f_p (v_2 - v_1)$$

$\Longrightarrow$ defines v_1, v_2

$\Longrightarrow$ energy spectrum of waves.

$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k k^2 |\phi_k|^2 \frac{\gamma_k}{\gamma_k^2 + (\omega_k - kv)^2} \frac{\partial f_0}{\partial v} = 0$$

$$\gamma_k = \frac{\pi}{2} \frac{\omega_{pe}^3}{k^2} \frac{1}{n_0} \frac{\partial f_0}{\partial v} \bigg|_{v_p}$$

Focus on resonant particles, taking γ_k small

$$\lim_{\gamma_k \to 0} \frac{\gamma_k}{\gamma_k^2 + (\omega_k - kv)^2} = \pi \, \delta(\omega_k - kv)$$

$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k k^2 |\phi_k|^2 \cancel{\pi} \, \delta(\omega_k - kv) \, \gamma_k \frac{2}{\cancel{\pi}} \frac{k^2}{\omega_{pe}^3} = 0$$

$$2\gamma_k |\phi_k|^2 = \frac{\partial}{\partial t} |\phi_k|^2$$

$$\frac{\partial}{\partial t} \left[f_0 - \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k \frac{k^4 |\phi_k|^2}{\omega_{pe}^3} \delta(\omega_k - kv) \right] = 0$$

$$f_0(v, 0) = f_p - \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k \frac{k^4 |\phi_k|^2}{\omega_{pe}^3} \delta(\omega_k - kv)$$

$$\sum_k k^4 |\varphi_k|^2\Big|_{t=\infty} \delta(\omega_k - k v) = \frac{m^2}{e^2}\frac{\omega_{pe}^3}{n_0}\int_{v_1}^{v} dv\,[f_p - f_0(v,0)]$$

Want to eliminate $\delta(\omega_k - kv)$ by carry out $\sum_k$

Discrete system: Length L

$$k = \frac{2\pi n}{L} \text{ with } n \text{ an integer}$$

$$\Delta k = \text{spacing of } k = \frac{2\pi}{L}$$

$$\frac{\Delta k L}{2\pi} = 1$$

$$\sum_k = \sum_k \frac{\Delta k L}{2\pi} = \int dk \frac{L}{2\pi}$$

$$\frac{L}{2\pi}\int dk\, k^4 |\varphi_k|^2\Big|_{t=\infty} \frac{\delta(\omega_k - kv)}{v}$$

$$= \frac{L}{2\pi}\frac{k^2 |E_k|^2_\infty}{v}\Big|_{k=\frac{\omega_{pe}}{v}}$$

$$\frac{L}{2\pi}\frac{\cancel{\omega_{pe}}}{v^3}\frac{|E_k|^2_\infty}{4\pi}\Big|_{k=\frac{\omega_{pe}}{v}} = \frac{m}{\cancel{\ }}\cancel{\omega_{pe}}\int_{v_1}^{v} dv\,[f_p - f_0(v,0)]$$

$$\frac{L}{2\pi}\frac{\omega_{pe}}{v}\frac{|E_k|^2_\infty}{4\pi}\Big|_{k=\frac{\omega_{pe}}{v}} = m v^2 \int_{v_1}^{v} dv\,[f_p - f_0(v,0)]$$

$$\frac{kL}{2\pi}\frac{|E_k|_\infty^2}{8\pi}\Bigg|_{k=\frac{\omega_{pe}}{v}} = \frac{mv^2}{2}\int_{v_1}^{v} dv\left[f_p - f_0(v,0)\right]$$

Note that $|E_k|_\infty^2 = 0$ for $k = \frac{\omega_{pe}}{v_1}$ and $\frac{\omega_{pe}}{v_2}$

Resonant versus non resonant diffusion

In exploring heating or diffusion of collisionless plasma it is important to distinguish between reversible and non-reversible processes.

$\Rightarrow$ consider nonresonant particles in bump-on-tail case

$$D(v) \simeq \frac{e^2}{m^2} \sum_k \frac{2\gamma_k |\alpha_k|^2 k^2}{2\,\omega_k^2}$$

$$= \frac{e^2}{m^2} \frac{\partial}{\partial t} \sum_k \frac{|\alpha_k|^2 k^2}{2\,\omega_k^2}$$

$$= \frac{\partial}{\partial t} \sum_k \frac{1}{2} |\tilde{v}_k|^2$$

$$\frac{\partial}{\partial t} \left(f_0 - \frac{\partial}{\partial v} \sum_k \frac{1}{2} |\tilde{v}_k|^2 \frac{\partial}{\partial v} f_0 \right) = 0$$

$\Rightarrow$ this is fake heating associated with the sloshing of nonresonant particles in unstable waves

$\Rightarrow$ this reverses as wave spectrum decays away.