Wave Particle Interactions

Plasma particles can exchange energy with waves even in the absence of classical dissipative processes through resonance interactions with waves

⟹ particles moving close to the phase speed of a wave effectively see a DC field and can therefore give or take energy from wave

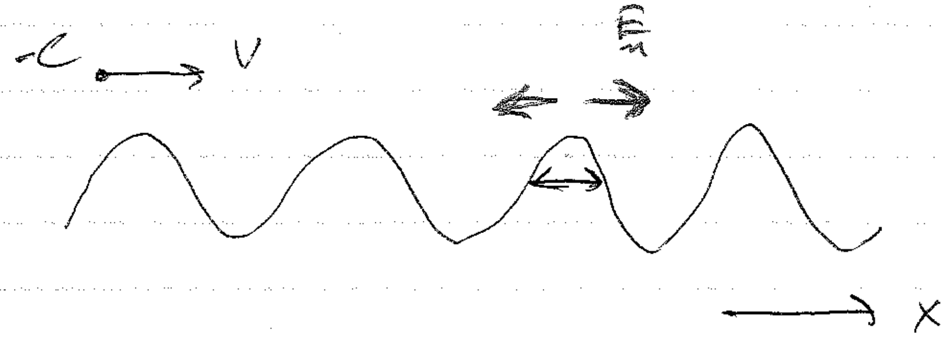
⟹ particles moving with very different velocities see an ocillutory field and there is typically no "irreversible" energy exchange.

The type of interaction takes on several forms. depending on the nature of the wave spectrum

~~① For a broad wave spectrum the~~

① For a narrow wave spectrum particles interact with a wave for a long time and can have particle trapping.

Consider a particle (electron) moving in the frame of the wave (wave potential is stationary) $\varphi(x)$



$\varphi(x) = \varphi_0 \cos kx$

In this frame the energy of the particle is conserved.

$$\frac{1}{2} m v^2 - e\varphi = \text{const} = E$$

$$\frac{1}{2} m v^2 = E + e\varphi$$

for $E < e\varphi_0$ particles are trapped.

What is the bounce time of deeply trapped particles?

$$\frac{1}{2} m v^2 = E + e\,\varphi_0\left(1 - \frac{1}{2}k^2x^2\right)$$

$$= \left(E + e\varphi_0\right) - \frac{1}{2}\, e\varphi_0 k^2 x^2$$

$$m\cancel{v}\dot{v} = -\cancel{\tfrac{1}{2}}\, e\varphi_0 k^2 x \cancel{\dot{x}}$$

$$\ddot{x} + \frac{e\varphi_0 k^2}{m}\, x = 0$$

$$\boxed{\omega_B^2 = \frac{k^2 e\varphi_0}{m}}$$

(2) For a broader wave spectrum the particle sees many waves and sees a coherent acceleration only for a time

$$\tau_c \sim \frac{2\pi}{\Delta\omega}$$

where $\Delta\omega$ is the spectral width. If

$$\omega_B \tau_c \ll 1$$

then there will be no particle trapping and the energy exchange with the waves can be treated in using a statistical approach. Often the nonlinearities of the system can then be considered weak and one can expand the equations of motion in powers of the wave amplitude

$\Longrightarrow$ quasilinear theory

When there is particle trapping this does not work. Motion is then fully nonlinear.

Quasilinear treatment of electromagnetic modes : see Sagdeev / Galeev
Davidson

Uniform magnetic field $\underline{B}_0 = B_0 \hat{z}$

initial distribution function

$$f_0 = f_0(v_\perp, v_\parallel, t)$$

For $T_\perp > T_\parallel$ have unstable whistlers

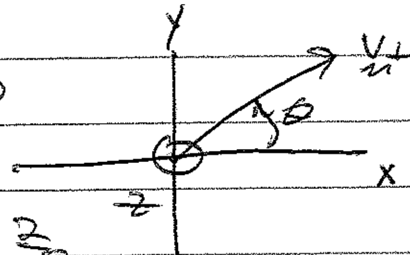
Assume parallel propagation

Linear theory:

$$f_1 = \mathrm{Re}\left(\hat{f}\, e^{ikz - i\omega t}\right)$$

$\underline{E}_1, \underline{B}_1$ transverse to $\underline{B}_0$

Faraday's Law



$$-\frac{i\omega \hat{\underline{B}}}{c} + ik\,\hat{z} \times \hat{\underline{E}} = 0$$

$$\hat{B} = \frac{kc}{\omega}\,\hat{z} \times \hat{E}$$

$$\frac{\partial}{\partial t} f_1 + v_z \frac{\partial}{\partial z} f_1 + \frac{eB_0}{mc} \overbrace{\underline{v} \times \hat{z} \cdot \frac{\partial}{\partial \underline{v}}}^{-v_\perp \frac{1}{v_\perp}\frac{\partial}{\partial \theta}} f_1$$

$$- \frac{e}{m}\left(\hat{\underline{E}} + \frac{1}{c}\,\underline{v} \times \hat{\underline{B}}\right) \cdot \frac{\partial}{\partial \underline{v}} f_0 = 0$$

$$\left(-i\omega + ikv_z + \Omega_0 \frac{\partial}{\partial \theta}\right)\hat{f} = \frac{e}{m}\left(\hat{\underline{E}} + \frac{1}{c}\,\underline{v} \times \hat{\underline{B}}\right) \cdot \frac{\partial}{\partial \underline{v}} f_0$$

$$\frac{1}{c}\underset{\sim}{v}\times \underset{\sim}{B}_1 = \cancel{\frac{1}{c}}\,\underset{\sim}{v}\times\left(\frac{kc}{\omega}\,\hat{z}\times \underset{\sim}{E}_1\right)$$

$$\left(\underset{\sim}{E}_1+\frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}_1\right)\cdot\frac{\partial}{\partial v}f_0 = 2\,\underset{\sim}{v}_\perp\cdot\underset{\sim}{E}_1\,\frac{\partial}{\partial v_\perp^2}f_0$$

$$+\cancel{\frac{1}{c}}\,\frac{k}{\omega}\,\underset{\sim}{v}_\perp\cdot\underset{\sim}{E}_1\,\frac{\partial}{\partial v_z}f_0 - \frac{k}{\omega}v_z\,2\,\underset{\sim}{v}_\perp\cdot\underset{\sim}{E}_1\,\frac{\partial}{\partial v_\perp^2}f_0$$

$$=\frac{1}{v_\perp}\,\underset{\sim}{v}_\perp\cdot\underset{\sim}{E}_1\left[\left(1-\frac{kv_z}{\omega}\right)\frac{\partial}{\partial v_\perp}f_0+\frac{kv_\perp}{\omega}\frac{\partial}{\partial v_z}f_0\right]$$

$$\underset{\sim}{E}_1\cdot\underset{\sim}{v}_\perp = v_\perp\left(E_{1x}\cos\theta+E_{1y}\sin\theta\right)$$

$$= v_\perp E_{1x}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+v_\perp E_{1y}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$$

$$= \frac{v_\perp}{2}e^{i\theta}\left(E_{1x}-iE_{1y}\right)+\frac{v_\perp}{2}e^{-i\theta}\left(E_{1x}+iE_{1y}\right)$$

$$\hat{E}_\pm = \hat{E}_x \mp i\hat{E}_y$$

$$\frac{v_\perp}{2}\left(e^{i\theta}\hat{E}_- + e^{-i\theta}\hat{E}_+\right)$$

$$\hat{f} = \hat{f}_+ e^{i\theta} + \hat{f}_- e^{-i\theta}$$

$$\hat{f}^\pm = \frac{e}{2m}\hat{E}_\mp\frac{\left[\left(1-\frac{kv_z}{\omega}\right)\frac{\partial}{\partial v_\perp}f_0+\frac{kv_\perp}{\omega}\frac{\partial}{\partial v_z}f_0\right]}{\cancel{i(\omega\pm\Omega_0)}\;-i(\bar{\omega}\mp\Omega_0)}$$

$$ik\,\hat{z}\times\left(\frac{kc}{\omega}\hat{z}\times\hat{E}\right) = \frac{4\pi}{c}\hat{\underset{\sim}{J}} - \frac{i\omega}{c}\hat{E}$$

$$\left(\frac{k^2c^2}{\omega^2}-1\right)\hat{\underset{\sim}{E}} = \frac{4\pi i}{\omega}\hat{\underset{\sim}{J}}$$

$$\hat{J} = -e\int d\underset{\sim}{v}\,\overbrace{\left(v_x\hat{x}+v_y\hat{y}\right)}^{v_\perp(\cos\theta\,\hat{x}+\sin\theta\,\hat{y})}\left(\hat{\epsilon}_+ e^{i\theta}+\hat{\epsilon}_- e^{-i\theta}\right)$$

$$\int d\underset{\sim}{v} = \int \cancel{d v_z\, v_\perp\, dv_\perp\, d\theta}$$

$$\hat{J} = -\frac{e}{2}\int d\underset{\sim}{v}\, v_\perp\, \hat{x}\left(\hat{\epsilon}_+ + \hat{\epsilon}_-\right) + \hat{y}\left(-\frac{\hat{\epsilon}_+}{i} + \frac{\hat{\epsilon}_-}{i}\right)$$

$$\hat{\underset{\sim}{J}} = -\frac{e}{2}\int d\underset{\sim}{v}\, v_\perp\left(\hat{\epsilon}_+(\hat{x}+i\hat{y}) + \hat{\epsilon}_-(\hat{x}-i\hat{y})\right)$$

$$\hat{J}_x + i\hat{J}_y = -\frac{e}{2}\int d\underset{\sim}{v}\, v_\perp\left[\cancel{\hat{\epsilon}_+} + \hat{\epsilon}_- + i(\cancel{i\hat{\epsilon}_+} - i\hat{\epsilon}_-)\right]$$

$$= -e\int d\underset{\sim}{v}\, v_\perp\,\hat{\epsilon}_- \sim \hat{E}_+$$

$$\left(\frac{k^2c^2}{\omega^2}-1\right)\hat{E}_+ = \frac{4\pi i}{\omega}\hat{J}_+$$

$$\frac{k^2c^2}{\omega^2}-1 = -\frac{4\pi i}{\omega}\frac{e^2}{2m}\int\frac{d\underset{\sim}{v}\,v_\perp}{-i}$$

$$\times\left[\frac{\left(1-\frac{kv_z}{\omega}\right)\frac{\partial}{\partial v_\perp}f_0 + \frac{kv_\perp}{\omega}\frac{\partial}{\partial v_z}f_0}{\omega\mp\Omega_0}\right]$$

$$\boxed{\frac{k^2c^2}{\omega^2}-1 - \frac{\omega_{pe}^2}{2\omega}\int d\underset{\sim}{v}\,v_\perp\left[\frac{\left(1-\frac{kv_z}{\omega}\right)\frac{\partial}{\partial v_\perp}\frac{f_0}{n_0} + \frac{kv_\perp}{\omega}\frac{\partial}{\partial v_z}\frac{f_0}{n_0}}{\omega\mp\Omega_0}\right] = 0}$$

Ordering for whistler:

consider $k^2 d_e^2 \sim 1$

$\omega \sim \Omega_e$

Compare: $\frac{k^2 c^2}{\omega^2}, 1$

$\frac{\omega_{pe}^2}{\Omega_e^2}, 1$

Take $\omega_{pe}^2 / \Omega_e^2 \gg 1 \Rightarrow$ discard displacement current

Whistler dispersion:

$$\frac{k^2 d_e^2}{\omega} = -\frac{1}{n_0}\int d\mathbf{v}\, \frac{v_\perp}{2} \frac{\left[\left(1-\frac{k v_z}{\omega}\right)\frac{\partial}{\partial v_\perp} f_0 + \frac{k v_\perp}{\omega}\frac{\partial}{\partial v_z} f_0\right]}{\omega - k v_z - \Omega_e}$$

$\Rightarrow$ kept lower sign which rotates in electron direction and resonates with electrons.

Consider weak damping:

$|\omega - \Omega_e| \sim \Omega_e \Rightarrow k v_t \sim \frac{v_t \omega_{pe}}{c}$

$\frac{v_t}{c} \ll \frac{\Omega_e}{\omega_{pe}}$

To lowest order

$$\frac{k^2 d_e^2}{\omega} = \frac{1}{2n_0} \int d\underset{\sim}{v} \frac{v_\perp \frac{\partial}{\partial v_\perp} f_0}{\omega - \Omega_e}$$

$$\int d\underset{\sim}{v}\, v_\perp \frac{\partial}{\partial v_\perp} f_0 = \int 2\pi v_\perp dv_\perp dv_z\, v_\perp \frac{\partial}{\partial v_\perp} f_0$$

$$= -2\int 2\pi v_\perp dv_\perp dv_z f_0$$

$$= -2\int d\underset{\sim}{v} f_0 = -2n_0$$

$$\frac{k^2 d_e^2}{\omega} = -\frac{1}{\omega - \Omega_e} \qquad (\omega - \Omega_e) k^2 d_e^2 = -\omega$$

$$\boxed{\omega = \frac{\Omega_e k^2 d_e^2}{1 + k^2 d_e^2}}$$ whistlers

~~Write dispersion relation as~~

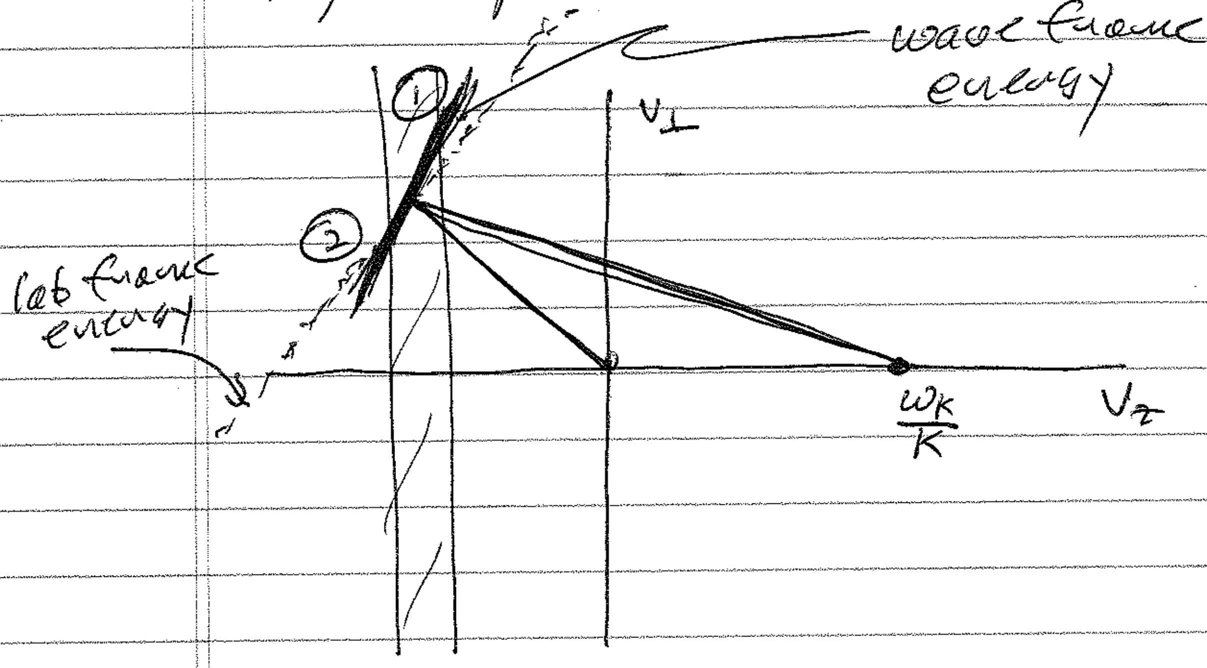
$$\cancel{\frac{\omega - \Omega_e}{\omega} k^2 d_e^2 = \frac{\omega - \Omega_e}{n_0} \int d\underset{\sim}{v} \frac{\frac{v_\perp}{2}[\quad]}{\omega - k v_z - \Omega_e}}$$

~~note for small v_z RHS is indep of ω~~

$$\cancel{\frac{\omega - \Omega_e}{\omega} k^2 d_e^2 = -1 +}$$

Want to keep corrections due to resonant contributions.

Physical picture of whistler



resonance: $\omega - kv_z = \Omega_e$

$\Rightarrow kv_z = \omega - \Omega_e < 0$

in wave frame energy of particle is conserved

$\Rightarrow$ no E field

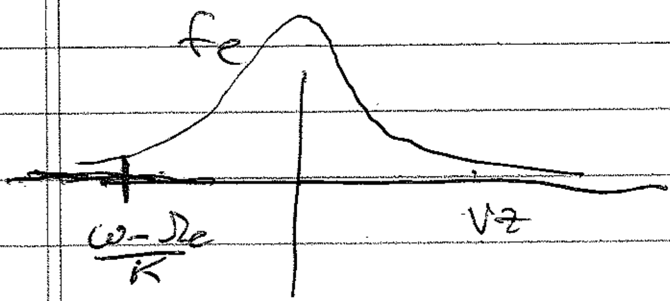
Particle motion ① to ② in wave frame
is constant energy
In lab frame particle energy reduced.
If more particles at ① than ②
have net energy reduction in lab
frame

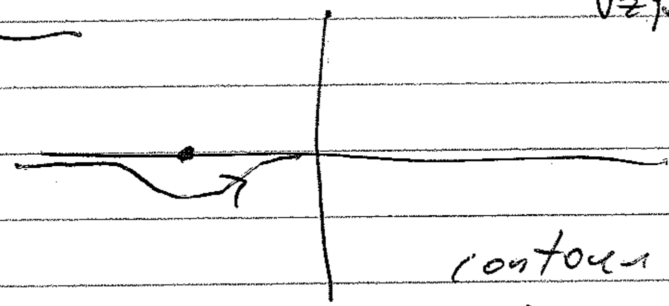
$\Rightarrow$ instability

$\Rightarrow$ requires $T_\perp > T_\parallel$

Res

$$\frac{K^2 d_e^2}{\omega} = - \frac{1}{\omega-\Omega_e} + \frac{1}{2n_0}\int dv \frac{v_\perp [\;]}{-K\left(v_z - \frac{\omega-\Omega_e}{K}\right)}$$





$$\frac{K^2 d_e^2}{\omega} = -\frac{1}{\omega-\Omega_e} \cancel{+} \frac{i\pi}{2n_0K}\int 2\pi v_\perp dv_\perp v_\perp [\;]\Big|_{\frac{\omega-\Omega_e}{K}}$$

$$K^2 d_e^2\left(\frac{\omega-\Omega_e}{\omega}\right) = -1 \underbrace{- \frac{i\pi}{2n_0K}(\omega-\Omega_e)\int 2\pi v_\perp dv_\perp v_\perp [\;]\frac{1}{\frac{\omega-\Omega_e}{K}}}_{\text{small}}$$

lowest order

$$K^2 d_e^2 \frac{\omega_K-\Omega_e}{\omega_K} = -1$$

first order

$$K^2 d_e^2 \frac{\Omega_e}{\omega_K^2} i\gamma = -i\frac{\pi(\omega_K-\Omega_e)}{2n_0K}\int 2\pi v_\perp^2 dv_\perp [\;]\Big|_{\frac{\omega_K-\Omega_e}{K}}$$

$$\Omega_e\left(\frac{\cancel{k^2 d_e^2} - 1 \cancel{+k^2 d_e^2}}{1+k^2 d_e^2}\right)$$

$$\cancel{\frac{k^2 d_e^2 \Omega_e}{\omega_k^2}} \; \gamma = -\frac{\pi\left(\omega_k - \Omega_e\right)}{2\, \cancel{n_0} k} \cancel{\omega_k} \frac{\cancel{k^2 d_e^2 \Omega_e}}{1+k^2 d_e^2}$$

$$\times \int \frac{2\pi v_\perp^2\, dv_\perp}{n_0}\left[\frac{k v_\perp}{\cancel{\omega_k}} \frac{\partial f_0}{\partial v_z} + \frac{\Omega_e}{\cancel{\omega_k}} \frac{\partial}{\partial v_\perp} f_0\right]$$

$$\gamma = \frac{\pi\, \Omega_e}{2\left(1+k^2 d_e^2\right)^2 k} \int_0^\infty \frac{2\pi v_\perp^2\, dv_\perp}{n_0}\left[k v_\perp \frac{\partial f_0}{\partial v_z} + \Omega_e \frac{\partial}{\partial v_\perp} f_0\right]_{v_z = \frac{\omega_k - \Omega_e}{k}}$$

$$\int d\underset{\sim}{v}\, v_\perp^2 f_0 = \frac{2}{m} \int d\underset{\sim}{v}\, \frac{1}{2} m v_\perp^2 f_0$$

$$T_\perp = \frac{1}{2} m v_{t\perp}^2 \qquad = \frac{2}{m}\, 2\left(\frac{T_\perp}{2}\right) n_0 = n_0 v_{t\perp}^2$$

$$\cancel{\int_0^\infty dv_\perp \int d\underset{\sim}{v} f_0 v_\perp = \int 2\pi v_\perp dv_\perp dv_z f_0}$$

$$\int d\underset{\sim}{v}\, v_\perp \frac{\partial f_0}{\partial v_\perp} = -2 n_0$$

$$\gamma = \frac{\pi\, \Omega_e}{2\left(1+k^2 d_e^2\right)^2 k}\left[k\, v_{t\perp}^2 \frac{\partial \bar{f}_0(v_z)}{\partial v_z} - \Omega_e\, 2 \bar{f}_0(v_z)\right] \qquad v_z = \frac{\omega - \Omega_e}{k}$$

define $\cancel{f_0 =}$ $$\bar{f}_0(v_z) = \frac{1}{\sqrt{\pi}\, v_{tz}} e^{-\frac{v_z^2}{v_{tz}^2}}$$

$$\gamma = \frac{\pi \Omega_e}{2k(1+k^2\lambda_e^2)^2}\Big[\underbrace{\cancel{k} V_{t\perp}^2\Big[-\frac{2}{V_{t\parallel}^2}\Big(\frac{\omega_k-\Omega_e}{k}\Big)\Big]-2\Omega_e}\Big]\bar{f}_0\Big|_{\frac{\omega_k-\Omega_e}{k}}$$

$$-2\Omega_e\Big[1-\frac{T_\perp}{T_\parallel}\frac{[redacted]}{1+k^2\lambda_e^2}\Big]\bar{f}_0\Big(\frac{\omega_k-\Omega_e}{k}\Big)$$

$$\gamma = -\frac{\pi\Omega_e^2}{[1+k^2\lambda_e^2]^2}\frac{1}{k}\Big[1-\frac{T_\perp}{T_\parallel}\frac{[redacted]}{1+k^2\lambda_e^2}\Big]\bar{f}_0\Big(\frac{\omega_k-\Omega_e}{k}\Big)$$

$\Rightarrow$ unstable for

$$\frac{[redacted]}{1+k^2\lambda_e^2}\frac{T_\perp}{T_\parallel} > 1 \quad \Rightarrow T_\perp > T_\parallel$$

Quasilinear theory

Assume that the dominant nonlinear process is from the resonant interaction of waves with particles

$\Rightarrow$ secular changes in f_0

$\Rightarrow$ find equation for spatially uniform component of $f(\underset{\sim}{x},\underset{\sim}{v},t)$

$\Rightarrow f_0(\underset{\sim}{v},t)$

$$\frac{\partial f_0}{\partial t} + \Omega_e\frac{\partial}{\partial\theta}f_0 = \frac{e}{m}\sum_k\Big(\hat{E}_{-k}+\frac{1}{c}\underset{\sim}{v}\times\hat{B}_{-k}\Big)\cdot\frac{\partial}{\partial\underset{\sim}{v}}\hat{f}_{+k}$$

Only want components of f_0 indep. of θ. $=0$

recall that $\hat{f}_k \sim \hat{f}_+ e^{i\theta}$

$$\left(\hat{E}_{-k} + \frac{1}{c} v \times \hat{B}_{-k}\right) \cdot \frac{\partial}{\partial v} \hat{f}_+ e^{i\theta}$$

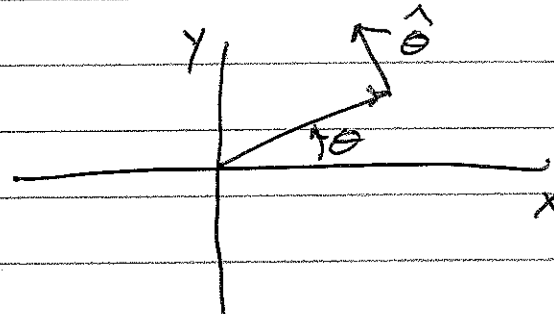
$$\hat{E}_{-k} + \frac{1}{c}\frac{(-k)c}{\omega_{-k}} v \times (\hat{z} \times E_{-k}) \qquad \omega_{-k} = -\omega_k^*$$

$$\hat{E}_{-k} = \hat{E}_k^*$$

$$\hat{E}_{-k}\left(1 - \frac{k v_z}{\omega_k^*}\right) + \hat{z}\frac{k}{\omega_k^*} \hat{E}_{-k} \cdot v$$

$$\frac{k}{\omega_k^*} \hat{E}_k^* \cdot v_\perp \frac{\partial}{\partial v_z} \hat{f}_+ e^{i\theta}$$

$$+ \underbrace{\left(1 - \frac{k v_z}{\omega_k^*}\right) \hat{E}_k^* \cdot \frac{\partial}{\partial v} \hat{f}_+ e^{i\theta}}_{\mathcal{L}}$$



$$v_\perp \cdot \hat{E}_k = \frac{v_\perp}{2}\left(e^{i\theta}\hat{E}_- + e^{-i\theta}\hat{E}_+\right)$$

$$\frac{k}{\omega_k^*}\frac{v_\perp}{2}\cancel{e^{-i\theta}}\, \hat{E}_- \frac{\partial}{\partial v_z} \hat{f}_+ \cancel{e^{i\theta}}$$

$$\mathcal{L} = \frac{\hat{E}_k^* \cdot v_\perp}{v_\perp} \frac{\partial}{\partial v_\perp} \hat{f}_+ e^{i\theta} + \frac{\hat{E}_k^* \cdot \hat{\theta}}{v_\perp} \hat{f}_+ (i) e^{i\theta}$$

$$= \frac{1}{2} \hat{E}_-^* \frac{\partial}{\partial v_\perp} \hat{f}_+ + \frac{i}{v_\perp} \hat{f}_+ e^{i\theta}\left(-\hat{E}_{kx}^* \sin\theta + \hat{E}_{ky}^* \cos\theta\right)$$

$$= \frac{1}{2} \hat{E}_-^* \frac{\partial}{\partial v_\perp} \hat{f}_+ + \frac{i}{v_\perp} \hat{f}_+ e^{i\theta}\left(-\hat{E}_{kx}^* \frac{(-e^{-i\theta})}{2i} + \hat{E}_{ky}^* \frac{e^{-i\theta}}{2}\right)$$

$$f = \frac{1}{2}\hat{E}_-^* \frac{\partial}{\partial v_\perp}\hat{f}_+ + \frac{1}{v_\perp}\frac{\hat{f}_+}{2}\Big(\underbrace{\hat{E}_{kx}^* + i\hat{E}_{ky}^*}_{\hat{E}_-^*}\Big)$$

$$= \frac{1}{2}\hat{E}_-^*\left(\frac{\partial}{\partial v_\perp}\hat{f}_+ + \hat{f}_+\frac{1}{v_\perp}\right)$$

$$= \frac{1}{2}\hat{E}_-^* \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp \hat{f}_+$$

$$\frac{\partial}{\partial t} f_0 - \frac{e}{m} \sum_k \frac{1}{2} \left[\hat{E}_-^* \frac{k v_\perp}{\omega_k^*} \frac{\partial}{\partial v_z} \hat{f}_+ + \left(1 - \frac{k v_z}{\omega_k^*}\right) \hat{E}_-^* \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp \hat{f}_+ \right] = 0$$

$$\frac{\partial}{\partial t} f_0 - \frac{e}{2m} \sum_k \hat{E}_-^* \left[\frac{k v_\perp}{\omega_k^*} \frac{\partial}{\partial v_z} + \left(1 - \frac{k v_z}{\omega_k^*}\right) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp \right] \hat{f}_+ = 0$$

$$\frac{\partial f_0}{\partial t} - \frac{e^2}{4m^2} \sum_k |\hat{E}_-|^2 \left[\frac{k v_\perp}{\omega_k^*} \frac{\partial}{\partial v_z} + \left(1 - \frac{k v_z}{\omega_k^*}\right) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp \right]$$

$$\otimes \quad \frac{\left[\left(1 - \frac{k v_z}{\omega_k}\right) \frac{\partial}{\partial v_\perp} + \frac{k v_\perp}{\omega_k} \frac{\partial}{\partial v_z} \right] f_0}{-i(\bar{\omega}_k - \Omega_e)} = 0$$

sum over $\pm k$ with γ_k small to consider resonant particles.

Note: have number conservation.

$$\frac{1}{-i\,2} \left(\frac{1}{\bar{\omega}_k - \Omega_e} + \frac{1}{\bar{\omega}_{-k} + \Omega_e} \right)$$

$$= \frac{1}{-i\,2} \left(\frac{1}{\bar{\omega}_k - \Omega_e} + \frac{1}{-(\bar{\omega}_k^* - \Omega_e)} \right)$$

$$= \frac{1}{-i\,2} \frac{\omega_k^* - \cancel{\Omega_e} - \omega_k + \cancel{\Omega_e}}{|\bar{\omega}_k - \Omega_e|^2} = \frac{\gamma}{|\bar{\omega}_k - \Omega_e|^2}$$

$$\Rightarrow \pi \delta(\bar{\omega}_k - \Omega_e)$$

for resonant particles

$$\frac{\partial f_0}{\partial t} - \frac{e^2}{4m^2}\sum_k |\hat{E}_-|^2 \left[\frac{k v_\perp}{\omega_k}\frac{\partial}{\partial v_z} + \left(1 - \frac{k v_z}{\omega_k}\right)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp\right]$$

$$\otimes\ \delta\left(\omega_k - k v_z - \Omega_e\right)\pi$$

$$\otimes\ \left[\left(1 - \frac{k v_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{k v_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right] f_0 = 0$$

If the spectrum of waves is not too broad, we can argue that the distribution will evolve until

$$\left[\left(1 - \frac{k v_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{k v_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right] f_0 = 0$$

$\Rightarrow$ equivalent to $\frac{\partial f}{\partial v} = 0$ for beam-plasma system.

$\Rightarrow$ also causes the growth rate to go to zero

In steady state the ~~ve~~ level curves of f_0 are ~~give~~ ~~functions~~ given by

$$\underbrace{\frac{1}{2}\left[v_\perp^2 + \left(v_z - \frac{\omega_k}{k}\right)^2\right]}_{H} = \text{const}$$

$f_0 = f_0(H)$ within the resonant region

$$\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial H}{\partial v_\perp}+\frac{kv_\perp}{\omega_k}\frac{\partial H}{\partial v_z}\right]\frac{\partial f_0}{\partial H}=0$$

$$\left(1-\frac{kv_z}{\omega_k}\right)\left(\cancel{v_\perp}\right)+\frac{k\cancel{v_\perp}}{\omega_k}\left(v_z-\frac{\omega_k}{k}\right)=0$$

$$\left(1-\frac{kv_z}{\omega_k}\right)+\left(\frac{kv_z}{\omega_k}-1\right)=0 \quad \text{ok}$$

$\Rightarrow$ this means that in the frame of the wave $\quad v_z'=v_z-\frac{\omega_k}{k}$

$$v_\perp^2+v_z'^2=\text{const}$$

$\Rightarrow$ energy is conserved.

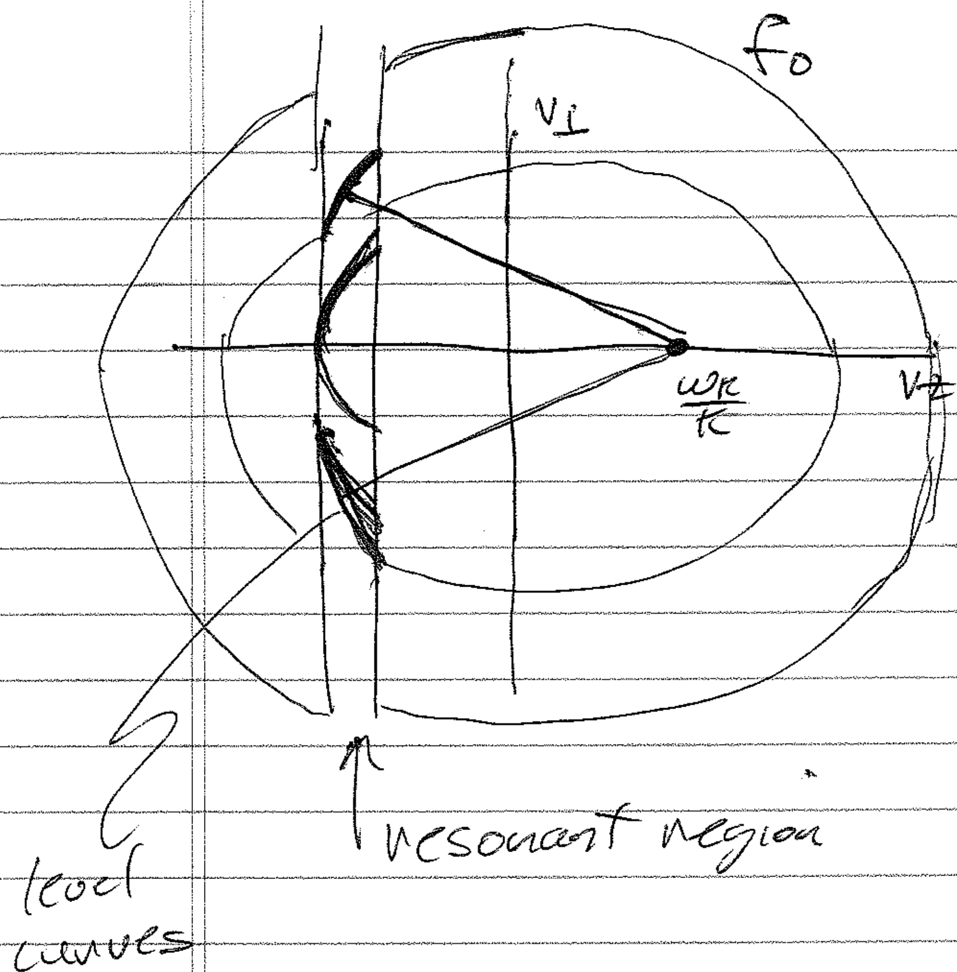
Can show this result rigorously by changing variables: ~~let H~~ from $v_\perp, v_z$ to H, v_z

$$\left(\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp}+\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right)f_0=\left(1-\frac{kv_z}{\omega_k}\right)\left(\frac{\partial H}{\partial v_\perp}\frac{\partial}{\partial H}+\cancel{\frac{\partial v_z}{\partial v_\perp}\frac{\partial}{\partial v_z}}\right)f_0$$

$$+\frac{kv_\perp}{\omega_k}\left[\frac{\partial H}{\partial v_z}\frac{\partial}{\partial H}+\frac{\partial v_z}{\partial v_z}\frac{\partial}{\partial v_z}\right]f_0$$

$$=\left[\cancel{\left(1-\frac{kv_z}{\omega_k}\right)v_\perp\frac{\partial}{\partial H}+\frac{kv_\perp}{\omega_k}\left(v_z-\frac{\omega_k}{k}\right)\frac{\partial}{\partial H}}+\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]f_0$$

$$=\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}f_0$$


f0
v⊥
$\frac{\omega_R}{k}$
v_z
resonant region
level curves

$\cancel{\frac{kv_\perp}{2}}$

$$\left[\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z} + \left(1-\frac{kv_z}{\omega_k}\right)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\right]$$

$$= \left[\underbrace{\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z} + \left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp}} + \left(1-\frac{kv_z}{\omega_k}\right)\frac{1}{v_\perp}\right]$$

$$\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}$$

$$\frac{\partial f_0}{\partial t} - \frac{e^2}{4m^2}\sum_k |E_-|^2\left[\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z} + \frac{1}{v_\perp}\left(1-\frac{kv_z}{\omega_k}\right)\right]$$

$$\otimes\ \delta(\omega_k - kv_z - \Omega_e)\,\pi \;\; \frac{k v_\perp}{\omega_k}\frac{\partial}{\partial v_z} f_0 = 0$$

$$\left[\frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\, v_\perp \overset{(\cancel{H}, v_z)}{+} \left(1-\frac{kv_z}{\omega_k}\right)\right]$$

$$= \frac{k}{2\omega_k}\frac{\partial}{\partial v_z}\left(v_\perp^2\right)_{\cancel{H}} + \frac{kv_\perp^2}{\omega_k}\frac{\partial}{\partial v_z} + \left(1-\frac{kv_z}{\omega_k}\right)$$

$$v_\perp^2 = 2H - \left(v_z - \frac{\omega_k}{k}\right)^2$$

$$\frac{k}{\omega_k}\left(\frac{\partial v_\perp^2}{\partial v_z}\right)_H = -2\left(v_z - \frac{\omega_k}{k}\right)\frac{k}{\omega_k} = +2\left(-\frac{kv_z}{\omega_k}+1\right)$$

$$\frac{k}{2\omega_k}\frac{\partial}{\partial v_z}\left(\cancel{v}_\perp^2\right)_H + \frac{1}{2}\frac{k}{\omega_k}\left(\frac{\partial v_\perp^2}{\partial v_z}\right)'_H + \frac{kv_\perp^2}{\omega_k}\frac{\partial}{\partial v_z}$$

$$= \frac{k}{\omega_k}\frac{\partial}{\partial v_z}\; v_\perp^2$$

$$\frac{\partial f_0}{\partial t} - \frac{e^2\pi}{4m^2}\sum_k \underbrace{|\hat{E}_-|^2 \frac{k^2}{\omega_k^2}}_{\hat{B}_k^2/c^2}\frac{\partial}{\partial v_z} v_\perp^2\, \delta(\omega_k - kv_z - \Omega_e)\frac{\partial}{\partial v_z} f_0 = 0$$

$$\sum_k = \int \frac{dk}{2\pi/L}$$

$$\boxed{\frac{\partial f_0}{\partial t} - \frac{e^2 L}{4m^2c^2}\frac{\partial}{\partial v_z}|\hat{B}_k|^2 \frac{1}{\left|v_z - \frac{d\omega_k}{dk}\right|}\frac{\partial}{\partial v_z} f_0 = 0}$$

$\omega_k - kv_z - \Omega_e = 0 \Rightarrow k$ determines k

growth rate

$\left.\frac{\partial f_0}{\partial v_z}\right|_H = 0$ in steady state

$$\gamma = \frac{\pi^2\Omega_e}{(1+k^2d_e^2)^2 k}\int_0^\infty \frac{dv_\perp\, v_\perp^2}{n_0}\underbrace{\left[kv_\perp \frac{\partial}{\partial v_z} f_0 + (\omega_k - kv_z)\frac{\partial}{\partial v_\perp} f_0\right]}_{k v_\perp \frac{\partial}{\partial v_z} f_0}$$

$$dH\big|_{v_z} = v_\perp dv_\perp \qquad H = \frac{1}{2}\left(v_\perp^2 + \left(v_z - \frac{\omega_k}{k}\right)^2\right)$$

$$\boxed{\gamma = \frac{\pi^2\Omega_e}{(1+k^2d_e^2)^2}\int_{H_{min}}^\infty dH\, \frac{v_\perp^2}{n_0}\frac{\partial}{\partial v_z} f_0} \Rightarrow \gamma \to 0$$

in steady state

$$H_{min} = \frac{1}{2}\left(v_z - \frac{\omega_k}{k}\right)^2 = \frac{1}{2}\frac{\Omega_e^2}{k^2}$$