Parametric Instabilities

Because of the nonlinearities in plasmas, waves propagating through the medium are typically unstable to the development of secondary instabilities. These can be important in understanding the mechanisms for wave absorption and dissipation

A simple example – nonlinear oscillators

Consider the Hamiltonian for three coupled oscillators

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + \omega_i^2 \frac{x_i^2}{2} + V x_1 x_2 x_3$$

The equations of motion are

$$\ddot{x}_1 + \omega_1^2 x_1 = -V x_2 x_3$$
$$\ddot{x}_2 + \omega_2^2 x_2 = -V x_1 x_3$$
$$\ddot{x}_3 + \omega_3^2 x_3 = -V x_2 x_1$$

To examine energy transfer consider an initial state in which almost all of the energy is in oscillator ① and oscillators ② and ③ can be treated perturbatively

$$\ddot{x}_1 + \omega_1^2 x_1 = 0$$

$$\ddot{x}_2 + \omega_2^2 x_2 = -U x_1 x_3$$

$$\ddot{x}_3 + \omega_3^2 x_3 = -U x_2 x_1$$

Oscillator x_1 simply oscillates at a frequency ω_1. Assuming that the amplitude of x_1 is not too large, the coupling of the oscillators x_2, x_3 will depend sensitively on the matching of the three frequencies, i.e.

$$\omega_1 = \omega_2 + \omega_3$$

That was for example the beat of $x_1 x_3$ will have a frequency which will resonate with the oscillator x_2. Generally we can write

$$x_1 = a_1 e^{i\omega_1 t} + a_1^* e^{-i\omega_1 t}$$

$\Rightarrow$ x_1 is now real

~~and~~ $\cancel{x_2 = a_2 e^{i\omega_2 t} + a_2^* e^{-i\omega_2 t}}$

$\cancel{x_3 = a_3 e^{i(\omega_2+\omega_3)t} + a_3^* e^{-i(\omega_2+\omega_3)t}}$

~~with~~ $\cancel{\omega \approx \omega_2}$

we take

$$X_2 = a_2(t)\, e^{i\omega_2 t} + a_2^*(t)\, e^{-i\omega_2 t}$$

$$X_3 = a_3(t)\, e^{i\omega_3 t} + a_3^*(t)\, e^{-i\omega_3 t}$$

where assume time dependence of a's is weak

$$\ddot{X}_2 + \omega_2^2 X_2 = -V\left(a_1 e^{i\omega_1 t} + a_1^* e^{-i\omega_1 t}\right)\left(a_3 e^{i\omega_3 t} + a_3^* e^{-i\omega_3 t}\right)$$

Have beats at $\omega_1 + \omega_3$, $\omega_1 - \omega_3$, $-\omega_1 + \omega_3$, $-\omega_1 - \omega_3$

$$\Longrightarrow \underbrace{\omega_1 - \omega_3}_{\omega_2},\ \underbrace{-\omega_1 + \omega_3}_{-\omega_2} \text{ are resonant}$$

$$\left(\frac{\partial}{\partial t} + i\omega_2\right)^2 a_2 + \omega_2^2 a_2 = -V\, a_1 a_3^*$$

$$\left(\cancel{\frac{\partial^2}{\partial t^2}} + 2i\omega_2 \frac{\partial}{\partial t}\right) a_2 = -V\, a_1 a_3^*$$

$$\Longrightarrow \frac{\partial}{\partial t} a_2 = \frac{iV}{2\omega_2}\, a_1 a_3^*$$

$$\ddot{X}_3 + \omega_3^2 X_3 = -V\left(a_2 e^{i\omega_2 t} + a_2^* e^{-i\omega_2 t}\right)\left(a_1 e^{i\omega_1 t} + a_1^* e^{-i\omega_1 t}\right)$$

want beat at $-\omega_3 = \omega_2 - \omega_1$

$$\left(\frac{\partial}{\partial t} - i\omega_3\right)^2 a_3^* + \omega_3^2 a_3^* = -V\, a_2 a_1^*$$

$$-2i\omega_3 \frac{\partial}{\partial t} a_3^* = -V\, a_2 a_1^*$$

$$\Rightarrow \quad \frac{\partial}{\partial t} a_3^* = -i \frac{V}{2\omega_3} a_2 a_1^*$$

$$\frac{\partial^2}{\partial t^2} a_2 = \frac{\cancel{i} V}{2\omega_2} a_1 \left(+\cancel{i} \frac{V}{2\omega_3}\right) a_2 a_1^*$$

$$\gamma^2 = |a_1|^2 \frac{V^2}{4\omega_2\omega_3}$$

$\Rightarrow$ growth for

$$\omega_2\omega_3 > 0$$

$$\omega_2(\omega_1 - \omega_2) > 0$$

$$\omega_1\omega_2 > \omega_2^2$$

take $\omega_1 > 0 \Rightarrow \omega_2 > 0$, $\omega_1 > \omega_2$

$$\Rightarrow \omega_3 > 0$$

$$\Rightarrow \boxed{\omega_1 > \omega_2, \omega_3}$$

Energy transfer only if oscillator with energy has highest frequency.

~~Rab~~

Raman scattering

Consider an incident electromagnetic wave E_0, B_0 ~~such that~~ with frequency ω_0 such that $\omega_0 \gg \omega_{pe}$

$$\Rightarrow k_0 c \approx \omega_0$$

Want to look at the scattering of this wave ~~off~~ of plasma waves

$\Rightarrow$ the incident wave amplifies the plasma waves and produces a scattered wave.

$\Rightarrow$ reflected EM wave must also to lowest order have $\omega_s \approx \omega_0$

So $\omega_s \approx k_s c \approx k_0 c \approx \omega_0$

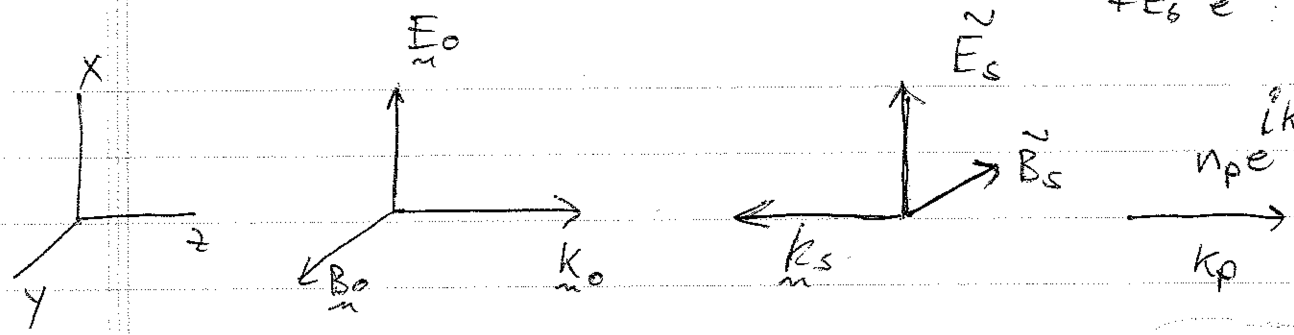
$$\Rightarrow k_p \approx 2k_0$$

$\longrightarrow k_0$

$\longrightarrow k_p$

$\longleftarrow k_s$

$E_0 e^{ik_0 z - i\omega_0 t} + E_0^* e^{-ik_0 z} e^{i\omega_0 t}$ $E_s e^{+ik_s z + i\omega_s t} + E_s^* e^{-ik_s z - i\omega_s t}$



$n_p e^{ik_p z - i\omega_p t} + cc$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$k_0 = k_p - k_s$$

$$\omega_0 = \omega_p + \omega_s$$

$$\frac{1}{c}\frac{\partial B}{\partial t} + \nabla \times E = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t} B_y + \frac{\partial}{\partial z} E_x = 0$$

$$-\frac{\partial}{\partial z} B_y = \frac{4\pi}{c} J_x + \frac{1}{c}\frac{\partial}{\partial t} E_x$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} n V_z = 0$$

$$\frac{\partial}{\partial t} V_z = -\frac{e}{m} E_z - \frac{e}{m}\frac{V_x}{c} B_y$$

$$\frac{\partial}{\partial z} E_z = -4\pi e (n - n_0)$$

$\cancel{J_x = -ne}$

$$\frac{\partial}{\partial t} V_x + V_z \frac{\partial}{\partial z} V_x = -\frac{e}{m} E_x + \frac{e}{m}\frac{V_z}{c} B_y$$

$$J_x = -n e V_x$$

pump

$$\frac{1}{c}\frac{\partial}{\partial t}B_0 + \frac{\partial}{\partial z}E_0 = 0$$

$$\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial z}B_0 = \frac{4\pi}{c}J_x + \frac{1}{c}\frac{\partial}{\partial t}E_0\right]$$

$$\frac{\partial}{\partial t}v_{x0} = -\frac{e}{m}E_0$$

$$J_x = -n_0 e v_{x0}$$

$$\cancel{-}\frac{\partial}{\partial z}\left(\cancel{-}c\frac{\partial}{\partial z}E_0\right) = \frac{4\pi}{c}\left(\cancel{-}n_0 e\left(\cancel{-}\frac{e}{m}\right)E_0\right) + \frac{1}{c}\ddot{E}_0$$

$$c^2 E_{0zz} = \omega_{pe}^2 E_0 + \ddot{E}_0$$

$$\Rightarrow \boxed{\omega_0^2 = k_0^2 c^2 + \omega_{pe}^2}$$

$$E_0 = E_0\, e^{ik_0 z - i\omega_0 t} + E_0^* e^{-ik_0 z + i\omega_0 t}$$

$$v_{x0} = \mathrm{Re}\left[-\frac{e}{m}\frac{E_0 e^{ikz} e^{-i\omega_0 t}}{-i\omega_0}\right]$$

~~Stochastic~~

~~Total~~

Scattered wave

$$-\frac{1}{c}\frac{\partial}{\partial t}B_s + \frac{\partial}{\partial z}E_s = 0 \Rightarrow -\frac{B_s}{c}i\omega_s + ik_s E_s = 0$$

$$B_s = \frac{k_s}{\omega_s}c E_s$$

$$+\frac{\partial}{\partial z}B_s = \frac{4\pi}{c}J_{xs} + \frac{1}{c}\frac{\partial}{\partial t}E_s$$

$$J_{xs} = -n_0 e V_{xs} - n_p e V_{x0}$$

compare nl part of these two terms

$$\frac{\partial}{\partial t}V_{xs} + \cancel{V_{zp}\frac{\partial}{\partial z}V_{x0}}^{\sim\frac{\omega_p}{\omega_0}} = -\frac{e}{m}E_s + \cancel{\frac{e}{m}\frac{V_{zp}}{c}B_0}^{\frac{\omega_p}{\omega_0}}$$

$$V_p k_0 \frac{e E_0}{m\omega_0} \sim V_p \frac{eE_0}{mc} \qquad \frac{e}{m}\frac{V_p B_0}{c}$$

$$V_{xs}^{nl} \sim V_p \frac{eE_0}{mc\omega_0}$$

$$J_{xs}^{nl} \sim \left[\cancel{n_0 e V_p \frac{e}{m\omega_0}\frac{E_0}{c}},\ \cancel{n_p e \frac{e}{m}\frac{E_0}{\omega_0}}\right]\frac{e^2E_0}{m\omega_0}$$

$$V_p \sim \frac{\omega_p}{k_0}\frac{n_p}{n_0} \text{ from continuity eqn.}$$

$$\omega_p n_p \sim n_0 k_p V_p$$

$$\sim \left(n_0 \frac{\omega_p}{n_0 c}\frac{n_p}{n_0},\ n_p\right)$$

$$\left(\frac{\omega_p}{\omega_0},\ 1\right)$$

small

~~$J_{xs} e^{i\omega_s t} = -n_0 e(-\frac{e}{m}E_s/i\omega_s) - n_p e V_{x0}$~~ ~~$e^{-i\omega_p t}$, $e^{i\omega_0 t}$, $e^{-i\omega_0 t}$~~

$$U_{xs} = -\frac{e}{m} E_s \frac{1}{i\omega_s}$$

$$J_{xs} = -n_0 e\left(-\frac{e}{m} E_s \frac{1}{i\omega_s}\right) - n_p e\, U_{x0}$$

note: $J_{xs} \sim e^{i\omega_s t} e^{+ik_s z} E_s$

$$n_p \sim e^{ik_p z} e^{-i\omega_p t}$$

$$U_{x0} \sim \cancel{E_0 e^{ik_0 z} e^{-i\omega_0 t}} \underbrace{+ E_0^* e^{-ik_0 z} e^{i\omega_0 t}}_{\text{this drives } J_{xs}}$$

$$J_{xs} = \frac{n_0 e^2}{m} \frac{E_s}{i\omega_s} - n_p e\left(\frac{-e E_0^*}{m\, i\omega_0}\right)$$

$$ik_s \frac{k_s c}{\omega_s} E_s = \frac{4\pi}{c}\left(\frac{n_0 e^2}{m}\frac{E_s}{i\omega_s} + \frac{n_p e^2}{m}\frac{E_0^*}{i\omega_0}\right) + \frac{i\omega_s}{c} E_s$$

$$\frac{k_s^2 c^2}{\omega_s} E_s = \frac{\omega_{pe}^2 E_s}{-\omega_s} + \omega_s E_s - \frac{\omega_{pe}^2}{\omega_0}\frac{n_p}{n_0} E_0^*$$

$$\left(k_s^2 c^2 - \omega_s^2 \epsilon_s\right) E_s = -\omega_{pe}^2 \frac{n_p}{n_0} E_0^*$$

$$\epsilon_s = 1 - \frac{\omega_{pe}^2}{\omega_s^2}$$

plasma wave

$$\frac{\partial}{\partial t} n_p + n_0 \frac{\partial}{\partial z} V_p = 0$$

$$\frac{\partial}{\partial z} E_p = -4\pi e\, n_p$$

$$\frac{\partial}{\partial t} V_p = -\frac{e}{m} E_p - \frac{e}{mc}\left(-V_{X0} B_S + V_{XS} B_0\right)$$

$\sim V_p\, e^{ik_p z}$ $\qquad \sim E_0 e^{ik_0 z}\; e^{ik_s z} B_s$

$$\frac{\partial}{\partial t} V_p = -\frac{e}{m} E_p - \frac{e}{mc}\left(-\left(-\frac{e}{m}\frac{E_0}{-i\omega_0}\right) \frac{k_s E_s c}{\omega_s} + -\frac{e}{m}\frac{E_s}{i\omega_s}\frac{E_0 k_0 c}{\omega_0}\right) e^{ik_0 z - i\omega_0 t} \otimes e^{ik_s z}\, e^{i\omega_s t}$$

$k_0 + k_s = k_p$

$\omega_0 = \omega_s + \omega_p$

$$= \left(-\frac{e}{m} E_p - \frac{e}{m}\, i k_p\, \frac{e E_0 E_s}{m\,\omega_0 \omega_s}\right) e^{ik_p z}\, e^{-i\omega_p t}$$

$$m \frac{\partial}{\partial t} V_p \simeq \left(-\frac{e}{\cdot} E_p - \frac{e^2}{m}\, i k_p\, \frac{E_0 E_s}{\omega_0^2}\right) e^{ik_p z}\, e^{-i\omega_p t}$$

Electrons driven out of region of high radiation pressure

$$\boxed{F_\omega = -\nabla\, \frac{e^2}{m}\frac{E_0 E_s}{\omega_0^2}}$$

ponderomotive force

$\Rightarrow$ generic result for low freq. response to beating of HF wave

$$-\cancel{i}\omega_p n_p + n_0 \cancel{i} k_p V_p = 0$$

$$n_p = n_0 \frac{k_p}{\omega_p} V_p$$

$$+\cancel{i}\omega_p m V_p = +e\left(+\frac{\cancel{i}^{\,?}\,4\pi e n_p}{\cancel{i} k_p}\right) + \cancel{i} k_p \frac{e^2}{m}\frac{E_0 E_s}{\omega_0^2}$$

$$n_p = \frac{\cancel{n_0 k_p}}{\cancel{m}\,\omega_p^2}\left[n_0 \frac{4\pi e^2}{m \cancel{k_p}} n_p + k_p^2 \frac{e^2 m}{m^2}\frac{E_0 E_s}{\omega_0^2}\right]$$

$$\boxed{n_p \underbrace{\left(1 - \frac{\omega_{pe}^2}{\omega_p^2}\right)}_{\epsilon_p} = \frac{k_p^2}{\omega_p^2}\frac{e^2}{m^2} n_0 \frac{E_0 E_s}{\omega_0^2}}$$

Dispersion Relation

$\cancel{k_s^2c^2 + \omega_{pe}^2}$

$$v_0 = \frac{eE_0}{m\omega_0}$$

$$\epsilon_p\left(K_s^2c^2 - \omega_s^2\epsilon_s\right) = -\omega_{pe}^2\,\frac{E_0^*}{\cancel{n_0}}\,\frac{K_p^2}{\omega_p^2}\,\frac{e^2\cancel{n_0}}{m^2}\,\frac{E_0}{\omega_0^2}$$

$$= -\frac{\omega_{pe}^2}{\omega_p^2}K_p^2\,|v_0|^2$$

$$\omega_0 = \omega_s + \omega_p$$

$$K_p = k_0 + K_s$$

$$\simeq 2k_0$$

$\cancel{\epsilon_p\left(k_s^2c^2 - \omega_s^2\epsilon_s\right)}$

$$\chi_e = -\frac{\omega_{pe}^2}{\omega_p^2}$$

$$\boxed{\epsilon_p\left(K_s^2c^2 - \omega_s^2\epsilon_s\right) = \chi_e K_p^2|v_0|^2}$$

generic form

$$\cancel{K_s^2c^2 - \omega_s^2\epsilon_s \cong k_0^2c^2 + \omega_{pe}^2 - (\omega_0 - \omega_p)^2}$$

$$\cancel{= k_0^2c^2 + \omega_{pe}^2 - \omega_0^2 + 2\omega_0\omega_p \mp \omega_p^2}$$

$$\simeq 2\omega_0\omega_p$$

$$K_p = 2k_0 + \Delta k$$

$$\omega_{pe} = \frac{\omega_0^2 - (k_p - k_0)^2 c^2}{2\omega_0}$$

$$2\omega_0 \omega_{pe} = \omega_0^2 - (k_0 + \Delta k)^2 c^2$$

$$= \cancel{\omega_0^2} - \cancel{k_0^2 c^2} - 2 k_0 \Delta k c^2$$

$$\omega_{pe} = - \Delta k c$$

$$\frac{\Delta k}{k_0} = - \frac{\omega_{pe}}{\omega_0}$$

$$\epsilon_p = 1 - \frac{\omega_{pe}^2}{\omega_p^2} \simeq \frac{\omega_p^2 - \omega_{pe}^2}{\omega_p^2}$$

$$\simeq \frac{(\omega_p - \omega_{pe})(2\cancel{\omega_{pe}})}{\omega_{pe}^{\cancel{2}}}$$

$$\epsilon_p \simeq \frac{2(\omega_p - \omega_{pe})}{\omega_{pe}}$$

$$K_s^2 c^2 - \omega_s^2 \epsilon_s = (k_p - k_0)^2 c^2 - \omega_s^2 + \omega_{pe}^2$$

$$= (K_p - k_0)^2 c^2 - (\omega_0 - \omega_p)^2 + \omega_{pe}^2$$

$$\simeq (K_p - k_0)^2 c^2 - \omega_0^2 + \underline{2\omega_0\omega_p} + \cancel{\omega_{pe}^2} \text{ small}$$

$$\equiv 2\omega_0(\omega_p - \Delta\omega)$$

$$\Delta\omega = \frac{\omega_0^2 - (K_p - k_0)^2 c^2}{2\omega_0}$$

$$\cancel{2}\frac{(\omega_p - \omega_{pe})}{\omega_{pe}} \cancel{2}\omega_0(\omega_p - \Delta\omega) = -\frac{\omega_{pe}^2}{\omega_p^2}(\cancel{2}k_0)^2 |V_0|^2$$

$$\omega_p = \omega_{pe} + i\gamma$$

$$\Delta\omega = \omega_{pe}$$

$$\gamma^2 = \frac{\omega_{pe} c^2 k_0^2}{\omega_0} \frac{|V_0|^2}{c^2}$$

$$\boxed{\gamma = (\omega_{pe}\omega_0)^{\frac{1}{2}} \frac{|V_0|}{c}}$$

can calculate K_p from $\Delta\omega = \omega_{pe}$

Raman Scattering

Non linear Energy Exchange

Now want to allow E_s and n_p to reach finite amplitude. This will deplete the pump E_0 and change the time behavior.

plasma wave

from earlier

$$\left(\omega_p^2 - \omega_{pe}^2\right) n_p = k_p^2 \frac{e^2}{m^2} n_0 \frac{E_0 E_s}{\omega_0^2}$$

$$\omega_p \to \omega_{pe} + i\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t} = -i\omega_p$$

$$\omega_p = i\frac{\partial}{\partial t}$$

$$\boxed{2i\omega_{pe}\frac{\partial}{\partial t} n_p = k_p^2 \frac{e^2}{m^2} n_0 \frac{E_0 E_s}{\omega_0^2}}$$

Scattered wave

$$\omega_s = \omega_0 - \omega_p$$

$$\left(k_s^2 c^2 - \omega_s^2 + \omega_{pe}^2\right) E_s = -\omega_{pe}^2 \frac{n_p}{n_0} E_0^*$$

$$\left[\left(k_p - k_0\right)^2 c^2 - \left(\omega_0 - \omega_{pe} - i\frac{\partial}{\partial t}\right)^2 + \omega_{pe}^2\right] E_s = -\omega_{pe}^2 \frac{n_p}{n_0} E_0^*$$

$$\boxed{2i\,\omega_0 \frac{\partial}{\partial t} E_s = -\omega_{pe}^2 \frac{n_p}{n_0} E_0^*}$$

pump As before but add nonlinear part to J_x

$$\Rightarrow J_x^{nL} = -n_p e\, V_{xs}^*$$

$$e^{ik_0 z} \quad e^{ik_p z} \quad e^{-ik_s z}$$

$$-\frac{\partial}{\partial z} B_0 = \frac{4\pi}{c} J_x + \frac{1}{c}\frac{\partial}{\partial t} E_0$$

$$\frac{1}{c}\frac{\partial}{\partial t} B_0 + \frac{\partial}{\partial z} E_0 = 0 \qquad J_x = -n_0 e V_{x0} + J_x^{nL}$$

$$\frac{\partial}{\partial t} V_{x0} = -\frac{e}{m} E_0$$

$$\frac{1}{c}\ddot{E}_0 = -ik_0(-ik_0 c)E_0 - \frac{4\pi}{c}\left(-n_0 e\left(-\frac{e}{m}E_0\right)\right)$$

$$-\frac{4\pi}{c}(-i\omega_0)\left(-n_p e\, V_{xs}^*\right)$$

$$\ddot{E}_0 + k_0^2 c^2 E_0 + \omega_{pe}^2 E_0 = -4\pi e\, i\omega_0\, n_p V_{xs}^*$$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega_0 + \frac{\partial}{\partial t} \qquad V_{xs}^* = -\frac{eE_s^*}{m}\frac{1}{-i\omega_s}$$

$$\cancel{\left(-\omega_0^2 + k_0^2 c^2\right)} E_0 - 2i\omega_0 \frac{\partial}{\partial t} E_0 = -4\pi e\, i\omega_0 n_p V_{xs}^*$$

$$\frac{\partial}{\partial t} E_0 = +\frac{4\pi e}{2}\frac{n_p}{n_0} n_0 \left(-\frac{e}{m}\frac{E_s^*}{(-i\omega_s)}\right)$$

$$\boxed{\frac{\partial E_0}{\partial t} = -i\frac{\omega_{pe}^2}{\omega_0}\frac{n_p}{n_0} E_s^*}$$

Want to normalize the amplitudes of the modes by defining the # of quanta in the mode as the wave energy divided by the frequency

$$N_0 = \left(\frac{|E_0|^2}{8\pi} + \frac{|B_0|^2}{8\pi}\right)\frac{1}{\omega_0}$$

$$= \frac{|E_0|^2}{4\pi}\frac{1}{\omega_0} = |a_0|^2$$

$$\Longrightarrow \quad a_0 = \frac{E_0}{\sqrt{4\pi\omega_0}} \Longrightarrow \text{pump}$$

$$\Longrightarrow \quad a_s = \frac{E_s}{\sqrt{4\pi\omega_0}} \Longrightarrow \text{scattered.}$$

for plasma wave

$\frac{\partial}{\partial t} v_p = -\frac{e}{m} E_p$

$$N_p = \left(\frac{1}{2} m n_0 |v_p|^2 + \frac{|E_p|^2}{8\pi}\right)\frac{1}{\omega_p}$$

$i k_p E_p = -4\pi e n_p$

$$= \frac{|E_p|^2}{4\pi}\frac{1}{\omega_p}$$

$$= \frac{(4\pi e)^2 |n_p|^2}{4\pi k_p^2 \omega_p}$$

$$\Longrightarrow \quad a_p = \frac{\sqrt{4\pi} e}{k_p}\frac{1}{\omega_{pe}^{1/2}} n_p \Longrightarrow \text{plasma wave}$$

$$2i\,\cancel{\omega_{pe}}\,\frac{\partial}{\partial t}\,\cancel{\frac{k_p\,\omega_{pe}^{1/2}}{\sqrt{4\pi e}}}\,a_p = k_p^{\cancel{2}}\,\frac{e^{\cancel{2}}}{m^{\cancel{2}}}\,\frac{\cancel{n_0}\,\omega_{pe}^{1/2}}{\omega_0}\,\cancel{\frac{\omega_{pe}^2}{4\pi}}\,a_0 a_s$$

$$i\frac{\partial}{\partial t}a_p = \frac{\sqrt{4\pi}\,e\,k_p}{2m}\,\omega_{pe}^{1/2}\,\frac{1}{\omega_0}\,a_0 a_s$$

$$+2i\omega_0\frac{\partial}{\partial t}\,\sqrt{4\pi\cancel{\omega_0}}\,a_s = -+\frac{\cancel{\omega_{pe}^2}}{\cancel{n_0}}\,\frac{\sqrt{4\pi\cancel{\omega_0}}\;e\,\omega_{pe}^{1/2}\,a_0^*\,k_p\,a_p}{m\,\cancel{4\pi e^2}}$$

$$\boxed{+i\frac{\partial}{\partial t}a_s = -\frac{\sqrt{4\pi}\,\omega_{pe}^{1/2}\,e\,k_p}{2m\,\omega_0}\,a_0^* a_p}$$

$$\frac{\partial}{\partial t}a_0 = -i\,\frac{\cancel{\omega_{pe}^2}}{m\,2}\,\frac{1}{\cancel{n_0}}\,\frac{e\,\omega_{pe}^{1/2}\,k_p}{\cancel{4\pi e}}\,a_p\,\frac{a_s^*}{\omega_0}\,\sqrt{4\pi}$$

$$\boxed{i\frac{\partial}{\partial t}a_0 = +\frac{\sqrt{4\pi}\,k_p\,e\,\omega_{pe}^{1/2}}{2m\,\omega_0}\,a_p a_s^*}$$

$$V = \frac{\sqrt{4\pi}\,k_p\,e\,\omega_{pe}^{1/2}}{2m\,\omega_0}$$

(1) $i\frac{\partial}{\partial t}a_p = V\,a_0 a_s$

(2) $i\frac{\partial}{\partial t}a_s = -V\,a_0^* a_p$

(3) $i\frac{\partial}{\partial t}a_0 = +V\,a_p a_s^*$

Completely general for decay type instabilities

Conservation Laws

mult. 1 by a_p^* and add conjugate eqn.

$$N_p = |a_p|^2$$

$$\dot{N}_p = -i\,V\,a_0 a_s a_p^* + cc$$

mult. 3 by a_0^* and add conj.

$$\dot{N}_0 = -i\,V\,a_p a_s^* a_0^* + cc$$

$$\dot{N}_p + \dot{N}_0 = 0$$

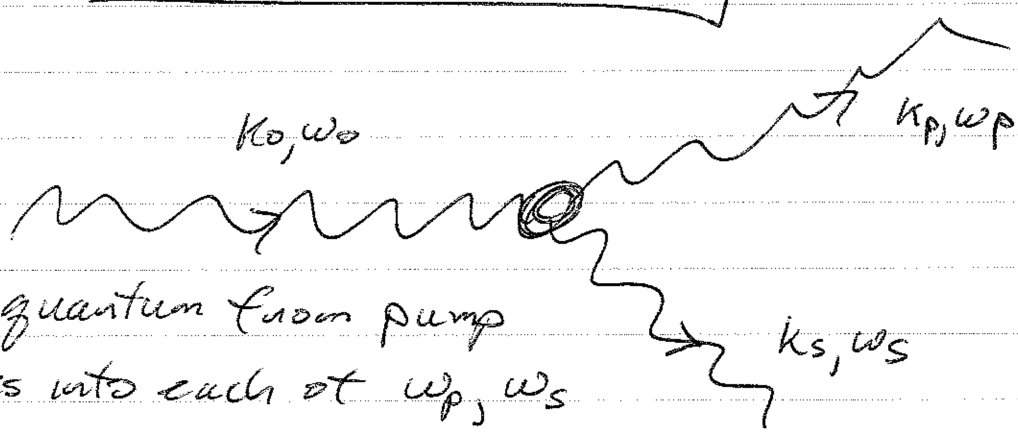
$$\boxed{N_p + N_0 = \text{const}}$$

similarly

$$\boxed{N_s + N_0 = \text{const.}}$$

$$\boxed{N_p - N_s = \text{const}}$$

Manley Rowe relations



a quantum from pump goes into each of ω_p, ω_s

Energy conservation

$$\frac{d}{dt} W = \frac{d}{dt}\left(N_0 \omega_0 + N_p \omega_p + N_s \omega_s\right)$$

$$= \frac{d}{dt}\left(N_0 \omega_0 + \left(-N_0 \omega_p\right) + \left(-N_0 \omega_s\right)\right)$$

$$= \frac{d}{dt} N_0\left(\omega_0 - \omega_p - \omega_s\right)$$

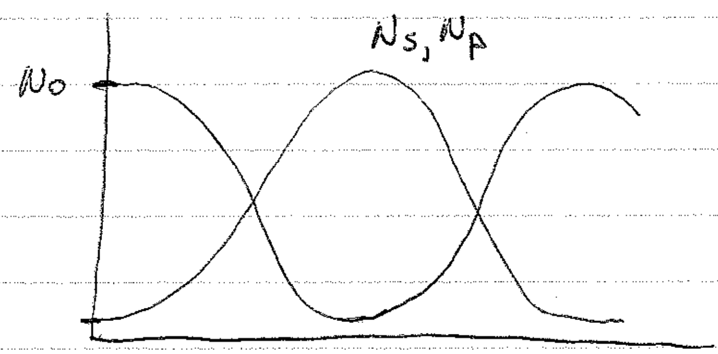
$$= 0$$

$$W = \text{const.}$$

$\Rightarrow$ total wave energy is conserved.

Full eqns can be solved exactly
$\Rightarrow$ see Sagdeev - Galeev

for decay unstable system



decay stable