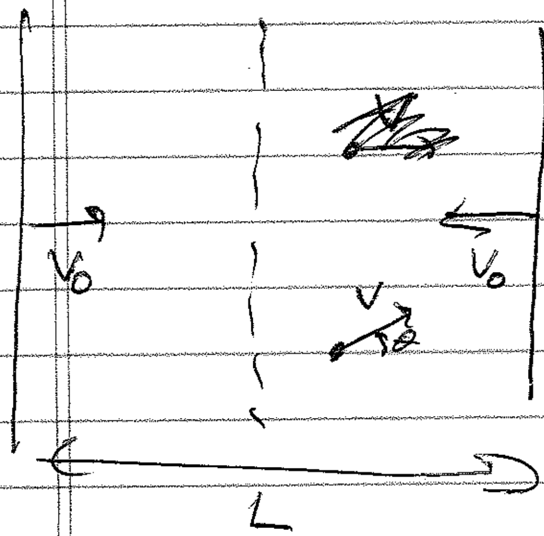
Parker's transport equation

Energy gain during converging flow



Consider a particle with velocity V and angle θ undergoing reflection.

$$V_t = \text{const}$$

$$V_n \rightarrow -V_n - 2V_0$$

$$\Delta \mathcal{E} = V_t^2 + (V_n + 2V_0)^2 - V^2$$

$$= V^2 \sin^2\theta + (V\cos\theta + 2V_0)^2 - v^2$$

$$\Delta \mathcal{E} \simeq V^2 \sin^2\theta + V^2\cos^2\theta + 4 V_0 V \cos\theta - V^2$$

$$\Delta \mathcal{E} = 4 V V_0 \cos\theta$$

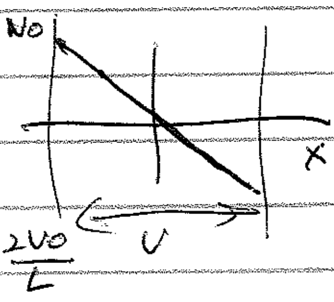
$$\Delta t = \frac{L}{2}\,\frac{1}{V\cos\theta}$$

$$\frac{\Delta \mathcal{E}}{\Delta t} = \frac{2V^2 V_0 \cos^2\theta}{\frac{L}{2}} = 4\,\frac{V^2 V_0 \cos^2\theta}{L}$$

$$\left\langle \frac{\Delta \mathcal{E}}{\Delta t} \right\rangle = \int_0^1 d\cos\theta \; \frac{4 V^2 V_0 \cos^2\theta}{L}$$

$$= \frac{4}{3}\,\frac{V^2 V_0}{L}$$

$$\left(\frac{d}{dt}\langle V^2 \rangle = -\frac{2}{3} V^2 V_0' \right) \qquad V_0' = -\frac{2V_0}{L}$$



More generally can write as

$$\frac{d}{dt} V = -\frac{1}{3} V \nabla \cdot \underset{\sim}{u}$$

where $\underset{\sim}{u}$ is now the fluid flow velocity.

Parker's transport Equation

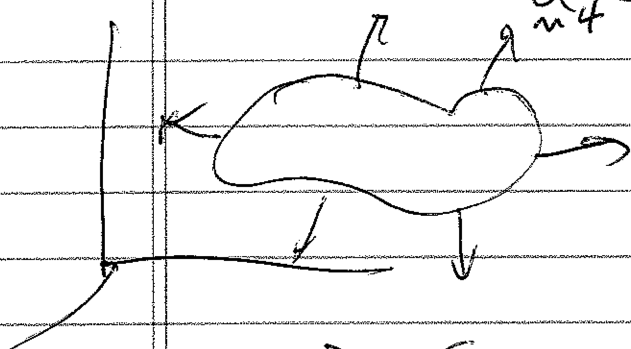
Consider a distribution $f(\underset{\sim}{x}, v, t)$ of energetic particles with $v \gg \underset{\sim}{u}$

The phase space volume is

$$d\underset{\sim}{x}\, v^2 dv \sim d\underset{\sim}{x}\, dv^3$$

The divergence of f in phase space is given by

$$\underset{\sim}{u}_4 = \left(\underset{\sim}{u}, \dot{v}^3\right)$$



$$\frac{\partial}{\partial t} \int_V \underbrace{d\underset{\sim}{x}\, dv^3}_{dV} f = -\int_S d\underset{\sim}{s} \cdot \left(\underset{\sim}{u}_4 f\right)$$

$$= -\int dV\, \nabla \cdot \left(\underset{\sim}{u}_4 f\right)$$

Valid for any v so

$$\frac{\partial f}{\partial t} = -\nabla\cdot\underset{\sim}{u} f - \underbrace{\frac{\partial}{\partial v^3}\dot{v}^3 f}_{\frac{1}{\cancel{3}v^2}\frac{\partial}{\partial v}\cancel{3}v^2\dot{v} f} + \nabla\cdot\underset{\approx}{K}\cdot\nabla f$$

$$\frac{\partial f}{\partial t} + \nabla\cdot\underset{\sim}{u} f - \nabla\cdot\underset{\approx}{K}\cdot\nabla f - \frac{1}{3}\nabla\cdot\underset{\sim}{u}\,\frac{1}{v^2}\frac{\partial}{\partial v}v^3 f = 0$$

define $v^2 f \equiv F$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}F + \nabla\cdot\underset{\sim}{u} F - \nabla\cdot\underset{\approx}{K}\cdot\nabla f - \frac{1}{3}\nabla\cdot\underset{\sim}{u}\,\frac{\partial}{\partial v} v F = 0}$$

Parker transport equation.

This equation is valid in the strong scattering limit

$\Rightarrow$ no energy gain if $\nabla\cdot\underset{\sim}{u} = 0$

$\Rightarrow$ has been used to calculate the energetic particle spectra in shocks.

Alternate form

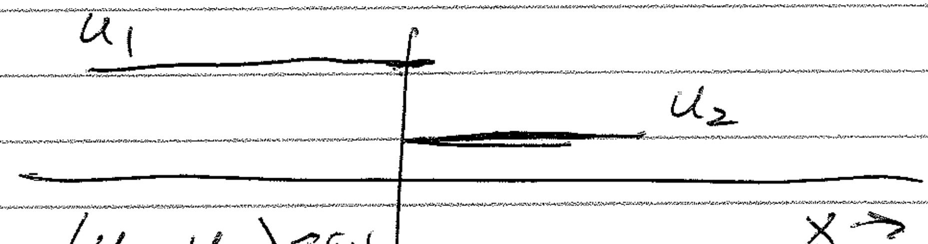
$$\frac{\partial f}{\partial t} + \cancel{f\,\nabla\cdot\underset{\sim}{u}} + \underset{\sim}{u}\cdot\nabla f - \nabla\cdot\underset{\approx}{K}\cdot\nabla f$$
$$\cancel{-\frac{1}{3}\nabla\cdot\underset{\sim}{u}\,\frac{1}{v^2}3v^2 f} - \frac{1}{3}\nabla\cdot u\, v\frac{\partial}{\partial v}f$$
$$=0$$

$$\boxed{\frac{\partial}{\partial t}f + u\cdot\nabla f - \nabla\cdot\underset{\approx}{K}\cdot\nabla f - \frac{1}{3}\left(\nabla\cdot\underset{\sim}{u}\right)\frac{\partial}{\partial(\ln v)}f = 0}$$

Particle Acceleration at a fast shock

1-D

steady state



$$\nabla\cdot\underset{\sim}{u} = -(u_1-u_2)\,\delta(x)$$

$$u\frac{\partial}{\partial x}f - \kappa\frac{\partial^2}{\partial x^2}f + \frac{(u_1-u_2)}{3}\delta(x)\frac{\partial}{\partial(\ln v)}f = 0$$

$x \simeq 0$

$$+\kappa\frac{\partial^2}{\partial x^2}f = +(u_1-u_2)\,\delta(x)\frac{\partial}{\partial(\ln v)}f$$

$$\frac{\partial}{\partial x}f\Big|_1^2 = \frac{(u_1-u_2)}{3\kappa}\frac{\partial}{\partial(\ln v)}f\Big|_{x=0}$$

$\underline{x > 0}$

$$u_2 \frac{\partial}{\partial x} f - \kappa \frac{\partial^2}{\partial x^2} f = 0$$

$$u_2 f - \kappa \frac{\partial}{\partial x} f = \text{const}$$

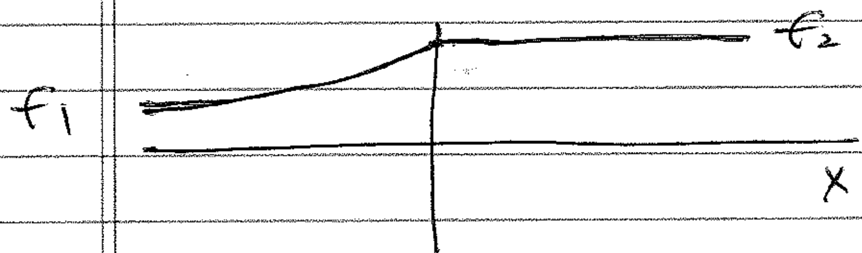
homogeneous $\sim e^{\frac{u_2}{\kappa} x} \Rightarrow$ diverges

$$\Rightarrow f = f_2 = \text{const}$$

$\underline{x < 0}$

$$u_1 f - \kappa \frac{\partial}{\partial x} f = \text{const}$$

$$f = f_1 + (f_2 - f_1) e^{\frac{u_1}{\kappa} x}$$



jump condition

$$-\frac{\partial}{\partial x} f \Big|_{x=0_-} = -(f_2 - f_1)\frac{u_1}{\kappa} = \frac{u_1 - u_2}{3\kappa} \frac{\partial f_2}{\partial \ln v}$$

$$f_2 + \frac{u_1 - u_2}{3 u_1} \frac{\partial}{\partial \ln v} f_2 = f_1$$

let $g = \ln v$

$$f_2 + \underbrace{\frac{u_1 - u_2}{3u_1}}_{= \gamma^{-1}} \frac{\partial}{\partial g} f_2 = f_1$$

$$\frac{\partial}{\partial g} f_2 + \frac{3u_1}{u_1 - u_2} f_2 = f_1 \frac{u_1 3}{u_1 - u_2}$$

$$\frac{\partial}{\partial g} f_2 \underbrace{e^{\frac{3u_1}{u_1-u_2} g}} = f_1 \frac{3u_1}{u_1 - u_2} e^{\frac{3u_1}{u_1-u_2} g}$$

$$\underbrace{e^{\frac{3u_1}{u_1-u_2} \ln v}}$$

$$\gamma = \frac{3u_1}{u_1 - u_2}$$

$$v^{\frac{3u_1}{u_1-u_2}}$$

$$v^{\gamma}$$

$$\cancel{v} \frac{\partial}{\partial v} (f_2 v^{\gamma}) = \gamma f_1 v^{\gamma - 1}$$

$$\boxed{f_2 = \gamma v^{-\gamma} \int_0^v dv' \, v'^{\gamma - 1} f_1(v')}$$

power law with

$$\gamma = \frac{3u_1}{u_1 - u_2} = \frac{n_1 u_1}{n_1 u_1 - \frac{n_1}{n_2} n_2 u_2} = \frac{3r}{r-1}$$

$$\boxed{\gamma = \frac{3r}{r-1}}$$

fast shock $r = 4$

$\gamma = 4$

Measurements of the cosmic rays are expressed as ~~fluxes~~.

$$J \sim \int \underbrace{dv\, v^2}_{\text{volume element}} f\, v$$

$$\varepsilon \sim v^2$$

$$\sim \int d\varepsilon\, f\, v^2$$

$$\frac{dJ}{d\varepsilon} \sim f v^2 \sim \frac{\varepsilon}{\varepsilon^{\gamma/2}} \sim \frac{1}{\varepsilon^{\frac{\gamma}{2}-1}}$$

$$\frac{\gamma}{2} - 1 = \frac{3r}{2(r-1)} - \frac{2(r-1)}{2(r-1)}$$

$$= \frac{r+2}{2(r-1)}$$

~~fast~~ shock

$$\sim \frac{6}{2(3)} \sim 1$$

$$J \sim \frac{1}{\varepsilon}$$

Relativistic limit

$$\mathcal{E} \sim pc$$

$$J \sim \int dp\, f\, c\, p^2$$

$$\frac{dJ}{d\mathcal{E}} \sim f p^2 \sim \frac{1}{p^2} \sim \frac{1}{\mathcal{E}^2}$$

Cosmic Ray Energy Spectrum

- Total energy density ~1eV/cm^3
  - Comparable to magnetic and thermal energy density of interstellar gas
    - Dynamical importance in the galaxy
- Supernova shocks remain the favored mechanism for producing cosmic rays
  - Fermi reflection across the shock front
    - Converging flow at shock
  - Energies up to ~ 10^{15}eV
    - Too small to contain higher energy particles
  - Powerlaw spectra close to observations
    - ~ $E^{-2.7}$
- Jets from active Galactic nuclei and associated radio lobes are large enough to produce particles above 10^{15}eV
  - Open issue
- GZK cutoff at around 10^{20}eV
  - Pion production due to scattering off the microwave background