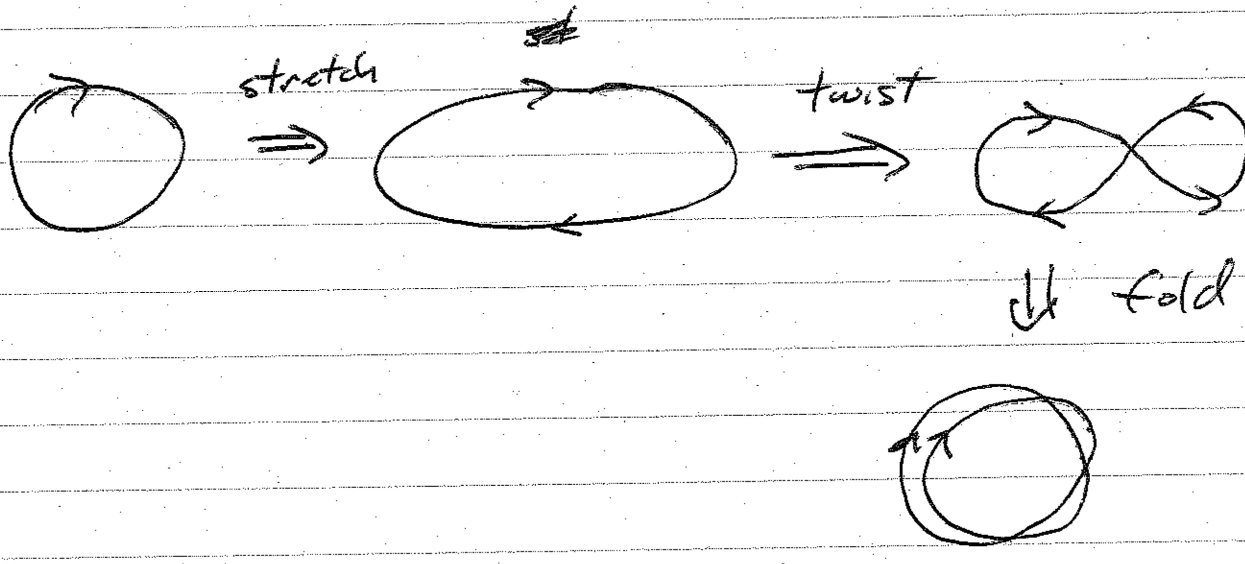
~~Dynamo~~

Generation of Magnetic Fields: the dynamo

Planets, stars and other objects self-generate magnetic fields. ~~which reverse in time~~
The generation is believed to result from the convection of magnetic fields. A simple picture of the amplification process is the stretch-twist-fold dynamo:

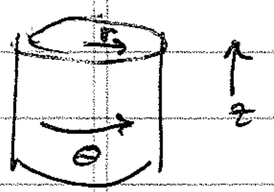


This doubles the poloidal flux. The energy to do this comes from the stretching process.

Cowlings Theorem: no self-generation in a 2-D system

Can a system with 2-D variation amplify $\underline{B}$?

To prove this we consider an axi-symmetric system



$\Rightarrow$ cylindrical geometry with $\frac{\partial}{\partial\theta}(\,) = 0$

$$\underset{\sim}{B}(r,z) = \underbrace{\underset{\sim}{B_p}}_{\text{poloidal}} + \underbrace{\underset{\sim}{B_\theta}}_{\text{toroidal}} \qquad \underset{\sim}{U}(r,z) = \underset{\sim}{U_p} + \underset{\sim}{U_\theta}$$

$$\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla\times\underset{\sim}{E} = 0 \qquad \nabla\times\underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J}$$

$$\underset{\sim}{E} = -\frac{1}{c}\underset{\sim}{u}\times\underset{\sim}{B} + \eta\underset{\sim}{J}$$

Want to separate the induction eqn into poloidal and toroidal components.

<u>Toroidal:</u>

Take $\nabla\theta\cdot\left[\frac{\partial}{\partial t}\underset{\sim}{B} + \cdots\right] = 0$

$$\frac{1}{c}\frac{\partial}{\partial t}\nabla\theta\cdot\underset{\sim}{B} + \underbrace{\nabla\theta\cdot\nabla\times\underset{\sim}{E}}_{-\nabla\cdot(\nabla\theta\times\underset{\sim}{E})} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\nabla\theta\cdot\underset{\sim}{B} + \nabla\cdot\frac{1}{c}\underbrace{\nabla\theta\times(\underset{\sim}{u}\times\underset{\sim}{B})}_{\nabla\theta\cdot\underset{\sim}{B}\,\underset{\sim}{u} - \nabla\theta\cdot\underset{\sim}{u}\,\underset{\sim}{B}} - \nabla\cdot(\nabla\theta\times\eta\underset{\sim}{J}) = 0$$

$$\frac{\partial}{\partial t}\nabla\theta\cdot\underset{\sim}{B} + \underset{\sim}{u}\cdot\nabla(\underset{\sim}{B}\cdot\nabla\theta) - \underset{\sim}{B}\cdot\nabla\,\underset{\sim}{u}\cdot\nabla\theta$$

$$+\underbrace{\eta c\nabla\theta\cdot\nabla\times\underset{\sim}{J}}_{-\frac{\eta c^2}{4\pi}\nabla\theta\cdot\nabla^2\underset{\sim}{B}} = -\frac{\eta c^2}{4\pi}\frac{1}{r}\underbrace{\left[\nabla^2 B_\theta - \frac{1}{r^2}B_\theta\right]}_{\frac{1}{r}\nabla^{*2} rB_\theta}$$

$$\nabla^{*2} = \frac{\partial^2}{\partial z^2} + r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}$$

$$\boxed{\frac{\partial}{\partial t}\frac{B_\theta}{r} + u_p\cdot\nabla\left(\frac{B_\theta}{r}\right) - B_p\cdot\nabla\frac{u_\theta}{r} - \frac{\eta c^2}{4\pi}\frac{1}{r^2}\nabla^{*2} r B_\theta = 0}$$

Poloidal: In a cylinder with $\frac{\partial}{\partial\theta} = 0$ only term that produces a poloidal component is E_θ

$$\frac{\partial}{\partial t}\underset{\sim}{B_p} = \nabla\times\left(\underset{\sim}{u_p}\times\underset{\sim}{B_p}\right) - \eta\,\nabla\times\underset{\sim}{J_\theta}$$

Can write $\underset{\sim}{B_p} = \nabla\psi\times\nabla\theta$ since $\nabla\cdot\underset{\sim}{B_\theta} = 0 \Rightarrow \nabla\cdot\underset{\sim}{B_p} = 0$

$$\Rightarrow \underset{\sim}{B_p} = \nabla\times\psi\nabla\theta$$

$\underset{\sim}{B_p} \equiv \underset{\sim}{G}\times\nabla\theta$

$\nabla\cdot B_p = 0 = \nabla\cdot\underset{\sim}{G}\times\nabla\theta = \nabla\theta\cdot\nabla\times\underset{\sim}{G}$

$\Rightarrow \underset{\sim}{G} = \nabla\psi$

lift $\nabla\times[\ ]$

$$\frac{\partial}{\partial t}\psi\nabla\theta - \underbrace{\underset{\sim}{u_p}\times\underset{\sim}{B_p}}_{\underbrace{\underset{\sim}{u_p}\times(\nabla\psi\times\nabla\theta)}_{(-u_p\cdot\nabla\psi)\nabla\theta}} + \eta c J_\theta + \underbrace{\nabla\Phi}_{\text{no }\nabla\theta\text{ component}} = 0$$

$$\cancel{\frac{4\pi}{c}J_\theta = \left[\nabla\times(\nabla\psi\times\nabla\theta)\right]_\theta = -(\nabla^2\psi)\nabla\theta - \nabla\psi\cdot\nabla\nabla\theta}$$

$$\cancel{= -\hat{\theta}\left[\frac{1}{r}\nabla^2\psi - \frac{\partial\psi}{\partial r}\frac{1}{r^2}\right]}$$

$$\frac{4\pi}{c}\,\nabla\theta\cdot\underset{\sim}{J} = \nabla\theta\cdot\left[\nabla\times\left(\nabla\psi\times\nabla\theta\right)\right]$$

$$= -\nabla\cdot\left[\nabla\theta\times\left(\nabla\psi\times\nabla\theta\right)\right]$$

$$= -\nabla\cdot|\nabla\theta|^2\nabla\psi$$

$$\frac{4\pi}{c}J_\theta = -r\nabla\cdot\frac{1}{r^2}\nabla\psi$$

$$\frac{\partial\psi}{\partial t} + u_p\cdot\nabla\psi - \frac{3c^2}{4\pi}\underbrace{r^2\nabla\cdot\frac{1}{r^2}\nabla\psi}_{\nabla^{*2}\psi} = 0$$

$$\boxed{\frac{\partial\psi}{\partial t} + u_p\cdot\nabla\psi - \frac{3c^2}{4\pi}\nabla^{*2}\psi = 0}$$

$$\frac{\partial \psi}{\partial t} + \underset{\sim}{u}_p \cdot \nabla \psi - \frac{3c^2}{4\pi} \nabla^2 \psi = 0$$

$\Rightarrow$ convective / diffusion equation

$\Rightarrow$ no coupling to B_θ

$\Rightarrow$ no source

$\Rightarrow$ B_p dies away.

$\Rightarrow$ source due to B_p in B_θ eqn $\rightarrow 0$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u}_p \cdot \nabla\right) \frac{B_\theta}{r} - 3 \frac{1}{r^2} \nabla^{*2} r B_\theta = 0$$

$\Rightarrow$ again convection

$\Rightarrow$ $B_\theta \Rightarrow 0$

need $B_p \rightarrow B_\theta \rightarrow B_p$ $\Rightarrow$ closed loop.

Note that $\Omega = \frac{u_\theta}{r}$ is rotation rate so that if Ω varies along $\underset{\sim}{B}_p$ the rotation will twist B_p into the θ direction

$\Rightarrow$ ~~not~~ feed back on B_p so once $B_p \rightarrow 0$ this ends

The α ~~$\alpha\omega$~~ Kinematic dynamo

Now consider the 3-D case. To make progress assume that the flows and magnetic field can be separated into small scale and large scale components

$$\underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{\tilde{B}}$$ ~~take $\tilde{B}$ to be small~~

$$\underset{\sim}{V} = \underset{\sim}{V}_0 + \underset{\sim}{\tilde{V}}$$ $\langle \tilde{V} \rangle = 0$, $\langle \tilde{B} \rangle = 0$

$\Rightarrow$ take $\tilde{B}, \tilde{V}$ to be small.

$\Rightarrow$ write down equations for $\tilde{B}, B_0$

$$\frac{\partial}{\partial t}\underset{\sim}{B}_0 = \nabla\times\left(\underset{\sim}{V}_0\times\underset{\sim}{B}_0\right) - \nabla\times\underset{\sim}{\mathcal{E}} + \eta\nabla^2\underset{\sim}{B}_0$$

$$\frac{\partial}{\partial t}\underset{\sim}{\tilde{B}} = \nabla\times\left(\underset{\sim}{V}_0\times\underset{\sim}{\tilde{B}} + \underset{\sim}{\tilde{V}}\times\underset{\sim}{B}_0\right) + \nabla\times\underset{\sim}{\tilde{G}} + \eta\nabla^2\tilde{B}$$

$\underset{\sim}{\mathcal{E}} = -\langle\underset{\sim}{\tilde{V}}\times\underset{\sim}{\tilde{B}}\rangle$ = average electric field from turbulence.

$$\underset{\sim}{\tilde{G}} = \underset{\sim}{\tilde{V}}\times\underset{\sim}{\tilde{B}} - \langle\tilde{V}\times\tilde{B}\rangle$$

$\Rightarrow$ neglect $\tilde{G}$ since small compared with other terms

$\Rightarrow$ keep $\underset{\sim}{\mathcal{E}}$ because only coupling term for generating $\underset{\sim}{B}_0$.

To evaluate $\underset{\sim}{\mathcal{E}}$ need to find the correlation between $\tilde{v}$ and $\tilde{B}$. This is obtained from the $\tilde{B}$ equation $\Rightarrow$ specifically the $\nabla \times (\tilde{\underset{\sim}{v}} \times \underset{\sim}{B_0})$ drive term.

~~Consider a local region where~~

Jump to a local frame where $\underset{\sim}{V_0} \approx 0$

$$\underset{\sim}{\tilde{B}} = \int^{t} dt' \, \nabla \times (\tilde{\underset{\sim}{v}}(x, t') \times \underset{\sim}{B_0}),$$

where the time dependence of B_0 is neglected.

$$\underset{\sim}{\mathcal{E}} = - \int^{t} dt' \langle \tilde{v}(x,t) \times (\nabla \times (\tilde{\underset{\sim}{v}}(x,t') \times \underset{\sim}{B_0})) \rangle$$

If the turbulence is isotropic ~~[illegible]~~, ~~reasonable for [illegible]~~ there are only two possible vectors which can define the direction for $\underset{\sim}{\mathcal{E}}$: $\underset{\sim}{B_0}$ and $\nabla \times \underset{\sim}{B_0}$

Thus.

$$\underset{\sim}{\mathcal{E}} = \alpha \, \underset{\sim}{B_0} + \beta \, \nabla \times \underset{\sim}{B_0}$$

~~α portion $\Rightarrow$ take B_0 constant along i~~

$$\cancel{\underset{\sim}{\mathcal{E}}_\alpha = - \int^{t} dt' \langle \tilde{\underset{\sim}{v}} \times B_0 \cdot \nabla \tilde{\underset{\sim}{v}}' \rangle}$$

i, j, k

$$\cancel{\mathcal{E}_{\alpha i} = - \int^{t} dt' \, B_{0k} \langle (\tilde{v} \times \nabla_i \tilde{v}')_i \rangle}$$

~~only ( )$_i$ survives because isotropic~~

$$= - B_{0i} \int^{t} dt' \langle \tilde{v}_j \nabla_i \tilde{v}_k' - \tilde{v}_k \nabla_i \tilde{v}_j \rangle$$

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α portion

$$\underset{\sim}{\mathcal{E}}_\alpha = -\int^t dt' \langle \underset{\sim}{\tilde{V}} \times \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{\tilde{V}}' \rangle$$

take $\underset{\sim}{B}_0$ to be locally in the i direction

$$\underset{\sim}{\mathcal{E}}_\alpha = -\int^t dt' \, B_{0i} \langle \underset{\sim}{\tilde{V}} \times \nabla_i \underset{\sim}{\tilde{V}}' \rangle$$

for isotropic $\tilde{V}$

$\langle \underset{\sim}{\tilde{V}} \times \nabla_i \underset{\sim}{\tilde{V}}' \rangle$ will be on average zero unless it is in the ∇_i direction

$\Rightarrow$

$$\mathcal{E}_{\alpha i} = -\int^t dt' \, B_{0i} \underbrace{\langle \underset{\sim}{\tilde{V}} \times \nabla_i \tilde{V}' \rangle_i}_{\langle \tilde{V}_j \nabla_i \tilde{V}_k' - \tilde{V}_k \nabla_i \tilde{V}_j' \rangle}$$

isotropy means can cycle indices

$$\Downarrow$$

$$\langle \tilde{V}_k \nabla_i \tilde{V}_j' \rangle = \langle \tilde{V}_j \nabla_k \tilde{V}_i' \rangle$$

$$\mathcal{E}_{\alpha i} = + B_{0i} \int^t \langle \tilde{V}_j (\nabla \times \tilde{V}')_j \rangle \, dt'$$

$$= \frac{1}{3} B_{0i} \int^t \langle \tilde{V} \cdot \nabla \times \tilde{V}' \rangle \, dt'$$

$$\alpha = \frac{1}{3} \int^t dt' \langle \tilde{V} \cdot \nabla \times \tilde{V}' \rangle$$

$$= \frac{1}{3} \int^t dt' \langle \tilde{V} \cdot \tilde{\omega}' \rangle$$

$$\boxed{\alpha \equiv \frac{1}{3} \tau \langle \tilde{V} \cdot \tilde{\omega} \rangle}$$

may have either sign - kinetic helicity

τ = correlation time

$$\mathcal{E}_\beta = + \int^t dt' \langle \tilde{V} \times (\tilde{V}' \cdot \nabla) B_0 \rangle$$

$\Rightarrow$ only $\langle \tilde{V}_i \tilde{V}_j' \rangle = 0$ for $i \neq j$

$$\mathcal{E}_\beta = \int^t dt' \langle \epsilon_{ijk} \hat{e}_i \tilde{V}_j \tilde{V}_p' \nabla_p B_{0k} \rangle$$

$$\langle \epsilon_{ijk} \hat{e}_i \tilde{V}_j \tilde{V}_j' \nabla_j B_{0k} \rangle$$

$$\frac{1}{3} \langle \epsilon_{ijk} \hat{e}_i \nabla_j B_{0k} \rangle \langle \tilde{V} \cdot \tilde{V}' \rangle$$

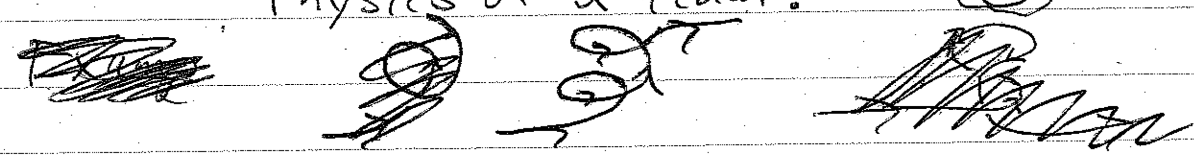
$$\frac{1}{3} \langle \tilde{V} \cdot \tilde{V}' \rangle \; \nabla \times B_0$$

$$\beta = \frac{1}{3} \int^t dt' \langle \tilde{V} \cdot \tilde{V}' \rangle = \boxed{\frac{1}{3} \tau \langle \tilde{V} \cdot \tilde{V} \rangle}$$

$$\frac{\partial}{\partial t} \underset{\sim}{B}_0 = \nabla \times (\underset{\sim}{V}_0 \times \underset{\sim}{B}_0) - \nabla \times \alpha B_0 + (\eta + \beta)\nabla^2 \underset{\sim}{B}_0$$

⇒ β produces an enhanced flux diffusion
⇒ anomalous resistivity

Physics of α term: $\underset{\sim}{\omega} \cdot \underset{\sim}{v} \neq 0$



<u>Flux Eqn</u> As before $\frac{\partial}{\partial t} \nabla \times \psi \nabla\theta + (\;) = -\nabla \times \alpha r B_\theta \nabla\theta$

$$\frac{d}{dt}\psi = \eta \nabla^{*2} \psi + \alpha r B_\theta$$

⇒ azimuthal field now generates poloidal field so Cowling's theorem fails

<u>example</u>

⇒ neglect V_0, $\alpha = \text{const}$

$$\frac{\partial}{\partial t} \underset{\sim}{B}_0 = - \nabla \times \alpha \underset{\sim}{B}_0 \qquad B_0 \sim e^{\gamma t}$$

$$\frac{\partial}{\partial t} \nabla \times B_0 = \alpha \nabla^2 \underset{\sim}{B}_0$$

$$\gamma \underset{\sim}{B}_0 = - \alpha \frac{1}{\gamma} \alpha (-k^2) \underset{\sim}{B}_0$$

$$\boxed{\gamma^2 = \alpha^2 k^2}$$ ⇒ amplification of field