Physics 762

Topics in Nonlinear Plasma Theory

Introduction

Most phenomena which take place in plasmas evolve to a state in which the non-linear behavior is important and ultimately must be understood to describe the dynamics. Why do these non-linear processes dominate plasma dynamics ~~and~~, what type of nonlinearities are important and what techniques have been developed to address these issues. These are some of the topics which will be addressed in this class.

What do we mean ~~by~~ by nonlinear?

Consider the sourceless ~~[illegible]~~

Maxwell equations ~~without~~:

$$\nabla \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{B} \qquad \nabla \cdot \underset{\sim}{E} = 0$$

We can solve these field equations in whatever geometry is of interest. Suppose that the fields have some solution $\underset{\sim}{E}_0, \underset{\sim}{B}_0$ then we know that multiplying these solutions by an arbitrary factor α will yield solutions to the equations because the equations are invariant under multiplacation by any constant. $\Rightarrow$ the equations are <u>linear</u> in the field amplitudes. If we now consider sources:

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J}(\underset{\sim}{x},t) + \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{B} \qquad \nabla \cdot \underset{\sim}{E} = 4\pi \rho(\underset{\sim}{x},t)$$

The sources $\underset{\sim}{J}$ and ρ ~~may not be~~ are generally not related to $\underset{\sim}{E}$, $\underset{\sim}{B}$ in a simple linear manner

$$\underset{\sim}{J} = \underset{\approx}{\mathcal{J}}^{L} \cdot \underset{\sim}{E} + \underset{\approx}{\mathcal{J}}^{NL} \cdot \underset{\sim}{E}\, \underset{\approx}{\mathcal{J}}^{NL} \cdot \underset{\sim}{E} + \ldots$$

where $\underset{\approx}{\mathcal{J}}^{L}$, $\underset{\approx}{\mathcal{J}}^{NL}$ are some integral, differential operators.

$\Rightarrow$ $\underset{\sim}{J}$ may also depend on $\underset{\sim}{E}$, $\underset{\sim}{B}$ in some fractional power relationship. If the fields are now doubled, the current density $\underset{\sim}{J}$ could increase very rapidly and it is evident that the larger amplitude fields do not satisfy the Maxwell eqns.

$\Rightarrow$ the equations are quadratic or have some other power of the field amplitude.

$\Rightarrow$ the equations are now <u>nonlinear</u>

Why are nonlinearities so important in plasmas? In plasmas the restoring forces which keep objects in equilibrium or close to an equilibrium position typically do not exist. In a solid, for example, the lattice of ions is relatively rigid. In a plasma a magnetic field can sometimes constrain the response of a plasma but even such a field does not usually suppress all motions.

What are some of the obvious ~~sources~~ source of nonlinearities ~~in plasmas~~ which can influence plasma dynamics?

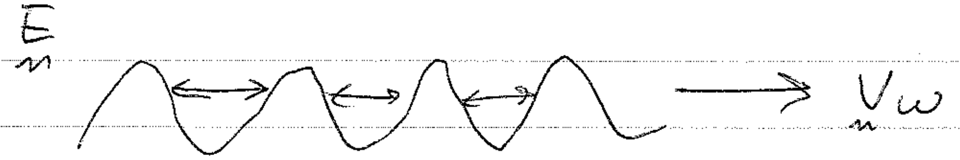
(1) Particle motion

$$m \frac{d\underset{\sim}{v}}{dt} = q\, \underset{\sim}{E}(\underset{\sim}{x}, t) + \frac{q}{c}\, \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{x}, t)$$

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v}$$

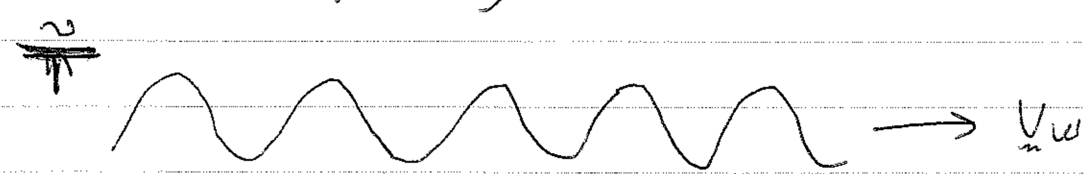
Integrating these equations in time requires that $\underset{\sim}{E}, \underset{\sim}{B}$ be evaluated at the particle position $\underset{\sim}{x}(t)$ but $\underset{\sim}{x}(t)$ is itself a function of $\underset{\sim}{E}, \underset{\sim}{B}$ so the equation of motion is nonlinear.

$\Longrightarrow$ phenomena such as particle trapping



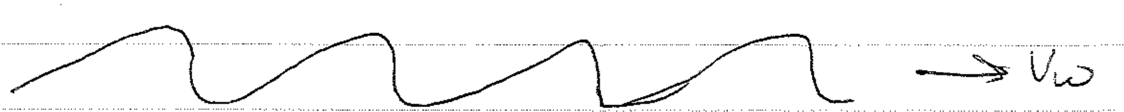
Particles can be trapped in the wave troughs, executing complicated orbits.

(2) Wave Steepening



In sound waves $\omega = kC_s$ so $V_w \sim C_s \sim T^{\frac{1}{2}}$

$\Longrightarrow$ peaks of wave propagate faster than troughs



$\Rightarrow$ leads to harmonic generation

$$\tilde{T} \sim e^{ikx} \Rightarrow e^{2ikx} \Rightarrow e^{4ikx}$$

$\Rightarrow$ forms shock waves if only dissipation limits the steepening process

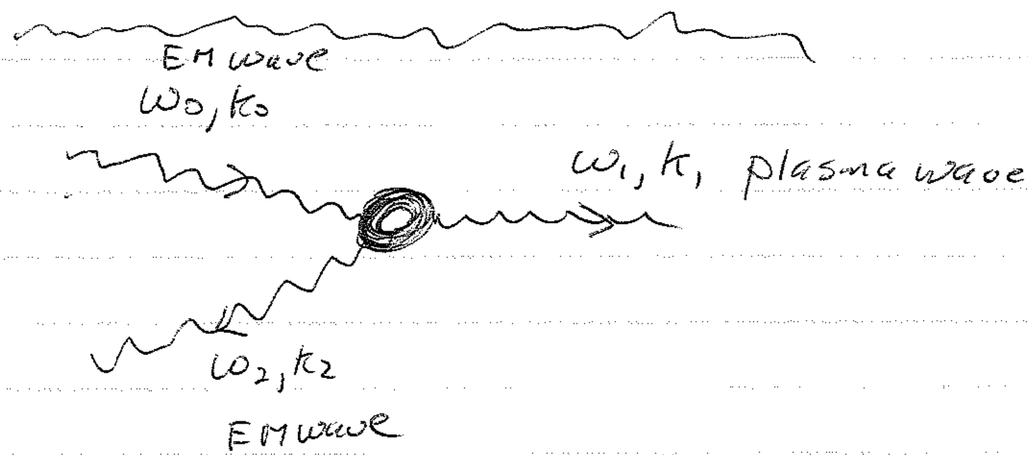
3 Wave scattering

Dependence of wave propagation on characteristics of plasma medium can lead to scattering processes

$\Rightarrow$ Raman scattering

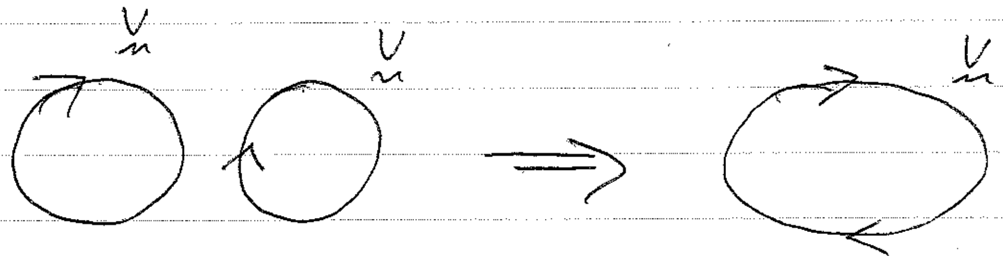
$$\omega^2 = \omega_{pe}^2 + k^2c^2 \quad \text{EM wave}$$

EM wave can scatter off plasma waves



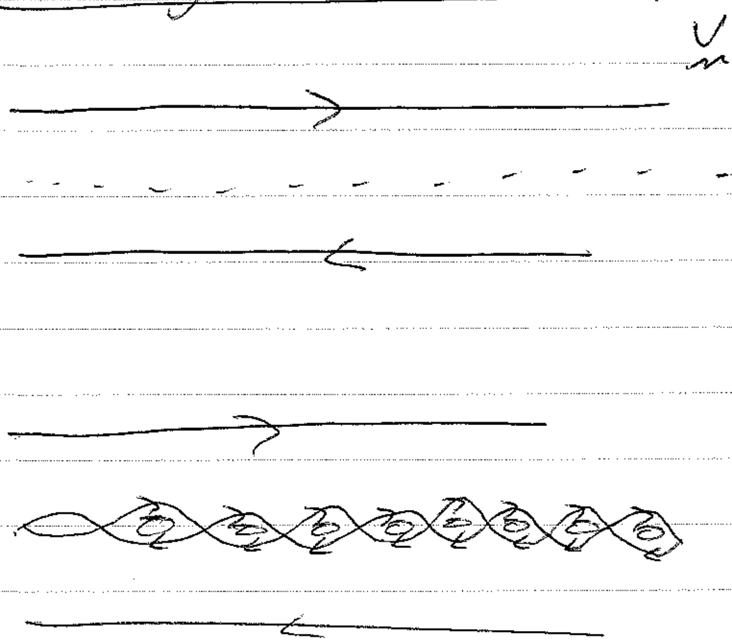
4 Convective Nonlinearity

Convective motions of fluids and plasmas leads to the merging of vortices or the self-generation of smaller scale vortices



Vortex merger

Vortex generation → Kelvin-Helmholtz



⟹ energy cascade to short scales where can be dissipated.

⇒ will study exampmples of all of these processes.

⇒ computation with particle and fluid models is becoming ~~of great~~ essential to the study of plasma dynamics.

⇒ understanding the physical processes which are at work in these codes is critical to ~~developing the~~ ~~use~~ exstracting ~~useful~~ information which has broad importance.

Nonlinear Waves / Structures

Nonlear structures are a common features of plasma observations,
solitons, cavitons, shocks etc.

Skip this
Nonlinear plasma waves

Plasma waves are high frequency compressional modes in magnetized or unmagnetized plasma. Frequency $\sim \omega_{pe}$. Ions can not respond to high frequency. What is characteristic ion response rate? $\sim \omega_{pi} = (4\pi n e^2/m_i)^{1/2}$.

Equations for plasma waves in the cold plasma limit

continuity

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} nV = 0$$

$$\frac{dV}{dt} = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)V = -\frac{eE}{m_e}$$

$$\frac{\partial}{\partial x} E = -4\pi e(n - n_0)$$

Linear wave analysis

$$n = n_0 + \tilde{n}, \quad v = \tilde{v}, \quad E = \tilde{E}$$

$$\frac{\partial}{\partial t}\tilde{n} + n_0 \frac{\partial}{\partial x}\tilde{v} = 0$$

$$\frac{\partial}{\partial t}\tilde{v} = -\frac{e}{m_e}\tilde{E}$$

$$\frac{\partial}{\partial x}\tilde{E} = -4\pi e \tilde{n}$$

$$\frac{\partial^2}{\partial t^2}\tilde{n} + n_0 \frac{\partial}{\partial x}\left(-\frac{e}{m_e}\right)\tilde{E} = 0$$

$$\frac{\partial^2}{\partial t^2}\tilde{n} + n_0\left(-\frac{e}{m_e}\right)\left(-4\pi e \tilde{n}\right) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2\right)\tilde{n} = 0$$

$$\tilde{n} \sim e^{-i\omega t}$$

$$\boxed{\omega^2 = \omega_{pe}^2} \qquad \omega_{pe} = \left(\frac{4\pi n e^2}{m_e}\right)^{\frac{1}{2}}$$

Why are plasma waves important?

① plasmas with electron beams drive plasma waves unstable

$\Rightarrow$ scattering of beams

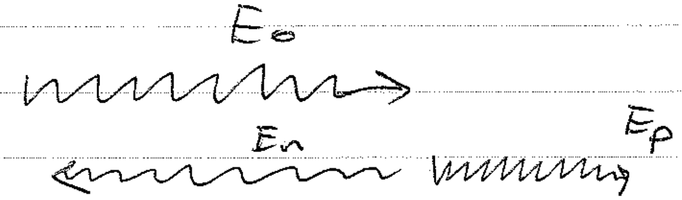
$\Rightarrow$ thermalization ⅋

$\Rightarrow$ type III radio bursts during solar flares

2. Absorption of high power lasers in plasma

$\Rightarrow$ incident laser light amplifies plasma waves and an associated scattered EM wave

$\Rightarrow$ ~~the~~ Raman scattering



3. Creation of large amplitude plasma waves by intense lasers can produce very large electric fields (plasma wave)

$\Rightarrow$ new generation of particle accelerators.

$$E \sim \frac{4\pi e n}{K} \sim \frac{4\pi e n c}{Kc} \sim \frac{4\pi e n c}{\omega}$$

$$\sim \frac{4\pi e^2 n c}{m_e \omega_{pe}} \frac{m_e}{e}$$

$$\sim \omega_{pe} \frac{m_e c}{e}$$

$$eE \sim \omega_{pe} m_e c$$

~~$\omega_{pe} \approx 5.6\times10^4\, n^{1/2}$ rad/sec~~

for $n \sim 10^{18}/cm^3$

$$\frac{eE}{m_e c^2} \sim \frac{\omega_{pe}}{c} \qquad \frac{c}{\omega_{pe}} \sim \frac{5.3\times10^5}{\sqrt{n}} \sim 5.3\times10^{-4}\ cm$$

$$eE \sim \frac{\frac{1}{2}\ MeV}{} \sim \boxed{GeV}$$

Take ~~second~~ derivative of momentum eqn.

$$\frac{d^2 v}{dt^2} = -\frac{e}{m_e}\left(\frac{\partial E}{\partial t} + v\frac{\partial}{\partial x} E \right)$$

From ~~Poisson~~ E Eqn.

$$\frac{\partial}{\partial x}\frac{\partial E}{\partial t} = -4\pi e\frac{\partial n}{\partial t} = -4\pi e\left(-\frac{\partial}{\partial x} nv\right)$$

$$\frac{\partial E}{\partial t} = 4\pi e\, nv$$

$$\frac{d^2}{dt^2} v = -\frac{e}{m_e}\left(\cancel{4\pi e nv} + v\left(-4\pi e(\cancel{n} - n_0)\right)\right)$$

$$= -\frac{4\pi n_0 e^2}{m_e} v \qquad \omega_{pe0}^2 = \frac{4\pi n_0 e^2}{m_e} = \underline{\underline{const}}$$

$$\frac{d^2}{dt^2} v + \omega_{pe0}^2 v = 0 \qquad \cancel{\frac{d}{dt}} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$$

$\Rightarrow$ nonlinearity from convective term.

Define ~~as~~ Lagrangian variables (x_0, τ) which move with the local fluid velocity.

$$x_0 \equiv x - \int_0^{\tau} d\tau' \, v(x_0, \tau')$$

$$\tau = t$$

$$\frac{\partial}{\partial t} = \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial x_0}{\partial t}\frac{\partial}{\partial x_0}$$

$$\frac{\partial T}{\partial t} = 1$$

$$\frac{\partial x_0}{\partial t} = \cancel{\frac{\partial x}{\partial t}}^{\,0} - \frac{\partial T}{\partial t} V(x_0, T) - \int_0^T dT' \frac{\partial V_0}{\partial x_0}\frac{\partial x_0}{\partial t}$$

$$\frac{\partial x_0}{\partial t} = -V(x_0, T)\frac{1}{1 + \int_0^T dT' \frac{\partial}{\partial x_0} V(x_0 T')}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T} - \frac{V}{1 + \int_0^T dT' \frac{\partial}{\partial x_0} V(x_0, T')}\frac{\partial}{\partial x_0}$$

$$\frac{\partial}{\partial x} = \frac{\partial T}{\partial x}\frac{\partial}{\partial T} + \frac{\partial x_0}{\partial x}\frac{\partial}{\partial x_0}$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial x_0}{\partial x} = 1 - \int_0^T dT' \frac{\partial V}{\partial x_0}\frac{\partial x_0}{\partial x}$$

$$\frac{\partial x_0}{\partial x} = \frac{1}{1 + \int_0^T dT' \frac{\partial V}{\partial x_0}}$$

$$\frac{\partial}{\partial x} = \frac{1}{1+\int_0^\tau d\tau' \frac{\partial V}{\partial x_0}} \frac{\partial}{\partial x_0}$$

Note that $\frac{d}{dt} x_0 = 0$

$$\frac{\partial}{\partial t} + V\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}$$

$$\boxed{\frac{\partial^2}{\partial \tau^2} V(x_0,\tau) + \omega_{pe_0}^2 V(x_0,\tau) = 0}$$

V oscillates at the plasma frequency ω_{pe0}

$\Rightarrow$ unaffected by the nonlinearity

density?

$$\frac{d}{dt} n + n\frac{\partial}{\partial x} V = 0$$

$$\frac{\partial n}{\partial \tau} + n \frac{1}{\left(1+\int_0^\tau d\tau' \frac{\partial}{\partial x_0} V\right)} \frac{\partial}{\partial x_0} V = 0$$

$$\frac{\partial}{\partial \tau}\left(n\left(1+\int_0^\tau d\tau' \frac{\partial}{\partial x_0} V\right)\right) = 0$$

$$n\left(1+\int_0^\tau d\tau' \frac{\partial}{\partial x_0} V\right) = \text{const} = n(x_0,0)$$

$$n(x_0,\tau) = \frac{n(x_0,0)}{1+\int^\tau d\tau' \frac{\partial}{\partial x_0} V}$$

initial conditions (two degrees of freedom)

$$n(x_0,0) = n_0(1+\Delta\cos k x_0), \quad V(x_0,0)=0$$

note at $t=0$, $X=x_0$ so n is same in Euler coordinates

$$\left.\frac{\partial V}{\partial \tau}\right|_0 = -\frac{e}{m_e}E(x_0,0)$$

$$\frac{\partial}{\partial x} = \frac{1}{1+\int_0^\tau()}\frac{\partial}{\partial x_0} \qquad \left.\frac{\partial}{\partial x}E\right|_0 = \left.\frac{\partial}{\partial x_0}E\right|_0 = -4\pi e\left[n(x_0,0)-n_0\right]$$

$$= -4\pi e n_0 \Delta\cos k x_0$$

$$\left.\frac{\partial V}{\partial \tau}\right|_0 = +\frac{e\, n_0 4\pi e}{m_e\, k}\Delta\sin k x_0 \qquad \left.E\right|_0 = -\frac{4\pi e}{k}\Delta n_0 \sin k x_0$$

$$V(x_0\tau) = \frac{\omega_{peo}^{\cancel{2}}}{k}\Delta\frac{\sin k x_0}{\cancel{\omega_{peo}}}\sin(\omega_{peo}\tau)$$

$$\boxed{V(x_0,\tau) = \frac{\omega_{peo}\Delta}{k}\sin k x_0 \sin(\omega_{peo}\tau)}$$

$$n(x_0,\tau) = \frac{n(x_0,0)}{1+\int_0^\tau d\tau'\,\omega_{peo}\Delta\cos k x_0\sin(\omega_{peo}\tau')}$$

$$= \frac{n_0(1+\Delta\cos k x_0)}{1 - \Delta\cos k x_0\left[\cos(\omega_{peo}\tau)-1\right]}$$

$$n(x_0;T) = \frac{n_0(1+\Delta\cos kx_0)}{1+\Delta\cos kx_0\left[1-\cos(\omega_{pe0}T)\right]}$$

transformation:

$$x_0 = x - \frac{\cancel{\omega_{pe0}}\Delta}{k}\,\frac{\sin kx_0}{\cancel{\omega_{pe0}}}\left[1-\cos(\omega_{pe0}T)\right]$$

$$x = x_0 + \frac{\Delta}{k}\sin kx_0\left[1-\cos(\omega_{pe0}T)\right]$$

To find $n(x,t)$: for any x_0 map to x given by transformation.

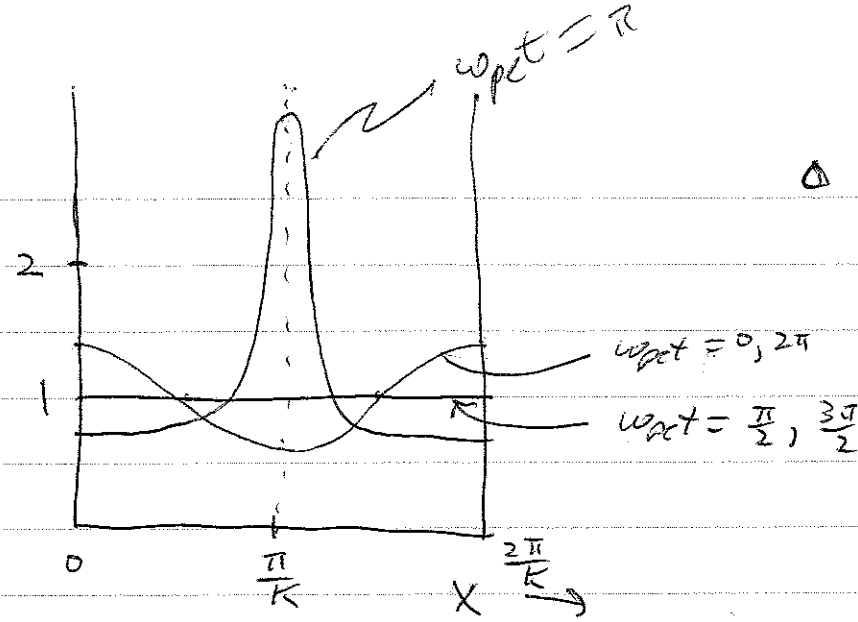
peak values of density:

$$\cos\omega_{pe0}T = -1$$

$$n(x_0,T) = \frac{n_0(1+\Delta\cos kx_0)}{1+2\Delta\cos kx_0}$$

$$\Rightarrow \cos kx_0 = -1 \qquad kx_0 = \pi$$

$$n = \frac{n_0(1-\Delta)}{1-2\Delta}$$ $\Rightarrow$ is singular for $\Delta \to \frac{1}{2}$



$\Delta \lesssim \frac{1}{2}$

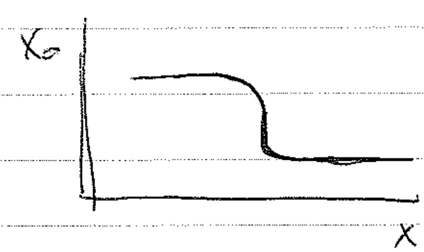
What happens if $\Delta \gtrsim 0.5$?

Lagrangian transformation breaks down

$$\frac{\partial X_0}{\partial X} = \frac{1}{1 + \Delta \cos kX_0 \left[\left(1 - \cos \omega_{pe0} T \right) \right]}$$

when $\Delta \to \frac{1}{2}$

$$\frac{\partial X_0}{\partial X} \to \infty$$



$\Rightarrow$ transformation not single valued.

do this

Nonlinear sound waves — KDV equation and solitons.

Sound waves develop and are important to the dynamics of many plasma systems. eg. weak electron beams excite ion acoustic waves which then scatter the beam to produce an effective anomalous resistivity. ~~Plasma waves~~ The electric fields from plasma waves can produce an effective pressure called the "ponderomotive force" which can create sound waves. Intense lasers do the same in Brillouin scattering. How do we describe the nonlinear development of sound waves?

$\Longrightarrow$ weak nonlinearity

nonlinear equations (fluid with $T_i = 0$)

For low frequency waves can neglect electron inertia and assume electrons are isothermal.

Electron momentum

$$0 = -eE - \frac{T_e}{n_e}\frac{\partial}{\partial x} n_e$$

Take $E = -\frac{\partial \varphi}{\partial x}$ $\Longrightarrow$ $n_e = n_0 e^{+\frac{e\varphi}{T_e}}$

Ion equations

① $$\frac{\partial}{\partial t} n_i + \frac{\partial}{\partial x} n_i V_i = 0$$

② $$\frac{\partial}{\partial t} V_i + V_i \frac{\partial}{\partial x} V_i = -\frac{e}{m_i} \frac{\partial}{\partial x} \varphi$$

Poisson's Eqn

$$\frac{\partial}{\partial x} E = -\frac{\partial^2}{\partial x^2}\varphi = 4\pi e(n_i - n_e)$$

③ $$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e\left(n_0 e^{\frac{e\varphi}{T_e}} - n_i\right)$$

Linear ~~the~~ waves

$$\frac{\partial}{\partial t}\tilde{n}_i + n_0 \frac{\partial}{\partial x}\tilde{V}_i = 0$$

$$\frac{\partial}{\partial t}\tilde{V}_i = -\frac{e}{m_i}\frac{\partial}{\partial x}\tilde{\varphi}$$

take $\tilde{n}_i, \tilde{V}_i, \tilde{\varphi} \sim e^{ikx - i\omega t}$

$$\frac{\partial^2}{\partial x^2}\tilde{\varphi} = 4\pi e\left(n_0 \frac{e\tilde{\varphi}}{T_e} - \tilde{n}_i\right)$$

$$-\omega\tilde{n}_i + n_0 k \tilde{V}_i = 0 \Rightarrow \tilde{n}_i = n_0 \frac{k}{\omega}\tilde{V}_i$$

$$-\omega\tilde{V}_i = -\frac{e}{m_i} k\tilde{\varphi} \Rightarrow \tilde{V}_i = \frac{e}{m_i}\frac{k}{\omega}\tilde{\varphi}$$

$$-k^2\tilde{\varphi} = 4\pi e\left(n_0 \frac{e\tilde{\varphi}}{T_e} - \tilde{n}_i\right) \Rightarrow (k^2 + k_{De}^2)\tilde{\varphi} = 4\pi e\tilde{n}_i$$

$$(k^2 + k_{De}^2)\cancel{\tilde{\varphi}} = 4\pi e\, n_0 \frac{k}{\omega}\frac{e}{m_i}\frac{k}{\omega}\cancel{\tilde{\varphi}}$$

$$k^2\left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = -k_{De}^2$$

$$\omega^2(k^2 + k_{De}^2) = k^2\omega_{pi}^2 \qquad c_s^2 = \frac{\omega_{pi}^2}{k_{De}^2} = \frac{T_e}{m_i}$$

$$\omega^2 = \frac{k^2\omega_{pi}^2}{k_{De}^2}\,\frac{1}{1 + \frac{k^2}{k_{De}^2}}$$

$$\boxed{\omega = k c_s \frac{1}{(1 + k^2/k_{De}^2)^{\frac{1}{2}}}}$$

$\Rightarrow$ propagation at speed c_s at long wavelength

$\Rightarrow$ propagation speed varies with k

$\Rightarrow$ wave dispersion

$\Rightarrow$ small k expansion

$$\omega = kc_s - \frac{1}{2}k^3\frac{c_s}{k_{De}^2}$$

~~Importance of dispersion~~

$\Rightarrow$ wave dispersion is important for systems in which steepening of wavefronts takes place.

$\Rightarrow$ steepening $\Rightarrow$ high k

$\Rightarrow$ altered propagation speed.

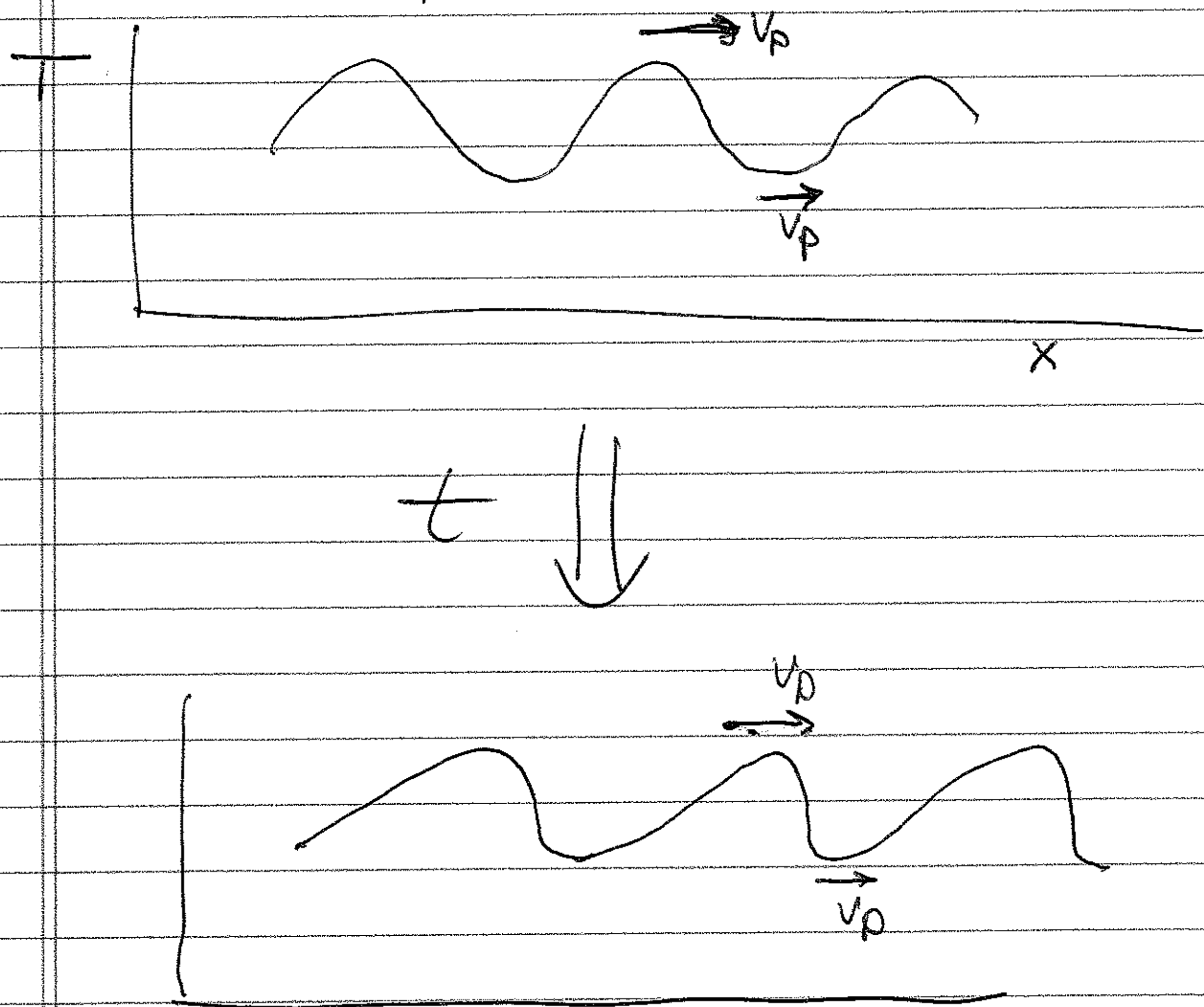
$\Rightarrow$ halt of steepening.

Why steepening?

Since $\omega \sim k C_s$ or $V_p \sim C_s \sim T^{1/2}$

$\Rightarrow$ a region of high pressure propagates faster than a region of low pressure

$\Rightarrow$ consider an initially periodic perturbation



steepening

step

consider a simple model

$$\frac{\partial}{\partial t} V + V \frac{\partial}{\partial x} V = 0$$

$\Rightarrow$ Lagrangian frame

$$\frac{\partial}{\partial \tau} V (x_0, \tau) = 0 \quad \Rightarrow \quad V(x_0, \tau) = V(x_0, 0)$$

$$\cancel{\Rightarrow V = V(x_0, 0)}$$

$$x_0 = x - V(x_0)\,\tau$$

$$\frac{\partial}{\partial x} V = \frac{\partial V}{\partial x_0} \frac{\partial x_0}{\partial x} \qquad \frac{\partial x_0}{\partial x} = 1 - \frac{\partial V}{\partial x_0} \tau \frac{\partial x_0}{\partial x}$$

$$\cancel{\alpha = \frac{\partial x}{\partial x_0} - \frac{\partial V}{\partial x_0} \frac{\partial x}{\partial x_0}}$$

$$\frac{\partial x_0}{\partial x} = \frac{1}{1 + \frac{\partial V}{\partial x_0} \tau}$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x_0} \frac{1}{1 + \frac{\partial V}{\partial x_0} \tau} \qquad \frac{\partial V}{\partial x_0} < 0$$

$$\Rightarrow \frac{\partial V}{\partial x} \to -\infty \text{ in a finite time.}$$

$\Rightarrow$ dispersion can balance the tendency to steepen.

$\Rightarrow$ normalize eqns.

$$\frac{n_i}{n_0} \rightarrow n, \quad \frac{e\varphi}{T_e} \rightarrow \varphi, \quad \frac{v_i}{c_s} \rightarrow v$$

$$\omega_{pi} t \rightarrow t \qquad k_{De} x \rightarrow x$$

(4) $$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} n v = 0$$

(5) $$\frac{\partial}{\partial t} v + v \frac{\partial}{\partial x} v = - \frac{\partial}{\partial x} \varphi$$

(6) $$\frac{\partial^2}{\partial x^2} \varphi = \left(e^{\varphi} - n\right)$$

$\Rightarrow$ go to ~~[illegible]~~ frame moving with sound speed.

$$x' = x - t \qquad \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'}$$
$$t' = t \qquad = \frac{\partial}{\partial t'} - \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$$

$$\begin{cases} \dfrac{\partial n}{\partial t'} - \dfrac{\partial n}{\partial x'} + \dfrac{\partial}{\partial x'} n v = 0 \\[2ex] \dfrac{\partial}{\partial t'} v - \dfrac{\partial}{\partial x'} v + v \dfrac{\partial}{\partial x'} v = - \dfrac{\partial \varphi}{\partial x'} \\[2ex] \dfrac{\partial^2}{\partial x'^2} \varphi = e^{\varphi} - n \end{cases}$$

In the moving frame time variation must be associated with wave dispersion or nonlinearity.

$\frac{\partial}{\partial t'} \ll \frac{\partial}{\partial t}$

$\Rightarrow$ note that this is __not__ true in lab frame

$\Rightarrow$ want these to balance

$$\frac{\partial}{\partial t'} \sim k^3 \sim V \frac{\partial}{\partial x'} \sim kV$$

$$k^2 \sim V \sim \epsilon \ll 1$$

$\Rightarrow$ weak nonlinearity

$$\frac{\partial}{\partial x'} \sim k \sim \epsilon^{1/2}$$

$$\frac{\partial}{\partial t'} \sim \epsilon^{3/2}$$

Expand n, v, φ as series in ϵ

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \cdots$$
$$v = \epsilon v_1 + \epsilon^2 v_2 + \cdots$$
$$\varphi = \epsilon \varphi_1 + \epsilon^2 \varphi_2 + \cdots$$

__lowest order__

$$-\frac{\partial}{\partial x'} n_1 + \frac{\partial}{\partial x'} v_1 = 0 \qquad n_1 = v_1$$

$$-\frac{\partial}{\partial x'} v_1 = -\frac{\partial}{\partial x'} \varphi_1 \qquad v_1 = +\varphi_1$$

$$0 = \varphi_1 - n_1 \qquad n_1 = \varphi_1$$

$$\Rightarrow \boxed{n_1 = v_1 = \varphi_1}$$

next order

$$\frac{\partial}{\partial t'} n_1 - \frac{\partial}{\partial x'} n_2 + \frac{\partial}{\partial x'} v_2 + \frac{\partial}{\partial x'} n_1 v_1 = 0$$

$$\left.\begin{array}{l} \frac{\partial}{\partial t'} v_1 - \frac{\partial}{\partial x'} v_2 + v_1 \frac{\partial}{\partial x'} v_1 = -\frac{\partial}{\partial x'} \phi_2 \\ \frac{\partial}{\partial x'}\left(\frac{\partial^2}{\partial x'^2}\phi_1 = \phi_2 + \frac{1}{2}\phi_1^2 - n_2\right) \end{array}\right\} \text{add.}$$

Let $n_1' = \frac{\partial}{\partial x'} n_1$ etc.

$$\dot{v}_1 = \frac{\partial}{\partial t'} v_1$$

add.

$$\dot{v}_1 - v_2' + v_1 v_1' + \phi_1''' = \phi_1 \phi_1' - n_2'$$

$$2\dot{n}_1 + 2 n_1 n_1' + \cancel{n_1 n_1'} + n_1''' - \cancel{n_1 n_1'} = 0$$

$$\boxed{\dot{n}_1 + n_1 n_1' + \tfrac{1}{2} n_1''' = 0}$$

Korteweg-deVries Eqn

$\Longrightarrow$ a very general equation which describes nonlinearities in dispersive systems.

$$\frac{\partial n}{\partial t} + n \frac{\partial}{\partial x} n + \frac{1}{2} \frac{\partial^3}{\partial x^3} n = 0$$

basic properties

① Galilean invariance

Let $\bar{x} = x + n_0 t$

$\bar{t} = t$

& $\bar{n} = n + n_0$

$\Longrightarrow$ reproduces equation

② Reversibility

$n(-x, -t)$ is a solution if

$n(x,t)$ is a solution

$\Longrightarrow$ non-dissipative

③ Conservation laws

$$\frac{d}{dt} \int_{-\infty}^{\infty} dx \, F(n, n', n'', \ldots) = 0$$

for a number of functions F

e.g.

$$\frac{d}{dt} \int_{-\infty}^{\infty} dx \, n = 0$$

$\Longrightarrow$ area under n preserved.

$$\frac{d}{dt} \int_{-\infty}^{\infty} dx \, n^2 = 0$$

$\vdots$

4 Soliton solutions

The KDV equation has soliton solutions

$\Rightarrow$ spatially localized structures whose velocity depends on the pulse amplitude

$\Rightarrow$ larger amplitude faster

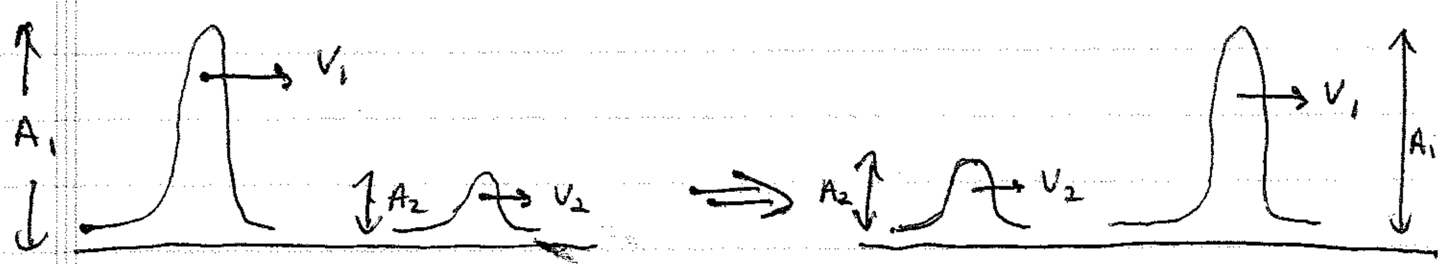
$\Rightarrow$ solitary structures pass through each other ~~without~~ and reform with no change

Single soliton

$$n(x-ct) = 3c\,\mathrm{sech}^2\left[\left(\frac{c}{2}\right)^{\frac{1}{2}}(x-ct)\right]$$

with $c > 0$.

two soliton solution



$\Rightarrow$ n soliton solution

$\Rightarrow$ soliton solution can be obtained analytically

$\Rightarrow$ arbitrary initial condition consists of solitons and ordinary waves

KDV equation solution check

$$\dot{u} + u u_x' + \tfrac{1}{2} u_x''' \qquad u = u(x-ct)$$

$$-cu' + uu' + \tfrac{1}{2}u''' = 0$$

$$-cu + \tfrac{1}{2}u^2 + \tfrac{1}{2}u'' = 0$$

$$u'\left(u'' + u^2 - 2cu\right) = 0$$

$$\frac{u'^2}{2} + \frac{u^3}{3} - \cancel{2}c\frac{u^2}{\cancel{2}} = 0$$

$$\cancel{u' = \sqrt{2cu^2 - \tfrac{2}{3}u^3}}$$

$$u'^2 + \tfrac{2}{3}u^3 - 2cu^2 = 0$$

$$u = 3c\,\mathrm{sech}^2\left[\left(\frac{c}{2}\right)^{\frac{1}{2}}(x-ct)\right]$$

$$\cancel{9c^2}\left(\frac{c}{\cancel{2}}\right)\cancel{4}\,\mathrm{sech}^2(\,)\left(\mathrm{sech}^2\tanh\right)^2$$

$$+\frac{\cancel{2}}{\cancel{3}}\,\cancel{27c^3}\,9\,\mathrm{sech}^6 - \cancel{2c}\,\cancel{9c^2}\,9\,\mathrm{sech}^4 = 0$$

$$\cancel{\mathrm{sech}^4}\tanh^2 + \mathrm{sech}^{\cancel{6}\,2} - \cancel{\mathrm{sech}^4} = 0$$

$$\tanh^2 + \mathrm{sech}^2 - 1 = 0$$

$$0 = 0$$