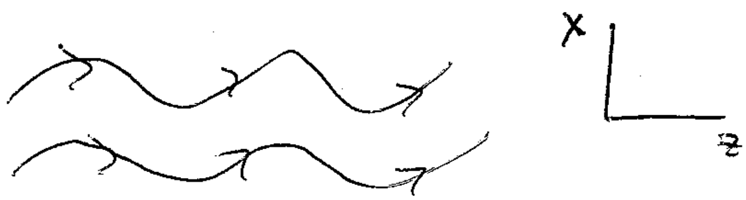
Homework #10 Solutions

(1) Calculate the dispersion relation for the firehose instability

In equilibrium $\underset{\sim}{B} = B_0 \hat{z}$, $\underset{\approx}{P} = \begin{pmatrix} P_\perp & & 0 \\ & P_\perp & \\ 0 & & P_\parallel \end{pmatrix}$

$$\underset{\approx}{P} = P_\parallel \underset{\sim}{b}\underset{\sim}{b} + P_\perp \left(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}\right)$$

$$\rho \frac{d}{dt} \underset{\sim}{v} = -\nabla \cdot \underset{\approx}{P} - \nabla \frac{B^2}{8\pi} + \frac{1}{4\pi} \underset{\sim}{B}\cdot\nabla \underset{\sim}{B}$$



take $V_{1x}, B_{1x} \neq 0$

$V_{1x} = \mathrm{Re}\left(\hat{V} e^{ikz - i\omega t}\right)$

$B_{1x} = \mathrm{Re}\left(\hat{B} e^{ikz - i\omega t}\right)$

$$-i\omega \rho_0 \hat{V} = -\left(\nabla \cdot \underset{\approx}{P}\right)_x + \frac{1}{4\pi} B_0 ik \hat{B}$$

$$\left(\nabla \cdot \underset{\approx}{P}\right)_x = \left(\underset{\sim}{B}\cdot\nabla \frac{P_\parallel \underset{\sim}{B}}{B^2}\right)_x - \left(\underset{\sim}{B}\cdot\nabla \frac{P_\perp}{B^2}\underset{\sim}{B}\right)_x$$

$$= \frac{\cancel{B_0}\, ik P_\parallel \hat{B}}{\cancel{B_0^2}\, B_0} - \cancel{B_0}\, ik \frac{P_\perp}{B_0^{\cancel{2}}} \hat{B}$$

$$-i\omega \rho_0 \hat{V} = -ik\frac{P_\parallel}{B_0}\hat{B} + ik\frac{P_\perp}{B_0}\hat{B} + \frac{B_0}{4\pi} ik \hat{B}$$

$$= ik \frac{B_0}{4\pi}\left(1 + \frac{\beta_\perp}{2} - \frac{\beta_\parallel}{2}\right)\hat{B}$$

$$\beta_\perp = \frac{P_\perp}{B^2/8\pi}, \quad \beta_\parallel = \frac{P_\parallel}{B^2/8\pi}$$

$$\Rightarrow \cancel{\frac{1}{c}} \frac{\partial}{\partial t} \underset{\sim}{B} - \cancel{\frac{1}{c}} \underbrace{\nabla \times (\underset{\sim}{V} \times \underset{\sim}{B})} = 0$$

$$-i\omega \hat{\underset{\sim}{B}} - ik \; \hat{z} \times (\underset{\sim}{V} \times \hat{z}) B_0 = 0$$

$$\cancel{+}\cancel{i}\omega \hat{\underset{\sim}{B}} + \cancel{i} K \underset{\sim}{V} B_0 = 0$$

$$\omega \hat{B} + k B_0 \hat{V} = 0$$

$$-\cancel{i}\omega \rho_0 \hat{V} = \cancel{i}k \frac{B_0}{4\pi}\left(1 + \frac{\beta_\perp}{2} - \frac{\beta_\parallel}{2}\right)\left(-\frac{kB_0}{\omega}\hat{V}\right)$$

$$\omega^2 = k^2 \frac{B_0^2}{4\pi\rho_0}\left(1 + \frac{\beta_\perp}{2} - \frac{\beta_\parallel}{2}\right)$$

$$\boxed{\omega^2 = k^2 C_A^2 \left(1 + \frac{\beta_\perp}{2} - \frac{\beta_\parallel}{2}\right)}$$ firehose relation

$\Rightarrow$ instability for $\boxed{\beta_\parallel - \beta_\perp > 2}$

Kinetic model of ~~[illegible]~~ whistler/cyclotron waves

Assume $\underset{\sim}{B}_0 = B_0 \hat{z}$ $\quad f_0 = f_0(v)$

$$f_1 = \mathrm{Re}\left(\hat{f} e^{ikz - i\omega t}\right), \quad \underset{\sim}{E}_1 = \mathrm{Re}\left(\hat{\underset{\sim}{E}} e^{ikz - i\omega t}\right)$$

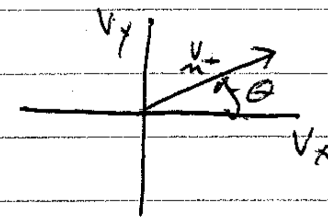
Use circularly polarized wave

$$\hat{\underset{\sim}{E}} = \cancel{[illegible]} \left(\hat{E}, i\hat{E}, 0\right)$$

From the linearized Boltzmann eqn.

$$-i(\omega - k v_\parallel)\hat{f} - \frac{e}{m_e}\left(\hat{\underset{\sim}{E}} + \cancelto{0}{\frac{1}{c} \underset{\sim}{v} \times \hat{\underset{\sim}{B}}}\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0$$

$$- \frac{e}{m_e}\frac{B_0}{c} \underbrace{v \times \hat{z} \cdot \frac{\partial}{\partial \underset{\sim}{v}}}_{-v_\perp \frac{1}{v_\perp}\frac{\partial}{\partial \theta}} \hat{f} = 0$$



$$\frac{\partial}{\partial \underset{\sim}{v}} f_0 = 2\underset{\sim}{v} \frac{\partial f_0}{\partial v^2}$$

$$\bar{\omega} \equiv \omega - k v_\parallel$$

~~$-i\omega$~~

$$\left(-i\bar{\omega} + \Omega \frac{\partial}{\partial \theta}\right)\hat{f} - \frac{e2}{m_e} \hat{E} \cdot \underset{\sim}{v} \frac{\partial f_0}{\partial v^2} = 0$$

$$\hat{E} \cdot \underset{\sim}{v} = \hat{E}_x v_x + \hat{E}_y v_y = \left(\hat{E} \cos\theta + i\hat{E}\sin\theta\right) v_\perp$$

$$= \hat{E} v_\perp e^{i\theta}$$

$$\left(-i\bar{\omega} + \Omega \frac{\partial}{\partial\theta}\right)\hat{f} = 2\frac{e}{m_e}\hat{E}\, v_\perp e^{i\theta} \frac{\partial f_0}{\partial v^2}$$

$$\Rightarrow \hat{f} \sim e^{i\theta}$$

$$\hat{f} = 2\frac{e}{m_e}\frac{\hat{E}\, v_\perp e^{i\theta}}{-i(\bar{\omega} - \Omega)}\frac{\partial f_0}{\partial v^2} \equiv \hat{A}\, e^{i\theta}\hat{E}$$

$$-\cancel{i}\frac{\omega}{c}\hat{\underline{B}} + \cancel{i}\,\underline{k}\times\hat{\underline{E}} = 0 \qquad \hat{B} = \frac{k\times\hat{E}}{1}\frac{c}{\omega}$$

$$i\,\underline{k}\times\hat{\underline{B}} = \frac{4\pi}{c}\hat{\underline{J}}$$

$$i\frac{c}{\omega}\underbrace{\underline{k}\times(\underline{k}\times\hat{\underline{E}})}_{-k^2\hat{\underline{E}}} = \frac{4\pi}{c}\hat{\underline{J}}$$

$$-i\frac{k^2 c}{\omega}\hat{\underline{E}} = \frac{4\pi}{c}\hat{\underline{J}}$$

$$\hat{\underline{J}} = -e\hat{E}\int d\underline{v}\,\hat{A}\, e^{i\theta} v_\perp(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

$$= -e\,\hat{E}\,\frac{1}{2}\int d\underline{v}\,\hat{A}\, v_\perp(\hat{x} + i\hat{y})$$

$$= -e\,\frac{1}{2}\int d\underline{v}\,\hat{A}\, v_\perp\,\hat{\underline{E}}$$

$$+i\frac{k^2 c}{\omega}\cancel{\hat{\underline{E}}} = \frac{4\pi}{c}\left(+\frac{e}{2}\int d\underline{v}\,\hat{A}\, v_\perp\right)\cancel{\hat{\underline{E}}}$$

$$\cancel{i}\frac{k^2c}{\omega} = \frac{4\pi}{c}\frac{e}{\cancel{\,}}\int d\underline{v}\, v_\perp \frac{\frac{\cancel{}e}{m_e} v_\perp \frac{\partial f_0}{\partial v^2}}{\cancel{+}\cancel{i}(\omega - \Omega)}$$

$$\frac{\partial f_0}{\partial v^2} = -\frac{f_0}{v_t^2} \qquad \int d\underline{v}\, v_\perp^2 f_0 = v_t^2$$

$$\frac{k^2c^2}{\omega} = -\frac{4\pi e^2}{m_e} n_0 \int \frac{dv_\parallel}{\sqrt{\pi}} \frac{e^{-v_\parallel^2/v_t^2}}{\omega - kv_\parallel - \Omega}$$

$$\boxed{k^2c^2 = -\omega_{pe}^2 \int \frac{dv_\parallel}{\sqrt{\pi}} \frac{e^{-v_\parallel^2/v_t^2}}{v_t} \frac{\omega}{\omega - kv_\parallel - \Omega}}$$

3 Cold plasma limit

$$k^2c^2 = -\omega_{pe}^2 \frac{\omega}{\omega - \Omega} \qquad (\omega - \Omega)k^2c^2 = -\omega\,\omega_{pe}^2$$

$$\boxed{\omega_k = \frac{k^2c^2}{\omega_{pe}^2 + k^2c^2}\Omega}$$

resonant velocity

$$v_r = \frac{\omega - \Omega}{k} = -\frac{\omega_{pe}^2}{\omega_{pe}^2 + k^2c^2}\frac{\Omega}{k}$$

Assume $|v_{\parallel r}| \gg v_t$

$$k^2c^2 = \omega_{pe}^2\left[-\frac{\omega}{\omega - \Omega} + \frac{i\sqrt{\pi}}{+kv_t}\,\omega\, e^{-\frac{v_r^2}{v_t^2}}\right]$$

$$\frac{k^2c^2}{\omega_{pe}^2}\frac{(\omega-\Omega)}{\omega} = -1 + i\sqrt{\pi}\,\frac{\omega-\Omega}{kv_t}\,e^{-\frac{v_r^2}{v_t^2}}$$

$$\frac{k^2c^2}{\omega_{pe}^2}\left(\frac{\Omega}{\omega_k^2}\Delta\omega\right) = +i\sqrt{\pi}\,\frac{v_r}{v_t}\,e^{-\frac{v_r^2}{v_t^2}}$$

$$\frac{\Delta\omega}{\omega_k} = -i\sqrt{\pi}\,\frac{|v_r|}{v_t}\,e^{-\frac{v_r^2}{v_t^2}}\,\frac{\omega_k}{\frac{k^2c^2}{\omega_{pe}^2}\Omega}$$

$$\boxed{\frac{\Delta\omega}{\omega_k} = -i\sqrt{\pi}\,\frac{|v_r|}{v_t}\,e^{-\frac{v_r^2}{v_t^2}}\,\frac{1}{1+k^2c^2/\omega_{pe}^2}}$$