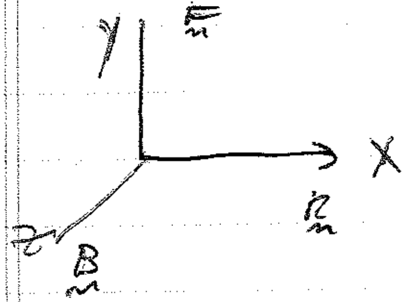
Electromagnetic Waves

Transverse waves



$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

VE

for Maxwellian

$$-i\omega \hat{f} + i\mathbf{k}\cdot\mathbf{v}\,\hat{f} + \frac{q}{m}\Big(\hat{\mathbf{E}} + \overset{0}{\cancel{\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}}}\Big)\cdot\frac{\partial}{\partial \mathbf{v}} f_0 = 0$$

$$\hat{f} = -\frac{q}{m}\,\frac{\hat{\mathbf{E}}\cdot\frac{\partial}{\partial \mathbf{v}} f_0}{i(\mathbf{k}\cdot\mathbf{v} - \omega)}$$

$$f_0 \sim \frac{e^{-\frac{v^2}{v_t^2}}\, n_0}{\pi^{3/2} v_t^3}$$

Calculate current

$$\hat{\mathbf{J}} = -\frac{q^2}{m}\int d\mathbf{v}\, \frac{\mathbf{v}\;\hat{\mathbf{E}}\cdot\frac{\partial}{\partial \mathbf{v}} f_0}{i(\mathbf{k}\cdot\mathbf{v} - \omega)}$$

$$= -\frac{q^2}{m}\int d\mathbf{v}\left(-\frac{2}{v_t^2}\right)\frac{\mathbf{v}\;\mathbf{v}\cdot\hat{\mathbf{E}}\; f_0}{i\,\mathbf{k}\cdot\mathbf{v} - \omega}$$

Only direction of $\hat{\mathbf{E}}$ survives

$$\hat{\mathbf{J}} = +\frac{2q^2}{m v_t^2}\int d\mathbf{v}\,\frac{v_E^2 f_0}{i(\mathbf{k}\cdot\mathbf{v} - \omega)}\,\hat{\mathbf{E}}$$

$$\int dv_y \, v_y^2 f_0 = \frac{1}{2} v_{te}^2 \int dv_y f_0$$

$$\hat{J} = + \frac{\cancel{k} q^2}{m v_{te}^2} \cancel{\frac{1}{2}} v_{te}^2 \hat{E} \int \frac{dv_x \; e^{-v_x^2/v_{te}^2}}{\sqrt{\pi}\, i v_{te} (k v_x - \omega)} n_0$$

$$= + \frac{q^2 \hat{E}}{m} \frac{1}{i k v_{te}} Z\left(\frac{\omega}{k v_{te}}\right)$$

$$i k \times \hat{B} = \frac{4\pi}{c} \frac{q^2}{m} \frac{n_0}{i k v_{te}} Z\left(\frac{\omega}{k v_{te}}\right) \hat{E} + \frac{i\omega}{c} \hat{E}$$

$$-\cancel{i}\omega \frac{\hat{B}}{c} + \cancel{i} k \times \hat{E} = 0$$

$$\hat{B} = c\, k \times \hat{E} \frac{1}{\omega}$$

$$\cancel{\frac{c}{\omega}} k \times (k \times \hat{E}) = -\frac{4\pi}{c^2} \frac{q^2 n_0}{m} \frac{\omega}{\cancel{i} k v_{te}} Z(\zeta_e) \hat{E} - \frac{\omega^2}{c^2} \hat{E}$$

$$-k^2 \hat{E} = + \frac{\omega_{pe}^2}{c^2} \zeta_e Z(\zeta_e) \hat{E} + \frac{\omega^2}{c^2} \hat{E} = + \frac{\omega^2}{c^2} \epsilon_\perp \hat{E}$$

$$\frac{k^2 c^2}{\omega^2} = \frac{\omega_{pe}^2}{\omega^2} \zeta_e Z(\zeta_e) + \cancel{\frac{\omega^2}{c^2}} 1 = \epsilon_\perp$$

$$\omega^2 = k^2 c^2 - \omega_{pe}^2 \zeta_e Z(\zeta_e)$$

large argument

$$Z(\zeta_e) \sim -\frac{1}{\zeta_e} \qquad \epsilon_\perp = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

$$\epsilon_\perp = 1 + \frac{\omega_{pe}^2}{\omega^2} \zeta_e Z(\zeta_e)$$

$$\omega^2 = k^2c^2 + \omega_{pe}^2$$

Light waves propage anywhere where
$\omega > \omega_{pe}$

$\Rightarrow$ cut off for $\omega < \omega_{pe}$

$$k^2c^2 = \omega^2 - \omega_{pe}^2 < 0$$

ham radio
can communicate
would wide

General Dielectric Function

$$i\underset{\sim}{k} \times \underset{\sim}{\hat{B}} = \frac{4\pi}{c}\hat{J} - \frac{i\omega}{c}\underset{\sim}{\hat{E}} \equiv -\frac{i\omega}{c}\underset{\approx}{\epsilon}\cdot\underset{\sim}{\hat{E}}$$

$$\underset{\sim}{k} \times \underset{\sim}{\hat{B}} = -\frac{\omega}{c}\underset{\sim}{\epsilon}\cdot\hat{E}$$

$$-\cancel{i}\frac{\omega}{c}\underset{\sim}{\hat{B}} + \cancel{i}\,\underset{\sim}{k}\times\hat{E} = 0 \qquad \hat{B} = \frac{c}{\omega}\underset{\sim}{k}\times\underset{\sim}{\hat{E}}$$

$$\frac{c^2}{\omega^2}\underbrace{\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\hat{E}})}_{(\underset{\sim}{k}\underset{\sim}{k} - \underset{\sim}{I}k^2)\cdot\underset{\sim}{\hat{E}}} + \underset{\approx}{\epsilon}\cdot\underset{\sim}{\hat{E}} = 0$$

dot with $\underset{\sim}{k}$

$$\underset{\sim}{k}\cdot\underset{\approx}{\epsilon}\cdot\hat{E} = 0$$

$$\epsilon_{\parallel} = 1 - \frac{k_{De}^2}{2k^2} Z'\left(\frac{\omega}{kv_{te}}\right)$$

cold limit

$$\epsilon_{\parallel} = \epsilon_{\perp} = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

$$\underset{\approx}{\epsilon} = \epsilon_{\parallel}\frac{\underset{\sim}{k}\underset{\sim}{k}}{k^2} + \epsilon_{\perp}\left(\underset{\sim}{I} - \frac{\underset{\sim}{k}\underset{\sim}{k}}{k^2}\right)$$

$$\epsilon_{\perp} = 1 + \frac{\omega_{pe}^2}{\omega^2}\zeta_e Z(\zeta_e)$$

$$\underline{\underline{G}} = \frac{c^2}{\omega^2}\left(\underline{k}\underline{k} - \underline{\underline{I}}k^2\right) + \underline{\underline{\epsilon}}$$

$$\underline{\underline{G}} \cdot \hat{\underline{E}} = 0$$

<u>Dispersion Relation</u>

$$\det|\underline{\underline{G}}| = 0$$

$\Rightarrow$ both ES, EM waves.

Wave Energy and Momentum

From Maxwell's Eqns know that

$$\frac{\partial}{\partial t} U + \nabla \cdot \underline{S} = -\underbrace{\underline{E} \cdot \underline{J}}_{\text{work done by fields on particles}}$$

$$U = \frac{E^2 + B^2}{8\pi}, \quad \underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$$

Also have momentum Eqn.

$$\frac{d}{dt}\left(\underline{P}_{part.} + \underline{P}_{field}\right) = \nabla \cdot \underline{\underline{T}}$$

$$\underline{P}_{field} = \frac{1}{4\pi c} \underline{E} \times \underline{B} \qquad \underline{P}_{part} = \sum_j m_j n_j \underline{u}_j$$

Want to evaluate energy and momentum as did previously for ES. waves.

Energy density of general wave

$\Rightarrow$ how much work do I do to create a wave of some amplitude?

$\Rightarrow$ introduce a test current and charge ρ_t, $\underset{\sim}{J}_t$

$$\nabla \cdot \underset{\sim}{E} = 4\pi \rho_{plasma} + 4\pi \rho_t$$

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} J_{plasma} + \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\underset{\sim}{J}_t$$

Continuity of test charge

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \underset{\sim}{J}_t = 0$$

$$-i\omega_k \rho_{kt} + i\underset{\sim}{k}\cdot \underset{\sim}{J}_{kt} = 0$$

$$\omega_k = \omega_r + i\gamma_k$$

$$\omega_{-k} = -\omega_r + i\gamma_k$$

$\dot{W}$ = rate of energy going into the wave

$\langle\,\rangle$ = space average.

$$\dot{W} = -\langle \underset{\sim}{J}_t \cdot \underset{\sim}{E} \rangle$$

$E_k = -ik\varphi_k$

$E_{-k} = ik\varphi_{-k}$
$= ik\varphi_k^*$
$= E_k^*$

= $-$work done by field on $\underset{\sim}{J}_t$ as before

$$= -\sum_k \underset{\sim}{J}_{tk} \cdot \underset{\sim}{E}_k^* \, e^{2\gamma_k t}$$

$$i\underset{\sim}{k}\times\left(\frac{\underset{\sim}{k}c}{\omega_k}\times \underset{\sim}{E}_k\right) + \frac{i\omega_k}{c}\underset{\approx}{\epsilon}\cdot \underset{\sim}{E}_k = \frac{4\pi}{c}\underset{\sim}{J}_{tk}$$

$$\frac{i\omega_k}{c}\underset{\approx}{G}\cdot \underset{\sim}{E}_k = \frac{4\pi}{c}\underset{\sim}{J}_{tk}$$

$$\dot{w} = -\sum_k \frac{i\omega_k}{4\pi}\underset{\sim}{E}_k^*\cdot \underset{\approx}{G}\cdot \underset{\sim}{E}_k\, e^{2\gamma_k t}$$

For small damping

$$\omega_k \underset{\approx}{G} \simeq \omega_r \underset{\approx}{G}(\omega_r) + i\gamma_k \frac{\partial}{\partial \omega_r}\left[\omega_r G(\omega_r)\right]$$

neglect dissipation

$$\mathrm{Im}\, G(\omega_r) = 0 \quad \Rightarrow \quad G_{-k} = G_k$$

$$\dot{w} = -\sum_k \frac{i}{4\pi}\underset{\sim}{E}_k^*\cdot\left(\cancel{\omega_r G(\omega_r)} + i\gamma_k \frac{\partial}{\partial \omega_r}\omega_r G(\omega_r)\right)\cdot E_k$$

$$= \sum_k \gamma_k \frac{1}{4\pi} E_k^*\cdot \frac{\partial}{\partial \omega_r}(\omega_r G)\cdot \underset{\sim}{E}_k\, e^{2\gamma_k t}$$

$$\bar{w} = \sum_k \frac{\underset{\sim}{E}_k^*}{8\pi}\cdot \frac{\partial}{\partial \omega_r}\left(\omega_r \underset{\approx}{G}\right)\cdot \underset{\sim}{E}_k$$

$$= \sum_k \frac{1}{8\pi} E_k^*\cdot \frac{\partial}{\partial \omega_r}\omega_r\Big[\underbrace{\underset{\sim}{k}\times(\underset{\sim}{k}\times \underset{\sim}{E}_k)\frac{c^2}{\omega_r^2}}_{|B_k|^2} + \underset{\approx}{\epsilon}\cdot \underset{\sim}{E}_k\Big]$$

$$= \sum_k \frac{1}{8\pi}\Big[|E_k|^2 + |B_k|^2 + E_k^*\frac{\partial}{\partial \omega_r}\omega_r(\underset{\approx}{\epsilon} - \underset{\approx}{I})\cdot E_k$$

$$\sum_k \frac{1}{8\pi} \underline{E}_k^* \cdot \frac{\partial}{\partial \omega_r} \omega_r \left(\underline{\underline{\epsilon}} - \underline{\underline{I}}\right) \cdot \underline{E}_k$$

is plasma energy

Momentum density

⇒ calculate rate of momentum transfer from test current to the wave.

$$\frac{d}{dt} \underline{P}_t = \left\langle e_t n_t \underline{E} + e_t n_t \frac{\underline{v}_t \times \underline{B}}{c} \right\rangle$$

$$\dot{P}_t = \left\langle \rho_t \underline{E} + \frac{1}{c} \underline{J}_t \times \underline{B} \right\rangle$$

$$= \sum_k \left(\underline{E}_k^* \rho_{tk} + \frac{\underline{J}_{tk} \times B_k^*}{c} \right)$$

cancel

$B_k = \frac{c}{\omega_k} \underline{k} \times \underline{E}_k$

$$-i\omega_k \rho_{tk} + i \underline{k} \cdot J_{tk} = 0$$

$$\frac{1}{c} \underline{J}_{tk} \times B_k^* = \underline{J}_{tk} \times \left(\underline{k} \times \underline{E}_k^* \right) \frac{1}{\omega_k^*}$$

$$= \frac{\underline{k}}{\omega_k^*} \underline{J}_{tk} \cdot \underline{E}_k^* - \frac{\underline{k} \cdot \underline{J}_{kt}}{\omega_k^*} \underline{E}_k^*$$

$\omega_r \gg \gamma$

$$= \frac{\underline{k}}{\omega_k^*} \underline{J}_{tk} \cdot \underline{E}_k^* - \cancel{\frac{\omega_k \rho_{kt}}{\omega_k^*}} \underline{E}_k^*$$

$$\dot{P}_t \simeq \sum_k \frac{\underline{k}}{\omega_k^*} J_{tk} \cdot \underline{E}_k^* \; e^{2\gamma_k t}$$

$$P_t^{*} = -\underset{\sim}{P}$$

$$\underset{\sim}{P}_{plasma} = \frac{\underset{\sim}{k}}{\omega_K} \bar{W}_{plasma}$$

$\Rightarrow$ momentum and energy are linked.

Group Velocity and Power Flux

Have the dispersion relation

$$\det|\underset{\approx}{G}| = 0$$

This yields $\omega(\underset{\sim}{k})$. The group velocity is given by $\underset{\sim}{V}_g = d\omega/d\underset{\sim}{k}$

$\Rightarrow$ velocity of a wavepacket
$\Rightarrow$ rate of energy transport.

$$\underset{\sim}{\Gamma} = V_g \bar{W}$$

$$\Gamma = \text{power flux} = \frac{\text{ergs}}{\text{s cm}^2}$$

To show this need to address the behavior of a wave packet. Consider

$$\underset{\approx}{G} \cdot \underset{\sim}{E} = 0$$

Let $\omega = \omega_0 + \delta\omega$, $k = k_0 + \delta k$, $\underset{\sim}{E} = \underset{\sim}{E}_0 + \delta \underset{\sim}{E}$

$\delta\omega, \delta k$ correspond to slow time and space variation.

Expand in $\delta\omega, \delta \underset{\sim}{k}$ $\quad \underset{\approx}{G}_0 = \underset{\approx}{G}_0(\omega_0, \underset{\sim}{k}_0)$

$$\cancel{\underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0} + \underset{\approx}{G}_0 \cdot \delta E + \delta\omega \frac{\partial}{\partial \omega_0}\left(G_0 \cdot \underset{\sim}{E}_0\right)$$

$$+ \delta \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{k}_0}\left(\underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0\right) = 0$$

Take dot product from left with E_0.
Assume $\underset{\approx}{G}$ is hermitian

$\frac{\partial}{\partial\omega_0}, \frac{\partial}{\partial k_0}$ act only on $\underset{\approx}{G}_0$

$$\Rightarrow \quad \underset{\sim}{E}_0^* \cdot \underset{\approx}{G}_0 = 0$$

$$\delta\omega \frac{\partial}{\partial \omega_0} E_0^* \underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0$$

$$= - \delta \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{k}_0} \underset{\sim}{E}_0^* \cdot \underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0$$

~~bring $\underset{\sim}{E}_0^* \cdot$ inside operators since $\underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0$ is zero unless take the derivative.~~

Multiply by ω_0 (bring inside $\frac{\partial}{\partial\omega_0}$ since $\underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0 = 0$

$$-i\delta\omega \underbrace{\left(\frac{\partial}{\partial \omega_0} \frac{\omega_0 \underset{\sim}{E}_0^* \cdot \underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0}{8\pi}\right)}_{\overline{\omega}} + i\delta \underset{\sim}{k} \cdot \underbrace{\left(-\frac{\partial}{\partial \underset{\sim}{k}_0} \frac{\omega_0 \underset{\sim}{E}_0^* \cdot \underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0}{8\pi}\right)}_{\underset{\sim}{\Gamma}} = 0$$

$$-i\delta\omega\, \overline{\omega} + i\delta \underset{\sim}{k} \cdot \underset{\sim}{\Gamma} = 0$$

$$\underset{\sim}{\Gamma} = -\frac{\omega_0}{8\pi} \frac{\partial}{\partial \underset{\sim}{k}_0} \underset{\sim}{E}_0^* \cdot \underset{\approx}{G}_0 \cdot \underset{\sim}{E}_0$$

Let $\Xi' = \omega_k \underset{\sim}{E}^*_0 \cdot \underset{\approx}{G} \cdot \underset{\sim}{E}_0$

$$\frac{d}{dk}\Xi' = \frac{\partial \Xi'}{\partial \underset{\sim}{k}} + \frac{\partial \Xi'}{\partial \omega_k}\frac{d\omega_k}{d\underset{\sim}{k}} = 0$$

$$\underset{\sim}{V}_g = -\frac{\dfrac{\partial \Xi'}{\partial \underset{\sim}{k}}}{\dfrac{\partial \Xi'}{\partial \omega_k}}$$

$$\frac{\partial \Xi'}{\partial \underset{\sim}{k}} = -\underset{\sim}{V}_g \frac{\partial \Xi'}{\partial \omega_k}$$

$$\underset{\sim}{\Gamma} = +\frac{1}{8\pi}\underset{\sim}{V}_g \frac{\partial \Xi'}{\partial \omega_k} = \underset{\sim}{V}_g \overline{W}$$

Wave energy propagates at the group velocity.

Flux

$$\underset{\sim}{\Gamma} = -\frac{\omega_0}{8\pi}\frac{\partial}{\partial \underset{\sim}{k}_0}\underset{\sim}{E}^*_0 \cdot \underset{\approx}{G} \cdot \underset{\sim}{E}_0$$

$$\underset{\approx}{G} = \frac{c^2}{\omega^2}\underset{\sim}{k}\times(\underset{\sim}{k}\times) + \underset{\approx}{\epsilon} - \underset{\approx}{I} + \underset{\approx}{I}$$

$$\frac{\partial}{\partial \underset{\sim}{k}_0}\underset{\sim}{E}^*_0 \cdot \underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}_0) = \frac{\partial}{\partial \underset{\sim}{k}}-(\underset{\sim}{k}\times\underset{\sim}{E}_0)\cdot(\underset{\sim}{k}\times\underset{\sim}{E}^*_0)$$

$$= -2\frac{\partial}{\partial k'}(\underset{\sim}{k}'\times\underset{\sim}{E}_0)\cdot(\underset{\sim}{k}\times\underset{\sim}{E}^*_0)$$

$$= -2\,\underset{\sim}{E}_0\times(\underset{\sim}{k}\times\underset{\sim}{E}^*_0)\cdot\underbrace{\frac{d}{d\underset{\sim}{k}}\underset{\sim}{k}'}_{\underset{\approx}{I}}$$

$$= -2\,\underset{\sim}{E}_0\times(\underset{\sim}{k}\times\underset{\sim}{E}^*_0)$$

$$\underset{\sim}{B}_0 = \frac{c}{\omega_0}\,\underset{\sim}{k}\times\underset{\sim}{E}_0$$

$$\underset{\sim}{\Gamma} = \frac{\omega_0}{8\pi}\left(E_0\times(\underset{\sim}{k}\times E_0^*)\right)\frac{c^2}{\omega_0^2}$$

$$-\ \frac{\omega_0}{8\pi}\ \frac{\partial}{\partial \underset{\sim}{k}_0}\, E_0^*\cdot\left(\underset{\approx}{\epsilon}-\underset{\approx}{I}\right)\cdot \underset{\sim}{E}_0$$

$$= \underbrace{c\,\frac{\underset{\sim}{E}_0\times \underset{\sim}{B}_0^*}{4\pi}}_{\underset{\sim}{S} = \text{Poynting flux}} - \underbrace{\frac{\omega_0}{8\pi}\ \frac{\partial}{\partial k_0}\, E_0^*\cdot\left(\underset{\approx}{\epsilon}-\underset{\approx}{I}\right)\cdot \underset{\sim}{E}_0}_{\text{particles}}$$

Since $\underset{\approx}{\epsilon} = \underset{\approx}{I}\left(1-\frac{\omega_{pe}^2}{\omega_0^2}\right)$ in cold limit,

$$\frac{\partial}{\partial k_0}\left(\underset{\approx}{\epsilon}-\underset{\approx}{I}\right) = 0$$

$\Rightarrow$ only Poynting flux.

Bellan - Fundamentals of Plasma Physics Ch 3. G and R Ch 3

Particle Orbits in Magnetic Fields

We are now going to start discussing the dynamics of plasmas with embedded magnetic fields. The dynamics are linked to the orbits of particles

$\Rightarrow$ determine where particles go? $\Rightarrow$ along or against electric fields

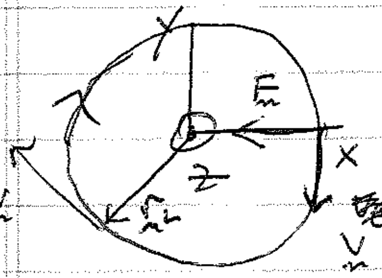
$\Rightarrow$ do particles gain or lose energy?

Uniform magnetic field

$$m \frac{d\underline{v}_\perp}{dt} = q\, \underline{v}_\perp \times \underline{B} \qquad m \frac{dv_\parallel}{dt} = 0$$

$$m \frac{d^2 \underline{v}_\perp}{dt^2} = \frac{q^2}{m} (\underline{v}_\perp \times \underline{B}) \times B = -\frac{q^2 B^2}{m} \underline{v}_\perp$$

$$\Omega = \frac{qB}{mc} \qquad \frac{d^2}{dt^2} \underline{v}_\perp + \Omega^2 \underline{v}_\perp = 0$$



$v_z = \text{const}$

$$m \frac{v_\theta^2}{r} = \frac{q v_\theta B}{c}$$

r_L = Larmor radius or gyro radius

$$r_L = \frac{v_\theta}{\Omega} \qquad \underline{r}_L = -\frac{\underline{v} \times \underline{\Omega}}{\Omega^2}$$