G+R Ch 23-24 ~~Choudhuri Ch 12.4~~

Electrostatic Waves in a Warm Plasma

When we were looking at plasma waves found that

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$$

$$\omega_{pe}^2 = \frac{4\pi n e^2}{m_e}, \quad \omega_{pi}^2 = \frac{4\pi n_i Z^2 e^2}{m_i}$$

take $\epsilon = 0$

$$\omega \simeq \pm \omega_{pe} \Rightarrow \omega_{pe} \gg \omega_{pi}$$

Want to explore what happens in a warm plasma. Do this by solving the VE. We take the electric field to be small.

~~[illegible]~~

Can expand $f = f_0 + f_1 + \cdots$ as a series in the small parameter. Neglect collisions

Zero order

$$\frac{\partial}{\partial t} f_0 + \underset{\sim}{v} \cdot \nabla f_0 = 0$$

take $\frac{\partial f_0}{\partial t}, \nabla f_0 = 0$ ~~[illegible]~~

$$a_0 = 0$$

Take f_0 to be a Maxwellian and require

$$-e n_e + Z e n_i = 0$$

$\Rightarrow$ charge neutrality

First Order

Take $\underset{\sim}{E}_1 = -\nabla \varphi_1$

$$\varphi_1 = \mathrm{Re}\left(\hat{\varphi}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$$

$\hat{\varphi}$ = complex amplitude

$\underset{\sim}{k}$ = wave vector, ω = frequency

$\Rightarrow$ can be complex

$$\cancel{f(\underset{\sim}{x}, \underset{\sim}{v}, t) = f_0(\underset{\sim}{v}) + \underbrace{\mathrm{Re}\left[\hat{f}\left(e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)\right]}_{\text{small}}}$$

$$\cancel{\frac{\partial}{\partial t} f_1 + v\cdot\nabla f_1 + \frac{q}{m}\cdot\frac{\partial}{\partial v}}$$

$$\frac{\partial}{\partial t} f_1 + \underset{\sim}{v}\cdot\nabla f_1 - \frac{q}{m}\nabla\varphi_1\cdot\frac{\partial}{\partial v} f_0 = 0$$

$\Rightarrow$ all terms the same order in the expansion parameter

$$f_1 = \mathrm{Re}\left(\hat{f}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right)\mathrm{Re}\left(\hat{f}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$$

$$- \frac{q}{m}\nabla\left(\mathrm{Re}\,\hat{\varphi}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)\cdot \frac{\partial}{\partial v} f_0(\underset{\sim}{v}) = 0$$

$$\mathrm{Re}\left[(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})\hat{f} - \frac{q}{m} i\hat{\varphi}\, \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0\right] = 0$$

$$(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})\hat{f} - \frac{q}{m}\hat{\varphi}\, i\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0 = 0$$

$$\hat{f} = \frac{\frac{q}{m}\hat{\varphi}\, \cancel{i}\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{-\cancel{i}\omega + \cancel{i}\underset{\sim}{k}\cdot\underset{\sim}{v}}$$

$$\hat{f} = -\frac{q}{m}\hat{\varphi}\,\frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}\,\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0$$

$\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}$ = frequency seen by moving particles.

Need to calculate the charge perturbation and put into Poisson's Eqn to calculate φ_1.

$$\rho_1 = \mathrm{Re}\left(\hat{\rho}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$$

$$\hat{\rho} = \sum_{i,e} e_{\alpha}\int d\underset{\sim}{v}\; \hat{f}_{e,i} \qquad \nabla\cdot\underset{\sim}{E}_1 = 4\pi\rho_1$$

$$-\nabla^2\varphi_1 = 4\pi\rho_1 \implies +k^2\hat{\varphi} = 4\pi\hat{\rho}$$

$$\hat{\rho} = -\sum_{i,e} \int d\underset{\sim}{v} \, \frac{q^2}{m} \, \hat{\varphi} \, \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

$$+k^2 \hat{\varphi} = -\sum_{e,i} \int d\underset{\sim}{v} \, \frac{4\pi q^2}{m} \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}} \, \hat{\varphi}$$

$$k^2 \underbrace{\left(1 + \sum_{e,i} \frac{1}{k^2} \frac{4\pi q^2}{m} \int d\underset{\sim}{v} \, \frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}} \, \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0\right)}_{\epsilon(\underset{\sim}{k},\omega)} \hat{\varphi} = 0$$

$$\boxed{\epsilon = 1 + \sum_{e,i} \frac{1}{k^2} \frac{4\pi q^2}{m} \int d\underset{\sim}{v} \, \frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}} \, \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}$$

dielectric function in warm plasma

First check to see that we obtain the cold plasma result

$\Rightarrow$ integrate by parts

$$\epsilon = 1 - \sum_{e,i} \frac{1}{k^2} \frac{4\pi q^2}{m} \int d\underset{\sim}{v} \, f_0 \, \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

$$\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}} = -\frac{1}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2} \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} \left(-\underset{\sim}{k}\cdot\underset{\sim}{v}\right) = \frac{k^2}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2}$$

$$\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} \underset{\sim}{k}\cdot\underset{\sim}{v} = k_i \frac{\partial}{\partial v_i} k_j v_j = k_i k_j \delta_{ij} = k_i^2 = k^2$$

$$\epsilon = 1 - \sum_{e,i} \frac{4\pi q^2}{m} \int d\underset{\sim}{v} \; \frac{f_0}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2}$$

In cold plasma limit where $k v_t \ll \omega$

$$\Rightarrow \quad v_t \ll \frac{\omega}{k} = v_p$$

$\Rightarrow$ thermal velocity small compared with the phase speed of the wave

$$\epsilon = 1 - \sum_{e,i} \frac{4\pi q^2}{m} \int d\underset{\sim}{v} \; f_0 \frac{1}{\omega^2}$$

$$= 1 - \sum_{e,i} \frac{4\pi n_0 q^2}{m} \frac{1}{\omega^2}$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \quad \text{as before}$$

Thermal corrections

Want to include thermal corrections to this dispersion relation

$$\frac{1}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2} \simeq \frac{1}{\omega^2}\left[1 + \frac{2\underset{\sim}{k}\cdot\underset{\sim}{v}}{\omega} + 3\frac{(\underset{\sim}{k}\cdot\underset{\sim}{v})^2}{\omega^2} + \cdots\right]$$

$$\int d\underset{\sim}{v} \; \frac{f_0}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2} \simeq \frac{1}{\omega^2}\left[n_0 + \frac{0}{\omega} + \frac{3}{\omega^2}\int d\underset{\sim}{v} \, (\underset{\sim}{k}\cdot\underset{\sim}{v})^2 f\right]$$

$$\frac{1}{2} m \int d\underset{\sim}{v} \, (\underset{\sim}{k}\cdot\underset{\sim}{v})^2 f = \frac{1}{2} n_0 T k^2$$

$$\epsilon = 1 - \frac{4\pi e^2}{m} \frac{n_0}{\omega^2}\left(1 + 3\frac{k^2 T_e}{m_e \omega^2}\right)$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3k^2\frac{T_e}{m_e\omega^2}\right)$$

$$T_e = \tfrac{1}{2} m_e v_{te}^2$$

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2 v_{te}^2}{\omega^2}\right)$$

Cold plasma assumption valid for

$$k v_{te} \ll \omega$$

Keep corrections as long as this is small

$$v_{te} \ll \frac{\omega}{k}$$ thermal velocity smaller than wave phase velocity.

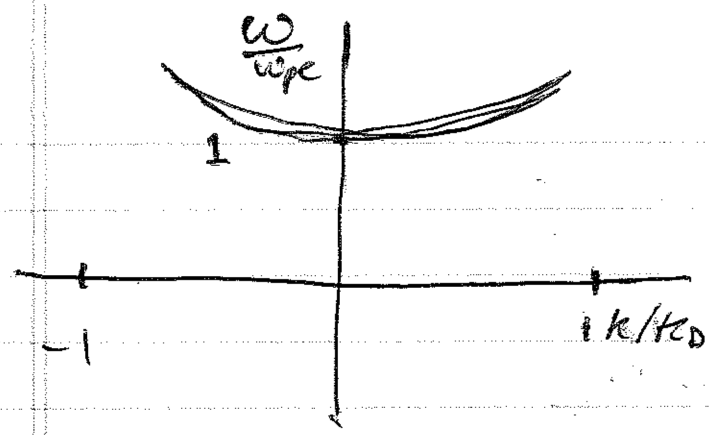
Solve for ω:

$$\omega^2 = \omega_{pe}^2\left(1 + \frac{3}{2}\frac{k^2 T_e}{m_e \omega^2}\right)$$

lowest order $\omega = \omega_{pe}$

$$\omega = \omega_{pe}\left(1 + \frac{3}{2}\frac{k^2 T_e}{m_e \omega_{pe}^2}\right)$$

$$= \omega_{pe}\left(1 + \frac{3}{2}\frac{k^2}{k_D^2}\right)$$



require $k/k_D \ll 1$ for expansion to be valid.

Can we obtain this from fluid equations?

<u>Fluid model</u> $\Rightarrow$ only electrons.

$$\frac{\partial}{\partial t} n + \nabla \cdot n\underset{\sim}{u} = 0 \qquad \nabla \cdot \underset{\sim}{E} = +4\pi e(n_i - n_e)$$

$$n_i = n_0$$

$$m_e n\left(\frac{\partial}{\partial t}\underset{\sim}{u} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\nabla P - en\underset{\sim}{E}$$

$$\underset{\sim}{E} = -\nabla\varphi$$

$$\frac{\partial}{\partial t}P + \underset{\sim}{u}\cdot\nabla P + \gamma_s P\,\nabla\cdot\underset{\sim}{u} = 0$$

Equilibrium — P_0, n_0, $\underset{\sim}{u}_0 = 0$,

$$P = P_0 + \mathrm{Re}\left(\hat{P}\,e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$$

$$\underset{\sim}{u} = \mathrm{Re}\left(\hat{u}\,e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$$

etc

$$-i\omega\hat{n} + n_0 i\underset{\sim}{k}\cdot\underset{\sim}{\hat{u}} = 0 \Rightarrow k\cdot\hat{u} = \omega\frac{\hat{n}}{n_0}$$

$$\underset{\sim}{k}\cdot\left(-i\omega n_0 m_e\hat{u} = -i\underset{\sim}{k}\hat{P} + e n_0 i\underset{\sim}{k}\hat{\varphi}\right)$$

$$-i\omega\hat{P} + \gamma_s P_0\, i\underset{\sim}{k}\cdot\hat{u} = 0 \Rightarrow \hat{P} = \frac{P_0 k\cdot\hat{u}}{\omega}\gamma_s$$

$$k^2\hat{\varphi} = -4\pi e\hat{n} \qquad \hat{P} = \frac{P_0}{n_0}\gamma_s\hat{n}$$

$$-\omega n_0 m_e\omega\frac{\hat{n}}{n_0} = -k^2\frac{P_0}{n_0}\gamma_s\hat{n} + n_0 e k^2\hat{\varphi}$$

$$\omega^2 m_e\frac{\hat{n}}{n_0} = k^2\frac{P_0}{n_0}\gamma_s\frac{\hat{n}}{n_0} + e\cancel{k^2}\left(+\frac{4\pi e\hat{n}}{\cancel{k^2}}\right)$$

$$\omega^2 - \omega_{pe}^2 = \frac{k^2 T_e\gamma_s}{m_e}$$

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{k^2 T_e\gamma_s}{m_e}\frac{1}{\omega^2}$$

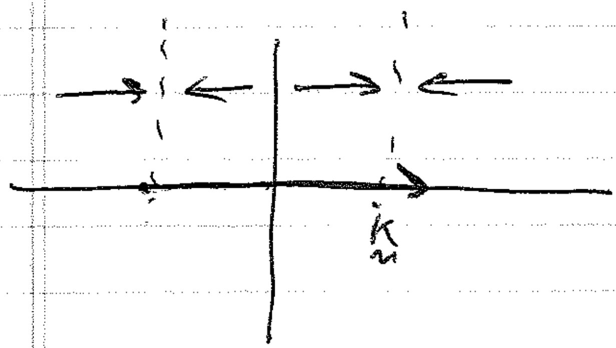
$$\epsilon \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{k^2 T_e\gamma_s}{m_e\omega^2}\right) \quad \text{fluid}$$

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2 T_e}{m_e\omega^2}\right) \quad \text{kinetic}$$

$\Rightarrow$ same result for $\gamma_s = \frac{n_f+2}{n_f} = 3$

$$\Rightarrow \boxed{n_f = 1}$$

Why $n_e = 1$?



oscillatory motion compresses plasma along k.

$\Rightarrow$ increases internal energy in x direction

$\Rightarrow$ not transverse

$\Rightarrow \frac{1}{2} m_e \Delta v_x^2$

In a fluid model with $n_e = 3$, any increase in $\frac{1}{2} m_e \Delta v_x^2$ spreads to $\frac{1}{2} m_e \Delta v_y^2$, $\frac{1}{2} m_e \Delta v_z^2$

Landau Damping

Need to address the question of what happens at the singularity where

$$\omega = \underset{\sim}{k} \cdot \underset{\sim}{v}$$

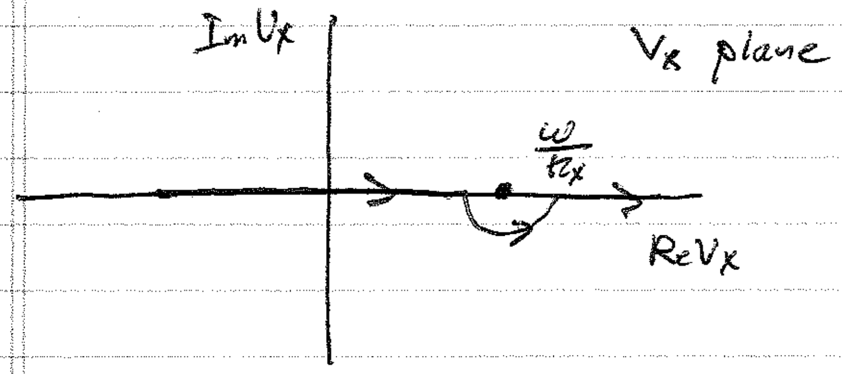
$$\epsilon = 1 + \frac{\omega_{pe}^2}{n_0 k^2} \int d\underset{\sim}{v} \, \frac{1}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0$$

$\Rightarrow$ where does the velocity space integral go with respect to the singularity?

$\Rightarrow$ must think about the velocity space integral as an integral in the complex plane

In case with $k = k_x$

$$\epsilon = 1 + \frac{\omega_{pe}^2}{n_0 k_x^2} \int dv_x \frac{1}{\omega - k_x v_x} k_x \frac{\partial}{\partial v_x} f_0$$



$\Rightarrow$ must integrate under the singularity

$\Rightarrow$ ϵ is complex

$\Rightarrow$ damping of plasma waves

$\Rightarrow$ ϵ above is defined for $\text{Im}\,\omega > 0$

To show that this is correct need to go back to the VE and do a more careful solution using Laplace transforms.

Landau damping:

Want to solve the Vlasov / Poisson equations by carrying out Fourier transforms in space and a Laplace transform in time.

$\Rightarrow$ linearize $\Rightarrow$ introduces direction in time into the problem

$$\frac{\partial}{\partial t} f_1 + v\cdot\nabla f_1 - \frac{e}{m} \underset{\sim}{E}_1 \cdot \frac{\partial}{\partial v} f_0 = 0$$

$$\nabla\cdot \underset{\sim}{E}_1 = -4\pi e\, n_1 \qquad E_1 = -\nabla \varphi_1$$

$$\frac{\partial}{\partial t} f_1 + v\cdot\nabla f_1 + \frac{e}{m} \nabla\varphi_1 \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0 = 0$$

$$\nabla^2 \varphi_1 = 4\pi e\, n_1$$

$\Rightarrow$ do FT

$$\int d\underset{\sim}{x}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} (\quad) = 0$$

$$\frac{\partial}{\partial t} f_{1k} + i\underset{\sim}{k}\cdot\underset{\sim}{v} f_{1k} + \frac{e}{m} \varphi_{1k} i\underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0 = 0$$

$$-k^2 \varphi_{1k} = 4\pi e\, n_{1k}$$

Carry out Laplace transform.

$$g(\omega) = \int_0^\infty dt\, g(t)\, e^{i\omega t}$$

where $\mathrm{Im}\,\omega$ must be sufficiently large so that the integral converges

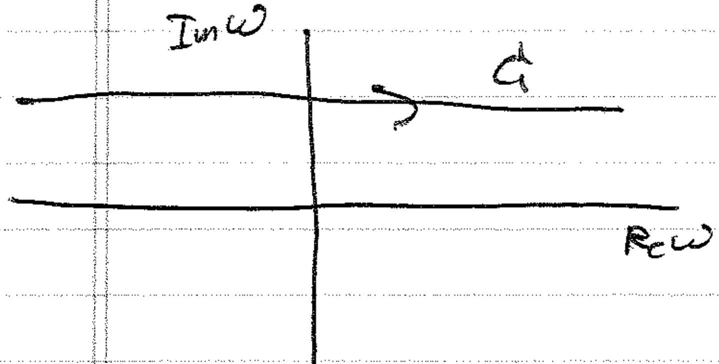
$$g(t)\, e^{i\omega t} \to 0$$

$\Rightarrow$ If g has exponentially growing roots γ must have

$$\mathrm{Im}\,\omega > \gamma$$

Inverse transform.

$$g(t) = \int_C \frac{d\omega}{2\pi}\, g(\omega)\, e^{-i\omega t}$$



C must lie above all singularities of $g(\omega)$.

Carry out the transform of the VE

$$\int_0^\infty dt\, e^{i\omega t} \frac{\partial f_{1k}}{\partial t} = f_{1k}(t=0) - i\omega \int_0^\infty dt\, e^{i\omega t} f_{1k}(t)$$

$$= f_{1k}(t=0) - i\omega f_{1k\omega}$$

$$-i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})\, f_{1k\omega} = -\frac{e}{m} \varphi_{1k\omega}\, i\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0 + f_{1k}(t=0)$$

$$k^2 \varphi_{1k\omega} = -4\pi e \int d\underset{\sim}{v}\, f_{1k\omega}$$

Substituting $f_{1k\omega}$ into P.E. yields.

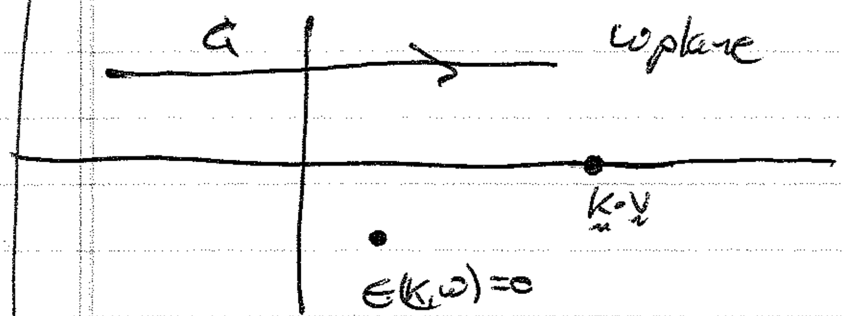
$$k^2 \epsilon(\mathbf{k},\omega)\,\phi_{1k\omega} = -4\pi e i \int d\mathbf{v}\, \frac{\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} f_0}{\omega - \mathbf{k}\cdot\mathbf{v}}$$

$$\epsilon(k\omega) = 1 + \overbrace{\frac{\omega_{pe}^2}{n_0 k^2} \int d\mathbf{v}\, \frac{\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} f_0}{\omega - \mathbf{k}\cdot\mathbf{v}}}^{4\pi\chi_e}$$

skip

Carrying out the inverse transform

$$\phi_{1k}(t) = -\int_{C_1} \frac{d\omega}{2\pi}\, 4\pi e i \int d\mathbf{v}\, \frac{f_{1k}(t=0)\, e^{-i\omega t}}{(\omega - \mathbf{k}\cdot\mathbf{v})\, k^2 \epsilon(\mathbf{k},\omega)}$$



For $t<0$ close contour in UHP but no singularities so find $\phi_{1k}(t)=0$.

For $t>0$ close in LHP and pick up contribution from two singularities.

$$\omega = \mathbf{k}\cdot\mathbf{v}\,, \quad \epsilon(\mathbf{k}, \omega_0) = 0$$

$\Longrightarrow$ $-2\pi i$ times residue skip

$t > 0$

$$\phi_{1k}(t) = -\frac{4\pi e}{k^2}\left[\frac{e^{-i\omega_0 t}}{\frac{\partial \epsilon}{\partial \omega_0}} \int d\underset{\sim}{v} \frac{f_{1k}(t=0)}{\omega_0 - \underset{\sim}{k}\cdot\underset{\sim}{v}} + \int d\underset{\sim}{v} \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}}{\epsilon(k, \underset{\sim}{k}\cdot\underset{\sim}{v})} f_{1k}(t=0) \right]$$

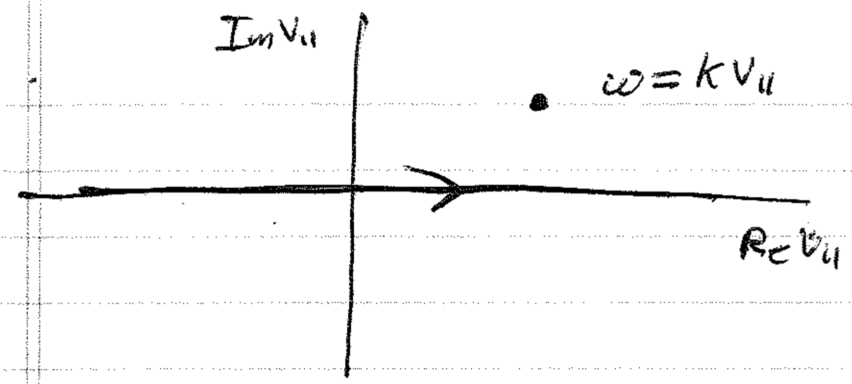
The first term is the natural mode of the system which evolves from the initial perturbation. The second arises from the free streaming of the particles. It is not a normal mode since $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) \neq 0$. For large t the rapid oscillations of $\exp(-i\underset{\sim}{k}\cdot\underset{\sim}{v}t)$ as the velocity integral is carried out cause this term to be small $\Rightarrow$ neglect this.

Lets focus on the normal mode of the system. Go back to $\epsilon(k, \omega)$

$$\epsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{n_0 k^2} \int d\underset{\sim}{v} \frac{\underset{\sim}{k}\cdot \frac{\partial f}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

Remember that ϵ is defined with $\mathrm{Im}\,\omega$ above all singularities. In velocity space have singularity when ~~this~~

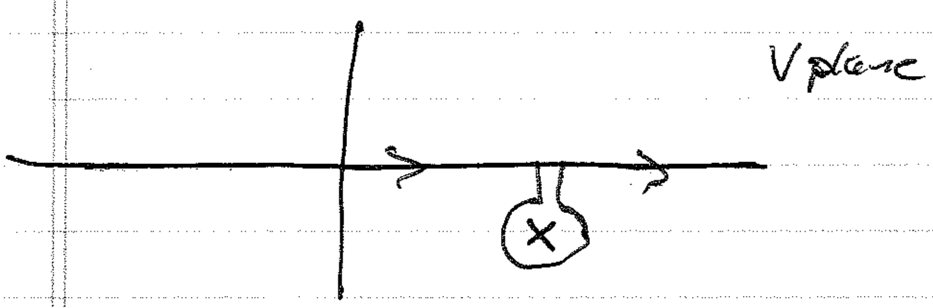
$$\omega = \underset{\sim}{k}\cdot\underset{\sim}{v} \equiv k v_{\parallel}$$



take $k > 0$

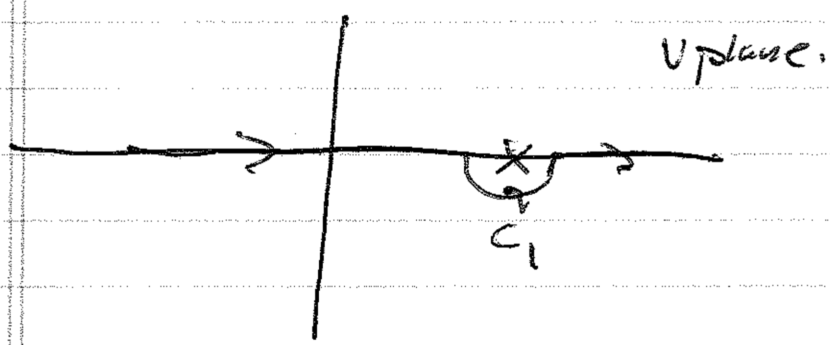
If the root of $\epsilon(k_\parallel, \omega_0)$ has $\underbrace{\text{Im}\,\omega_0 > 0}_{\text{growing mode}}$ then leave contour as is.

If $\text{Im}\,\omega_0 < 0$ need to be careful. Need to analytically continue ϵ into the lower half plane



$\Rightarrow$ contour stays below singularity

If $\text{Im}\,\omega_0 < 0$ then have



For $\operatorname{Im}\omega_0 \simeq 0$ can separate semi-circle from remaining integral

$$\cancel{\epsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{n_0 k^2}\left[P \int d\underset{\sim}{v}\, \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}\right.}$$

$$\int_{C_1} d\underset{\sim}{v}\, \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}} = - \int d\underset{\sim}{v}_\perp\, dv_\parallel \frac{\cancel{k}\, \frac{\partial}{\partial v_\parallel} f_0}{\cancel{k}\left(v_\parallel - \frac{\omega}{k}\right)}$$

$$= -i\pi \int d\underset{\sim}{v}_\perp \left.\frac{\partial f_0}{\partial v_\parallel}\right|_{v_\parallel = \frac{\omega}{k}}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{n_0 k^2}\left[P \int d\underset{\sim}{v}\, \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}} - \frac{i\pi k}{|k|}\int d\underset{\sim}{v}_\perp \frac{\partial f_0}{\partial v_\parallel}\right]$$

$\frac{k}{|k|}$ changes sign if $k < 0$.

$\Rightarrow$ same prescription

$\operatorname{Im}\omega > 0$.

Assume have a solution ω_0 with $\frac{\omega_0}{k} \gg v_{te}$

$$P \int d\underset{\sim}{v}\, \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

integrate by parts ignoring resonant particles

$$= - \int d\underset{\sim}{v}\, f_0\, \underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

$$= - \int d\underset{\sim}{v}\, f_0\, k^2 \frac{1}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2} \simeq - \frac{k^2 n_0}{\omega^2}$$

$$\epsilon = 1 + \frac{\omega_{pe}^2}{n_0 k^2}\left[-\frac{k^2 n_0}{\omega^2} - i\pi \frac{k}{|k|} \frac{\partial f_0(v_\parallel)}{\partial v_\parallel}\bigg|_{\frac{\omega}{k}} \right]$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2} - i\pi \frac{k}{|k|}\frac{1}{k^2}\frac{\omega_{pe}^2}{n_0}\frac{\partial f_0(v_\parallel)}{\partial v_\parallel}\bigg|_{\frac{\omega}{k}}$$

$$= \epsilon_R + i\,\epsilon_I$$

Last term

$$\sim \frac{1}{k^2}\,\omega_{pe}^2\,\frac{v_p}{v_{te}^2}\,\frac{e^{-\frac{v_p^2}{v_{te}^2}}}{v_{te}}$$

$$\sim \frac{v_p^3}{v_{te}^3}\,e^{-\frac{v_p^2}{v_{te}^2}} \ll 1$$

since $v_p \gg v_{te}$

<u>Can write</u>

$$\omega = \omega_0 + \delta\omega$$

lowest order

$$\epsilon = \epsilon_0 + \epsilon_1 + \cdots$$

$$\epsilon_0 = \epsilon_R(\omega_0) = 1 - \frac{\omega_{pe}^2}{\omega_0^2} = 0 \qquad \omega_0 = \omega_{pe}$$

$$\epsilon_1 = \frac{\partial \epsilon_0}{\partial \omega_0}\,\delta\omega + i\,\epsilon_I$$

$$\delta\omega = \frac{-i\,\epsilon_I}{\partial \epsilon_R / \omega_0}$$

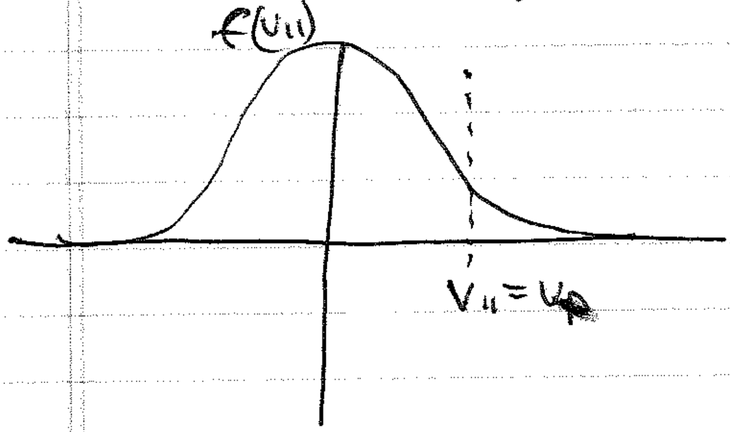
~~[illegible]~~

$$\frac{\partial \epsilon_0}{\partial \omega_0} = \frac{2\,\omega_{pe}^2}{\omega_0^3} = \frac{2}{\omega_{pe}}$$

$$\delta\omega = +\,i\,\frac{\omega_{pe}}{2}\,(+)\,\pi\,\frac{k}{|k|}\,\frac{1}{k^2}\,\frac{\omega_{pe}^2}{n_0}\,\frac{\partial f_0}{\partial v_{\parallel}}\Big|_{v_p}$$

$$= i\,\frac{\pi}{2}\,\omega_{pe}\,\frac{k}{|k|}\,\frac{v_p^2}{n_0}\,\frac{\partial f_0}{\partial v_{\parallel}}\Big|_{v_p}$$

wave is damped if slope is negative.



Damping occurs because particles with $v_{\parallel} \simeq v_p$ see a nearly DC electric field. Particles slightly slower than the wave. will gain energy from the wave while those slightly faster than the wave will slow down and give energy to the wave. More particles slower so wave damps.

The Plasma Dispersion Function

It is useful to define a standard function that ~~describes the kinetic dispersion~~ can be used to represent the kinetic plasma dispersion relation for a Maxwellian distribution.

$$\epsilon(\underset{\sim}{k},\omega) = 1 + \frac{4\pi q^2}{m k^2}\int d\underset{\sim}{v}\, \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

Let $\underset{\sim}{k} = k\hat{z}$

$$f_0 = \frac{n_0}{(2\pi T/m)^{3/2}}\, e^{-\frac{1}{2}m(v_\perp^2+v_z^2)/T}$$

$$\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0 = -\frac{v_z k m}{T} f_0$$

$\Rightarrow$ do $v_\perp$ integration

$$\epsilon(\underset{\sim}{k},\omega) = 1 \cancel{+} \frac{4\pi q^2}{\cancel{m} k^2} n_0 \frac{1}{(2\pi T/m)^{1/2}}\int dv_z\, \frac{\frac{k v_z}{T} \cancel{m}\, e^{-\frac{v_z^2}{v_t^2}}}{\omega - k v_z}$$

$$\frac{1}{2} m v_t^2 = T$$

$$\epsilon = 1 - \frac{4\pi n_0 q^2}{T k^2}\int \frac{dv_z}{\sqrt{\pi}\, v_{t_z}} \frac{k v_z - \omega + \omega}{\omega - k v_z}\, e^{-\frac{v_z^2}{v_t^2}}$$

$$= 1 + \frac{k_D^2}{k^2}\left[1 + \int \frac{dv_z}{\sqrt{\pi}\, v_t} \frac{e^{-v_z^2/v_t^2}}{k v_z - \omega}\,\omega\right]$$

normalize $v_z \Rightarrow s = v_z/v_t$, $\xi = \frac{\omega}{k v_t}$

$$\epsilon = 1 + \frac{k_D^2}{k^2}\left[1 + \xi \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s-\xi}\right]$$

Define

$$Z(\xi) = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s-\xi}.$$

defined for $\mathrm{Im}\,\xi > 0$

$\Longrightarrow$ plasma dispersion function.

Differential equation:

$$Z' = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{(s-\xi)^2} = -2\left(1 + \xi Z(\xi)\right)$$

HWK

$$\boxed{\epsilon = 1 - \frac{k_D^2}{2k^2} Z'\left(\frac{\omega}{k v_t}\right)}$$

Large argument $\xi \gg 1$

$$\frac{1}{s-\xi} = -\frac{1}{\xi - s} = -\frac{1}{\xi}\left(1 + \frac{s}{\xi} + \frac{s^2}{\xi^2} + \cdots\right)$$

$$Z(\xi) = -\int \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{\xi}\left(1 + \cancelto{0}{\frac{s}{\xi}} + \frac{s^2}{\xi^2} + \cancelto{0}{\frac{s^3}{\xi^3}} + \right)$$

$$= -\frac{1}{\xi}\left[1 + \frac{1}{2\xi^2} + \frac{3}{4}\frac{1}{\xi^4}\right]$$

$$Z'(\xi) = \frac{1}{\xi^2} + \frac{3}{2}\frac{1}{\xi^4}.$$

$$\cancel{Z' = -2\left[\xi + \xi\left(\xi + \frac{1}{2\xi^2} + \frac{3}{4}\frac{1}{\xi^4}\right)\right]}$$

$$\cancel{= \frac{1}{\xi^2} + \frac{3}{2}\frac{1}{\xi^4}}$$

$$\epsilon = 1 - \frac{4\pi n e^2 \cancel{m}}{\cancel{T}\, \cancel{k^2}\, m}\frac{\cancel{k^2} v_t^2}{\omega^2}\left[1 + \frac{3}{2}\frac{k^2 v_t^2}{\omega^2}\right]$$

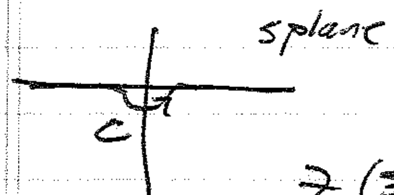
$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 T}{m\omega^2}\right)$$

$$\Longrightarrow \text{as before}$$

<u>Small argument</u>

$$Z(\xi) = Z(0) + Z'(0)\,\xi + \cdots$$

$$Z(0) = \int_C \frac{ds}{\sqrt{\pi}}\frac{e^{-s^2}}{s} = \frac{i\pi}{\sqrt{\pi}} = i\sqrt{\pi}$$



$$Z'(0) \simeq -2$$

$$Z(\xi) = i\sqrt{\pi} - 2\xi$$

$$\mathrm{Im}\, Z \text{ for real } \xi = \sqrt{\pi}\, e^{-\xi^2}$$

$$\epsilon = 1 - \frac{k_D^2}{2k^2}(-2) = 1 + \frac{k_D^2}{k^2}$$

$\Rightarrow$ electron Boltzmann response

$\Rightarrow$ shielding.

Wave energy

Want to discuss Landau damping in terms of energy transfer but first need to discuss how a wave carries energy in a plasma. Go back to the Vlasov equation.

$$\left(\frac{\partial}{\partial t} + v\cdot\nabla + \frac{q}{m}\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}}\right)f = 0$$

~~Have solved for the first order value~~

Generally can write f as a power series in E

$$f = f_0 + f_1 + f_2$$

To find the change in energy need to go the second order

$$\frac{\partial}{\partial t} f_2 + v\cdot\nabla f_2 + \frac{q}{m}\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_1 = 0$$

Do a space average. $\langle\rangle = \int dx \frac{1}{L^3}$

$$\frac{\partial}{\partial t}\langle f_2\rangle + \frac{q}{m}\left\langle \underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_1\right\rangle = 0$$

Calculate energy change

$$\int d\underset{\sim}{v}\, \frac{1}{2}mv^2 (\quad)$$

$$\dot{w}_p + \frac{q}{m} \left\langle \int d\underline{v}\, \tfrac{1}{2} m v^2\, \underline{E}\cdot\frac{\partial}{\partial \underline{v}} f_1 \right\rangle = 0$$

$$w_p = \int d\underline{v}\, \tfrac{1}{2} m v^2 f_2$$

Integrate by parts with $\underline{E} = -\nabla\varphi$

$$\dot{w}_p + q \left\langle \int d\underline{v}\, f_1\, \underline{v}\cdot\nabla\varphi \right\rangle = 0$$

$$\dot{w}_p = -q \int \frac{d\underline{x}}{L^3} \int d\underline{v}\, f_1\, \underline{v}\cdot\nabla\varphi$$

$$f_1 = -\sum_{k} e^{i\underline{k}\cdot\underline{x}}\, e^{-i\omega_k t}\, \frac{q}{m}\, \frac{\varphi_k}{\omega_k - \underline{k}\cdot\underline{v}}\, \underline{k}\cdot\frac{\partial}{\partial \underline{v}} f_0$$

$$\underline{v}\cdot\nabla\varphi = \sum_{k'} i\underline{k}'\cdot\underline{v}\, \varphi_{k'}\, e^{i\underline{k}'\cdot\underline{x}}\, e^{-i\omega_{k'} t}$$

$$\dot{w}_p = i\frac{q^2}{m} \int \frac{d\underline{x}}{L^3} \int dv \sum_{k,k'} e^{i(\underline{k}+\underline{k}')\cdot\underline{x}}\, e^{-i(\omega+\omega')t}\, \frac{\underline{k}'\cdot\underline{v}\; \underline{k}\cdot\frac{\partial}{\partial v} f_0}{\omega_k - \underline{k}\cdot\underline{v}}$$

$$\otimes\ \varphi_{k}\varphi_{k'}$$

$$\int d\underline{x}\, e^{i(\underline{k}+\underline{k}')\cdot\underline{x}} = (2\pi)^3\, \delta(\underline{k}+\underline{k}'), \quad \sum_{k'} \left(\frac{2\pi}{L}\right)^3 \equiv \int d\underline{k}'$$

sum over $\pm k$

$$\dot{w}_p = -i\frac{q^2}{m} \int d\underline{v} \sum_{k} e^{-i(\omega_k + \omega_{-k})t}\, \frac{\underline{k}\cdot\underline{v} - \omega_k + \omega_k}{\omega_k - \underline{k}\cdot\underline{v}}\, \underline{k}\cdot\frac{\partial}{\partial \underline{v}} f_0$$

$$\otimes\ \varphi_{-k}\varphi_{k}$$

86

$$\varphi(\underset{\sim}{x},t) = \varphi_k e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega_k t} + \varphi_{-k} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega_{-\underset{\sim}{k}} t}$$

$$\omega_{-\underset{\sim}{k}} = -\omega_k^* \qquad \Rightarrow \text{real } \varphi(\underset{\sim}{x},t)$$

$$\varphi_{-\underset{\sim}{k}} = \varphi_k^*$$

$$\dot{w}_p = -i \frac{q^2}{m} \int d\underset{\sim}{v} \sum_k \frac{\omega_k}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}} \underset{\sim}{k}\cdot\frac{\partial}{\partial\underset{\sim}{v}} f_0$$

$$\otimes\ e^{-i(\omega_k - \omega_k^*)t} |\varphi_k|^2$$

$$\omega_k = \omega_R + i\gamma_k \qquad e^{2\gamma_k t}|\varphi_k|^2 = \frac{|E_k|^2}{k^2}(t)$$

$$\omega_k - \omega_k^* = 2i\gamma_k$$

$$\dot{w}_p = -i \frac{q^2}{m} \sum_k \omega_k \int d\underset{\sim}{v} \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial\underset{\sim}{v}} f_0}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}} \frac{|E_k|^2}{k^2}$$

$$\epsilon = 1 + 4\pi\chi$$

$$\chi = \int d\underset{\sim}{v} \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial\underset{\sim}{v}} f_0}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}} \frac{q^2}{m} \frac{1}{k^2}$$

$$\dot{w}_p = -i \sum_k \omega_k \chi(k) |E_k|^2$$

see next page.

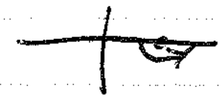
$$\omega_{-k}\chi_{-k} = -\omega_k^* \chi^*(\underset{\sim}{k}) = -(\omega_R - i\gamma_k)(\chi_R - i\chi_I)$$

Only Im part of $\omega_k\chi$ survives

$$\mathrm{Im}(\omega_k \chi) \simeq \mathrm{Im}\left[\frac{\partial(\omega_k\chi)}{\partial\omega}\bigg|_{\omega_R} i\gamma_k + \omega_R i\chi_I\right]$$

$$\chi_q(k) = \int d\underset{\sim}{v}\; \frac{\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}}\; \frac{q^2}{mk^2}$$

$$= \int dv_\perp \int dv_\parallel\; \frac{k \frac{\partial}{\partial v_\parallel} f_0}{\omega_k - k v_\parallel}\; \frac{q^2}{mk^2}$$



$$\chi_q = -\int dv_\perp dv_\parallel\; \frac{\frac{\partial}{\partial v_\parallel} f_0}{v_\parallel - \frac{\omega_k}{k}}\; \frac{q^2}{mk^2}$$

$$\mathrm{Im}\,\chi_q = -\int d\underset{\sim}{v}_\perp \frac{q^2}{mk^2} \frac{\partial}{\partial v_\parallel} f_0 \Big|_{v_p} \pi \qquad v_p = \frac{\omega_k}{k}$$

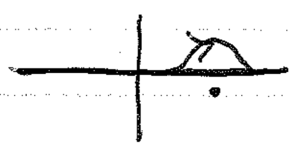
$$+ \frac{\partial \chi_e}{\partial \omega}\Big|_{\omega_R} \gamma_k$$

$$\chi_q(-k) = -\int dv_\perp dv_\parallel\; \frac{\frac{\partial}{\partial v_\parallel} f_0}{v_\parallel + \frac{\omega_{-k}}{k}}\; \frac{q^2}{mk^2}$$

$$\omega_{-k} = -\omega_k^*$$

$$= -\int d\underset{\sim}{v}_\perp dv_\parallel\; \frac{\frac{\partial}{\partial v_\parallel} f_0}{v_\parallel - \frac{\omega_k^*}{k}}\; \frac{q^2}{mk^2}$$

$\mathrm{Im}\,\chi_q(-k)$



$$= -\int d\underset{\sim}{v}_\perp \frac{\partial f_0}{\partial v_\parallel}\Big|_{v_p} \frac{q^2}{mk^2}(-\pi)$$

$$+ \frac{\partial \chi_q}{\partial \omega}\Big|_{\omega_R}(-\gamma_k) = -\,\mathrm{Im}\,\chi_q(k)$$

$\cancel{\mathrm{Re}\,\chi(k)}$ $\cancel{\mathrm{Re}\,\chi(-k)}$

$$\mathrm{Re}\,\chi_q(-k) = \mathrm{Re}\,\chi_q(k) \quad\Rightarrow\quad \chi_q(-k) = \chi_q(k)^*$$

$$\dot{w}_p = \sum_k \left[\frac{\partial}{\partial \omega_k}(\omega_k \chi_R)\Big|_{\omega_k} 2\gamma_k + 2\omega_R \chi_{RI} \right] \frac{|E_k|^2}{2}$$

now include the ~~wave~~ electric field energy

$$\dot{w}_E = 2\gamma_k \frac{|E_k|^2}{8\pi} = 2\gamma_k \frac{|E_k|^2}{8\pi} \frac{\partial \omega_k}{\partial \omega_k}$$

$$\epsilon = 1 + 4\pi \chi$$

$$\dot{w} = \sum_k \Big[\underbrace{\frac{\partial}{\partial \omega_k}(\omega_k \epsilon)\, 2\gamma_k}_{\text{change of wave energy}} + \underbrace{2\omega_R \epsilon_I}_{\text{change of resonant particle energy}} \Big] \frac{|E_k|^2}{8\pi}$$

$$\boxed{W_{wave} = \frac{|E_k|^2}{8\pi} \frac{\partial}{\partial \omega_k}(\omega_k \epsilon)}$$

for plasma waves

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega_k^2} \qquad \frac{\partial}{\partial \omega_k}(\omega_k \epsilon) = \frac{\partial}{\partial \omega_k} \omega_k \left(1 - \frac{\omega_{pe}^2}{\omega_k^2}\right)$$

$$= 1 + \frac{\omega_{pe}^2}{\omega_k^3} \omega_k = 2$$

$$W_{wave} = \frac{|E_k|^2}{8\pi} \, 2$$

$\Rightarrow$ equally split between electric field and sloshing particles