The fluid equations Bellan 2.4 - 2.5

The Vlasov equation with a collision operator can describe the dynamics of a plasma from the collisionless limit all the way to a system with a very high collision rate as long as the plasma parameter Γ is small. In the limit of large collisions we ~~expect~~ the dynamics to be fluid-like. Even in the limit of weak collisions fluid like concepts can help to ~~explain~~ understand how a plasma ~~evolves~~ behaves.

We will therefore proceed to construct a set of fluid equations by taking moments of the VE. We will be careful to include collisions so we start with the VE with a collision operator.

$$\frac{\partial}{\partial t} f_\alpha + \underset{\sim}{v} \cdot \nabla f_\alpha + \frac{\underset{\sim}{F}}{m_\alpha} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_\alpha = \sum_\beta C_{\alpha\beta}(f_\alpha)$$

$C_{\alpha\beta}$ = collisions acting on α due to species β.

zero moment

First integrate over the velocity and use fact that $\frac{\partial}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{F} = 0$ and that collisions don't create or destroy particles

$$\sum_\beta \int d\underset{\sim}{v}\, C_{\alpha\beta}(f_\alpha) = 0$$

and $f_\alpha \to 0$ as $v \to \infty$.

$$\frac{\partial}{\partial t}\int d\underset{\sim}{v}\, f_\alpha + \nabla\cdot\int d\underset{\sim}{v}\, \underset{\sim}{v}\, f_\alpha = 0$$

$$\Rightarrow \boxed{\frac{\partial n_\alpha}{\partial t} + \nabla\cdot n_\alpha \underset{\sim}{u}_\alpha = 0}$$ Continuity Equ.

$$n_\alpha \equiv \int d\underset{\sim}{v}\, f_\alpha$$

$$n_\alpha \underset{\sim}{u}_\alpha \equiv \int d\underset{\sim}{v}\, \underset{\sim}{v}\, f_\alpha \quad \Rightarrow \text{defines } \underset{\sim}{u}_\alpha$$

change in number density of particles at a location arises from the flux of particles in or out of the region.

<u>first moment</u>

Multiply the VE by $\underset{\sim}{v}$ and integrate. Note the $\underset{\sim}{v}$ is an independent variable so passes through ∇, $\partial/\partial t$ operators.

$$\frac{\partial}{\partial t}\int d\underset{\sim}{v}\, f_\alpha \underset{\sim}{v} + \nabla\cdot\int d\underset{\sim}{v}\, \underset{\sim}{v}\,\underset{\sim}{v}\, f_\alpha$$

$$-\int d\underset{\sim}{v}\left[\frac{\underset{\sim}{F}}{m_\alpha}\cdot\frac{\partial}{\partial \underset{\sim}{v}}\underset{\sim}{v}\right] f_\alpha = -\frac{1}{m_\alpha}\underset{\sim}{R}_{\alpha\beta}$$

$$\frac{\partial}{\partial \underset{\sim}{v}}\underset{\sim}{v} = \hat{e}_i\hat{e}_j\frac{\partial}{\partial v_i}v_j = \hat{e}_i\hat{e}_j\delta_{ij} = \underset{\approx}{I}$$

$\underset{\sim}{R}_{\alpha\beta}$ = drag between species α and β

$\underset{\sim}{R}_{\alpha\alpha} = 0$ since within a species no drag

Generally $\underset{\sim}{R}_{ei} = \nu_{ei} m_e n_e (\underset{\sim}{u}_e - \underset{\sim}{u}_i)$

$$\underset{\sim}{R}_{ie} = \nu_{ie} m_i n_i (\underset{\sim}{u}_i - \underset{\sim}{u}_e)$$

$$\underset{\sim}{R}_{ei} + \underset{\sim}{R}_{ie} = 0$$

$\Rightarrow$ total momentum of plasma is conserved by collisions.

$$\underbrace{\frac{\partial}{\partial t} n_\alpha \underset{\sim}{u}_\alpha}_{\text{local momentum}} + \underbrace{\nabla \cdot \int d\underset{\sim}{v}\, \underset{\sim}{v}\underset{\sim}{v} f_\alpha}_{\text{flux of momentum}} - \underbrace{\frac{q_\alpha}{m_\alpha} n_\alpha \left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{u}_\alpha \times \underset{\sim}{B}\right)}_{\text{change of momentum from EM field}}$$

$$= \underbrace{-\frac{1}{m_\alpha} \underset{\sim}{R}_{\alpha\beta}}_{\text{momentum transfer to other species}}$$

Can ~~want~~ extract average velocity from integral

$$\underset{\sim}{v} = \underset{\sim}{u}_\alpha + \underset{\sim}{v}'$$

and note that $\int d\underset{\sim}{v}'\, \underset{\sim}{v}' f_\alpha = 0$

$$\nabla \cdot \int d\underset{\sim}{v}\, \underset{\sim}{v}\underset{\sim}{v} f_\alpha = \nabla \cdot n_\alpha \underset{\sim}{u}_\alpha \underset{\sim}{u}_\alpha + \int d\underset{\sim}{v}'\, \underset{\sim}{v}'\underset{\sim}{v}' f_\alpha$$

Define the pressure tensor

$$\underset{\approx}{P}_\alpha = m_\alpha \int d\underset{\sim}{v}'\, \underset{\sim}{v}'\underset{\sim}{v}' f_\alpha$$

$$m_\alpha \left(\frac{\partial}{\partial t} n_\alpha \underset{\sim}{u}_\alpha + \nabla \cdot n_\alpha \underset{\sim}{u}_\alpha \underset{\sim}{u}_\alpha \right) = n_\alpha q_\alpha \left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{u}_\alpha \times \underset{\sim}{B} \right)$$

$$- \nabla \cdot \underset{\approx}{P}_\alpha - \underset{\sim}{R}_{\alpha\beta}$$

Use the continuity eqn to ~~ ~~ extract n_α from the momentum eqn.

$$\boxed{n_\alpha m_\alpha\left(\frac{\partial}{\partial t}\underset{\sim}{u}_\alpha + \underset{\sim}{u}_\alpha\cdot\nabla\underset{\sim}{u}_\alpha\right) = n_\alpha q_\alpha\left(\underset{\sim}{E}+\frac{1}{c}\underset{\sim}{u}_\alpha\times\underset{\sim}{B}\right) - \nabla\cdot\underset{\approx}{P}_\alpha - \underset{\sim}{R}_{\alpha\beta}}$$

This is the fluid momentum equation. Note that it is not complete since it depends on the unknown momentum tensor $\underset{\approx}{P}_\alpha$.

General pattern

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot n_\alpha u_\alpha = 0$$

$$\frac{\partial}{\partial t} m_\alpha n_\alpha \underset{\sim}{u}_\alpha + (\;) + \nabla\cdot\underset{\approx}{P}_\alpha = (\;)$$

$$\frac{\partial}{\partial t}\underset{\approx}{P}_\alpha + (\;) + \nabla\cdot \text{third moment} = (\;)$$

$\Rightarrow$ no closure of equations.

<u>Closure with collisions</u>

We know that collisions drive f_α toward a Maxwellian distribution. Thus if collisions are ~~too~~ strong we expect.

$$f_\alpha = \frac{n_\alpha}{(2\pi T_\alpha/m_\alpha)^{3/2}} e^{-\frac{(\underset{\sim}{v}-\underset{\sim}{u}_\alpha)^2 m_\alpha}{2T_\alpha}} + \delta f_\alpha$$

where $\delta f_\alpha \sim \frac{1}{\nu}$ $\Rightarrow$ small

$$P_{\alpha ij} \cong m_\alpha \int d\underset{\sim}{v}' \, v_i' v_j' \, e^{-\frac{v'^2 m_\alpha}{2T_\alpha}} \frac{n_\alpha}{(2\pi T_\alpha/m_\alpha)^{3/2}}$$

$\Rightarrow$ $P_{\alpha ij}$ is diagonal

$\Rightarrow$ diagonal terms **equal**

$$P_{\alpha ii} = m_\alpha \int dv_i' \, v_i'^2 \frac{e^{-\frac{v_i'^2 m_\alpha}{2T_\alpha}}}{(2\pi T_\alpha/m_\alpha)^{1/2}} n_\alpha$$

$$s^2 = \frac{v_i'^2 m_\alpha}{2T_\alpha}$$

$$P_{\alpha ii} = m_\alpha n_\alpha \left(\frac{2T_\alpha}{m_\alpha}\right)^{3/2} \frac{1}{(2\pi T_\alpha/m_\alpha)^{1/2}} \int_{-\infty}^{\infty} ds \, s^2 e^{-s^2}$$

$$\int_{-\infty}^{\infty} ds \, s^2 e^{-s^2} = -\frac{d}{d\epsilon} \int ds \, e^{-\epsilon s^2} = -\frac{d}{d\epsilon} \frac{1}{\epsilon^{1/2}} \underbrace{\int dp \, e^{-p^2}}_{\sqrt{\pi}}$$

$$= \frac{\sqrt{\pi}}{2}.$$

$$P_{\alpha ii} = n_\alpha T_\alpha = P_\alpha$$

$$\nabla \cdot \underset{\approx}{P}_\alpha = \nabla P_\alpha + \nabla \cdot \delta \underset{\approx}{P}_\alpha$$

$$n_\alpha m_\alpha \left(\frac{\partial}{\partial t} \underset{\sim}{u}_\alpha + \underset{\sim}{u}_\alpha \cdot \nabla \underset{\sim}{u}_\alpha \right) = n_\alpha q_\alpha \left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{u}_\alpha \times \underset{\sim}{B} \right)$$

$$- \nabla P_\alpha \underbrace{- \nabla \cdot \underset{\approx}{\delta P}_\alpha}_{\text{viscous stress}} - \underset{\sim}{R}_{\alpha\beta}$$

<u>Second Moment</u> $\Rightarrow$ energy equation

Multiply VE by $\frac{1}{2} m_\alpha v^2$

$$\frac{\partial}{\partial t} \int d\underset{\sim}{v} \, \frac{m_\alpha v^2}{2} f_\alpha + \nabla \cdot \int d\underset{\sim}{v} \, \frac{m_\alpha v^2}{2} \underset{\sim}{v} \, f_\alpha$$

$$+ q_\alpha \int d\underset{\sim}{v} \, \frac{v^2}{2} \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B} \right) f_\alpha$$

$$= \sum_\beta \int d\underset{\sim}{v} \, \frac{m_\alpha v^2}{2} C_{\alpha\beta}(f_\alpha)$$

Again write $\underset{\sim}{v} = \underset{\sim}{u}_\alpha + \underset{\sim}{v}'$

$$\int d\underset{\sim}{v} \, \frac{m_\alpha v^2}{2} f_\alpha = \frac{1}{2} m_\alpha u_\alpha^2 + \int d\underset{\sim}{v} \, \frac{m_\alpha v'^2}{2} f_\alpha$$

$$= \frac{1}{2} m_\alpha u_\alpha^2 + \frac{3}{2} n_\alpha T_\alpha$$

$$\int d\underset{\sim}{v} \, \frac{m_\alpha v^2}{2} \underset{\sim}{v} \, f_\alpha = \frac{1}{2} m_\alpha u_\alpha^2 \underset{\sim}{u}_\alpha n_\alpha + \int d\underset{\sim}{v}' \, \frac{m_\alpha v'^2}{2} \underset{\sim}{v}' f_\alpha$$

$$+ \int d\underset{\sim}{v}' \, \frac{1}{2} m_\alpha v'^2 \underset{\sim}{u}_\alpha f_\alpha + \int d\underset{\sim}{v}' \frac{1}{2} m_\alpha \, 2 \underset{\sim}{v}' \cdot \underset{\sim}{u}_\alpha \underset{\sim}{v}' f_\alpha$$

$$= \frac{1}{2} m_\alpha u_\alpha^2 \underset{\sim}{u}_\alpha n_\alpha + \frac{3}{2} P_\alpha \underset{\sim}{u}_\alpha + P_\alpha \underset{\sim}{u}_\alpha$$

$$+ \underset{\sim}{Q}_\alpha \qquad \underset{\sim}{Q}_\alpha = \int d\underset{\sim}{v}' \frac{m_\alpha v'^2}{2} \underset{\sim}{v}' f_\alpha$$

$$= \frac{1}{2} m_\alpha u_\alpha^2 \underset{\sim}{u}_\alpha n_\alpha + \frac{5}{2} P_\alpha \underset{\sim}{u}_\alpha + \underbrace{\underset{\sim}{Q}_\alpha}_{\substack{\text{heat flux} \\ \sim \frac{1}{\nu}}}$$

$$q_\alpha \int d\underset{\sim}{v} \frac{v^2}{2} \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B} \right) f_\alpha$$

$$= - \int d\underset{\sim}{v}\, \underset{\sim}{v} \cdot \left(\underset{\sim}{E} + \cancelto{0}{\frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}} \right) f_\alpha q_\alpha$$

$$= - q_\alpha \underset{\sim}{E} \cdot \underset{\sim}{u}_\alpha n_\alpha$$

Putting it all together,

$$\frac{\partial}{\partial t}\left(\frac{1}{2} m_\alpha u_\alpha^2 + \frac{3}{2} n_\alpha T_\alpha \right) + \nabla \cdot \left(\frac{1}{2} m_\alpha u_\alpha^2 \underset{\sim}{u}_\alpha n_\alpha + \frac{5}{2} P_\alpha \underset{\sim}{u}_\alpha + \underset{\sim}{Q}_\alpha \right)$$

$$= q_\alpha n_\alpha \underset{\sim}{u}_\alpha \cdot \underset{\sim}{E} - \underbrace{\left(\frac{\partial W}{\partial t} \right)_{\alpha\beta}}_{\substack{\text{rate energy transfer from} \\ \alpha \text{ to } \beta}}$$

Note that $\left(\frac{\partial}{\partial t} W \right)_{\alpha\beta} = - \left(\frac{\partial}{\partial t} W \right)_{\beta\alpha}$

and $\left(\frac{\partial W}{\partial t} \right)_{\alpha\alpha} = 0$

$\Rightarrow$ collisions conserve total energy

Can simplify by using continuity and momentum equations.

$$\frac{3}{2}\left(\frac{\partial}{\partial t} + \underset{\sim}{u}_\alpha \cdot \nabla\right) P_\alpha + \frac{5}{2} P_\alpha \nabla \cdot \underset{\sim}{u}_\alpha$$

$$= - \nabla \cdot \underset{\sim}{Q}_\alpha + R_{\alpha\beta} \cdot \underset{\sim}{u}_\alpha - \left(\frac{\partial W}{\partial t}\right)_{\alpha\beta}$$

In a 3-D system $\frac{3}{2} P_\alpha$ is the internal energy.

$\underset{\sim}{R}_{\alpha\beta} \cdot \underset{\sim}{u}_\alpha$ is frictional heating of α due to drag with β.

Can also consider systems with other degrees of freedom.

$$3 \longrightarrow n_f$$

$$5 \longrightarrow 2 + n_f$$

$$\frac{n_f}{2}\left(\frac{\partial}{\partial t} + \underset{\sim}{u}_\alpha \cdot \nabla\right) P_\alpha + \frac{n_f + 2}{2} P_\alpha \nabla \cdot \underset{\sim}{u}_\alpha$$

$$= - \nabla \cdot \underset{\sim}{Q}_\alpha + \underset{\sim}{R}_{\alpha\beta} \cdot \underset{\sim}{u}_\alpha - \left(\frac{\partial W}{\partial t}\right)_{\alpha\beta}$$

No heat flux or collisional heating or energy loss.

$\Longrightarrow$ adiabatic

$$\frac{n_\ell}{\cancel{2}}\left(\frac{\partial}{\partial t}+u_\alpha\cdot\nabla\right)p_\alpha + \frac{n_\ell+2}{\cancel{2}}\, p_\alpha\, \nabla\cdot u_\alpha = 0$$

$$\frac{1}{p_\alpha}\frac{d}{dt}p_\alpha + \gamma_s\, \nabla\cdot u_\alpha = 0$$

$$\gamma_s = \frac{n_\ell+2}{n_\ell}$$ ratio of specific heats.

from continuity

$$\frac{\partial n_\alpha}{\partial t} + u_\alpha\cdot\nabla n_\alpha + n_\alpha\, \nabla\cdot u_\alpha = 0$$

$$\gamma_s\left(\frac{1}{n_\alpha}\frac{d}{dt}n_\alpha + \nabla\cdot u_\alpha\right) = 0$$

Subtract equations

$$\cancel{\frac{1}{p_\alpha}\frac{d}{dt}p_\alpha - \gamma_s}$$

$$\frac{1}{p_\alpha}\frac{d}{dt}p_\alpha = \gamma_s\frac{1}{n_\alpha}\frac{d}{dt}n_\alpha$$

$$\frac{d}{dt}\left(\ln p_\alpha - \ln n_\alpha^{\gamma_s}\right) = 0$$

$$\frac{d}{dt}\left(\frac{p_\alpha}{n_\alpha^{\gamma_s}}\right) = 0$$

$$\Longrightarrow \quad \frac{p_\alpha}{n_\alpha^{\gamma_s}} = \text{const in frame of moving plasma.}$$