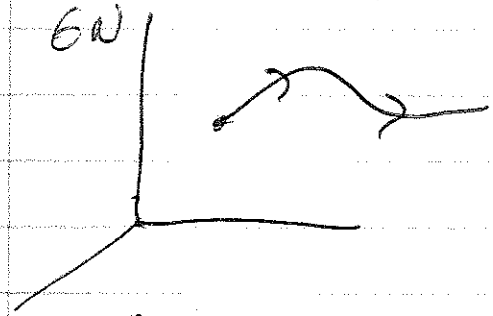
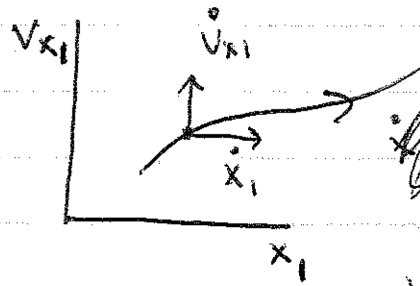
Liouville's Equ

Choudhuri Ch 1
G-R Ch. 22

We would like to construct a set of equations that can be used to explore the dynamics of a plasma. Suppose that the system consists of N particles. The complete state of a system is given by 6N variables $(\underset{\sim}{x}_1, \underset{\sim}{v}_1, \underset{\sim}{x}_2, \underset{\sim}{v}_2 \ldots \underset{\sim}{x}_N, \underset{\sim}{v}_N)$

The ~~trajectory~~ behavior of the system is given by a trajectory in 6N space





Generally would want to consider a statistical description represented by an ensemble of systems ~~(with differing phase information)~~ with the same mean properties. The local probability of finding the system in a state is

$$dP = F(\underset{\sim}{x}_1, \underset{\sim}{v}_1 \ldots \underset{\sim}{x}_N, \underset{\sim}{v}_N, t)\, d^3x_1\, d^3v_1 \ldots d^3x_N\, d^3v_N$$

$$\int dP = 1$$

Liouville's theorem says that F is a constant if you move along a trajectory defined by a member of the ensemble.

To show that this is correct, need to ~~...~~ use ~~...~~ the continuity equation for particles. Consider a volume $\mathcal{V}$ in $6N$ phase space

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} d\mathcal{V} F = - \int_{S} F \, \underset{\sim}{V}_{6N} \cdot d\underset{\sim}{S}$$

where $\underset{\sim}{V}_{6N}$ is the $6N$ velocity

$$\underset{\sim}{V}_{6N} \equiv (\dot{\underset{\sim}{X}}_1 , \dot{\underset{\sim}{V}}_1 , \cdots \dot{\underset{\sim}{X}}_N , \dot{\underset{\sim}{V}}_N)$$

Use the divergence theorem ~~...~~

$$\int_{\mathcal{V}} d\mathcal{V} \left(\frac{\partial F}{\partial t} + \nabla \cdot \underset{\sim}{V}_{6N} F \right) = 0$$

This is true for any $\mathcal{V}$ so

$$\frac{\partial F}{\partial t} + \sum_s \frac{\partial}{\partial \underset{\sim}{X}_s} \cdot \left(\dot{\underset{\sim}{X}}_s F \right) + \sum_s \frac{\partial}{\partial \underset{\sim}{V}_s} \cdot \left(\dot{\underset{\sim}{V}}_s F \right) = 0$$

$$\frac{\partial}{\partial \underset{\sim}{X}_s} \cdot \left(\dot{\underset{\sim}{X}}_s F \right) = \underset{\sim}{V}_s \cdot \frac{\partial}{\partial \underset{\sim}{X}_s} F + F \frac{\partial}{\partial \underset{\sim}{X}_s} \cdot \underset{\sim}{V}_s$$

$\frac{\partial}{\partial \underset{\sim}{X}_s} \cdot \underset{\sim}{V}_s = 0$ since $\underset{\sim}{X}_s , \underset{\sim}{V}_s$ are taken to be independent variables.

$$\dot{v}_s = \frac{1}{m} \underset{\sim}{F}_s$$

$$\frac{\partial}{\partial \underset{\sim}{v}_s} \cdot \left(\dot{\underset{\sim}{v}}_s F \right) = \frac{1}{m} \underset{\sim}{F}_s \cdot \frac{\partial}{\partial \underset{\sim}{v}_s} F + \frac{F}{m} \frac{\partial}{\partial \underset{\sim}{v}_s} \cdot \underset{\sim}{F}_s$$

$$\underset{\sim}{F}_s = q \left(\underset{\sim}{E}_s + \frac{1}{c} \underset{\sim}{v}_s \times \underset{\sim}{B}_s \right)$$

$$\frac{\partial}{\partial \underset{\sim}{v}_s} \cdot \underset{\sim}{F}_s = \frac{q}{c} \frac{\partial}{\partial \underset{\sim}{v}_s} \cdot \left(\underset{\sim}{v}_s \times \underset{\sim}{B}_s \right)$$

$$\frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\underset{\sim}{v} \times \underset{\sim}{B} \right) = \underbrace{\frac{\partial}{\partial v_x} \left(v_y B_z - v_z B_y \right)}_{0} + - - - -$$

$$= 0$$

$$\frac{\partial}{\partial t} F + \sum_s \underset{\sim}{v}_s \cdot \frac{\partial}{\partial \underset{\sim}{x}_s} F + \sum_s \frac{\underset{\sim}{F}_s}{m_s} \cdot \frac{\partial}{\partial v_s} F = 0$$

Note that

$$\nabla_{6N} \cdot \underset{\sim}{V}_{6N} = 0$$

$\Longrightarrow$ flow in 6N phase space is incompressible.

Insert

The Liouville equation is an essentially exact equation but it contains too much information. It is valid for Γ large or small but we will be most interested in the limit $\Gamma \ll 1$ where individual particles are only

For a Hamiltonian system have

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\nabla_{6N} \cdot \underline{v}_{6N} \sim \frac{\partial}{\partial p_i} \dot{p}_i + \frac{\partial}{\partial q_i} \dot{q}_i$$

$$\sim -\frac{\partial^2 H}{\partial q_i \partial p_i} + \frac{\partial^2 H}{\partial q_i \partial p_i} = 0$$

Can write the Liouville eqn using all canonical variables

$\Rightarrow$ e.g. cylindrical or spherical coordinates weakly correlated. In the limit where particles are uncorrelated,

$$F(\underset{\sim}{x}_1, \underset{\sim}{v}_1, \ldots \underset{\sim}{x}_N, \underset{\sim}{v}_N, t) = F_1(\underset{\sim}{x}_1, \underset{\sim}{v}_1, t)\, F_1(\underset{\sim}{x}_2, \underset{\sim}{v}_2, t) \cdots F_1(\underset{\sim}{x}_N, \underset{\sim}{v}_N, t)$$

$\Rightarrow$ product of single particle pdfs

What we want to evaluate is the single particle distribution function $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$. This is ~~related~~ related to the previous 6D distribution function $F_1(\underset{\sim}{x}, \underset{\sim}{v}, t)$

$$f(\underset{\sim}{x}, \underset{\sim}{v}, t) = N\, F_1(\underset{\sim}{x}, \underset{\sim}{v}, t)$$

where $\int_V d^3x\, d^3v\, F_1 = 1$.

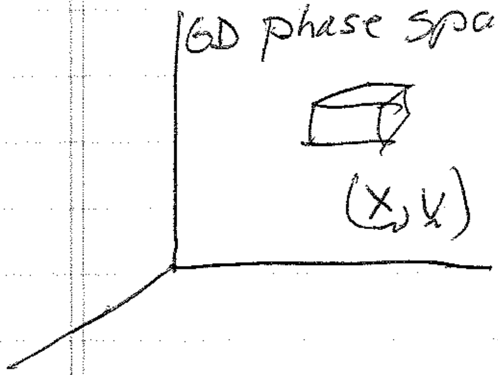
The equation for $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is the Vlasov equation.

Can derive an equation for f by integrated LE over $d^3x_2 d^3v_2 \ldots d^3x_N d^3v_N$

$$\frac{\partial}{\partial t} F_1 + v_1 \cdot \nabla_1 F_1 + \frac{\underset{\sim}{F}_1}{m_1} \cdot \frac{\partial}{\partial v_1} F_1 = 0$$

The Vlasov Equation (Collisionless Boltzmann Equation)

We want to be able to describe the dynamics of a plasma in the weakly coupled regime in which $\Gamma \ll 1$.
Want to obtain an equation for $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$



$dN = f(\underset{\sim}{x}, \underset{\sim}{v}, t)\, d^3x\, d^3v$
= # of particles in volume element.

Want to show:

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} f + \frac{q}{m}\left(\underset{\sim}{E} + \frac{1}{c}\, \underset{\sim}{v} \times B\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f = 0$$

Note that in this description $\underset{\sim}{x}, \underset{\sim}{v}, t$ are independent variables.

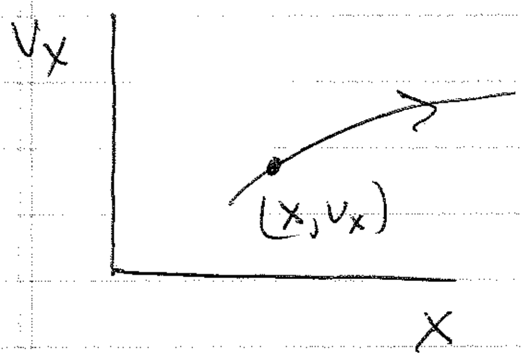
$\Rightarrow$ misses collisions

$\Rightarrow$ spontaneous radiation
Eg Synchrotron radiation
Bremsstrahlung.

Derivation similar to that of Liouville's Eqn.

$\Rightarrow$ consider motion in 6D phase space.

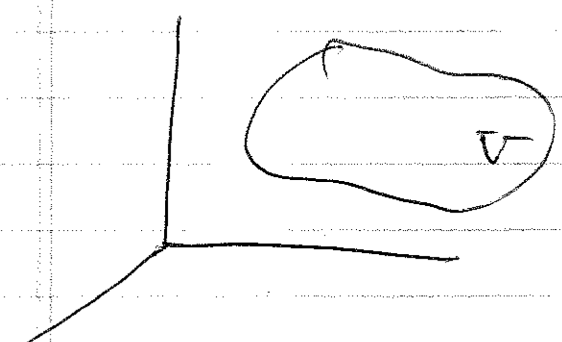
$\Rightarrow$ define a 6D velocity $\underset{\sim}{U}$



$$\left.\begin{aligned} \dot{x} &= v_x \\ \dot{v}_x &= \frac{F_x}{m} \end{aligned}\right\} \text{x components of } U$$

$$\underset{\sim}{U} = \left(v_x, \frac{F_x}{m}, v_y, \frac{F_y}{m}, v_z, \frac{F_z}{m} \right)$$

Use conservation of particles in 6D phase space



$$\frac{d}{dt} \int_V dV \, f = - \int_S d\underset{\sim}{s} \cdot \underset{\sim}{U} \, f$$

S = surface in 6D space (5D)

$$\int_{S'} d\vec{s}' \cdot \underset{\sim}{u}\, f = \int_{V} dv\, \nabla_6 \cdot \bar{u}\, f$$

from divergence theorem. ~~As before~~ $\cancel{\nabla_6 \cdot \underset{\sim}{u} = 0}$

$$\cancel{\frac{d}{dt}} \int_{V} dv \left(\frac{\partial f}{\partial t} + \nabla_6 \cdot \bar{u}\, f \right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \nabla_6 \cdot \bar{u}\, f = 0$$

As before, $\nabla_6 \cdot \bar{u} = 0$

$$\frac{\partial f}{\partial t} + \underset{\sim}{u} \cdot \nabla_6\, f = 0$$

What about collisions — Choudhuri Ch 2, GR Ch 13

In deriving the Boltzman equation we have ignored collisions. If collisions are sufficiently weak this is justified. Under what conditions can collisions be neglected?

Since

$$\frac{\partial f}{\partial t} \sim v \cdot \nabla f \sim \frac{F}{m} \cdot \frac{\partial}{\partial x_i} f$$

Assume that $\nabla \sim \frac{1}{L}$, $\frac{\partial}{\partial \underset{\sim}{v}} \sim \frac{1}{v_{th}}$

$$\frac{\partial}{\partial t} \sim \frac{V_t}{L}, \frac{F}{m V_t}$$

Compare these rates with typical scattering ~~time~~ rates $\sim \nu$, we can ignore collisions for

$$\frac{V_t}{L}, \frac{F}{m V_t} >> \nu .$$

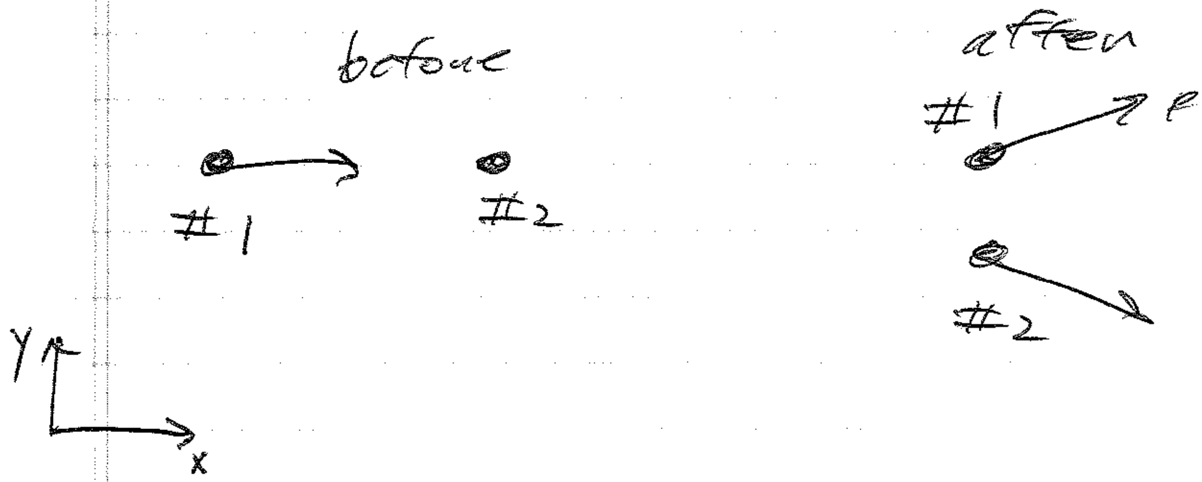
mean-free path

$\lambda = \frac{V_t}{\nu}$ $\quad \lambda >> L$ can neglect collisions

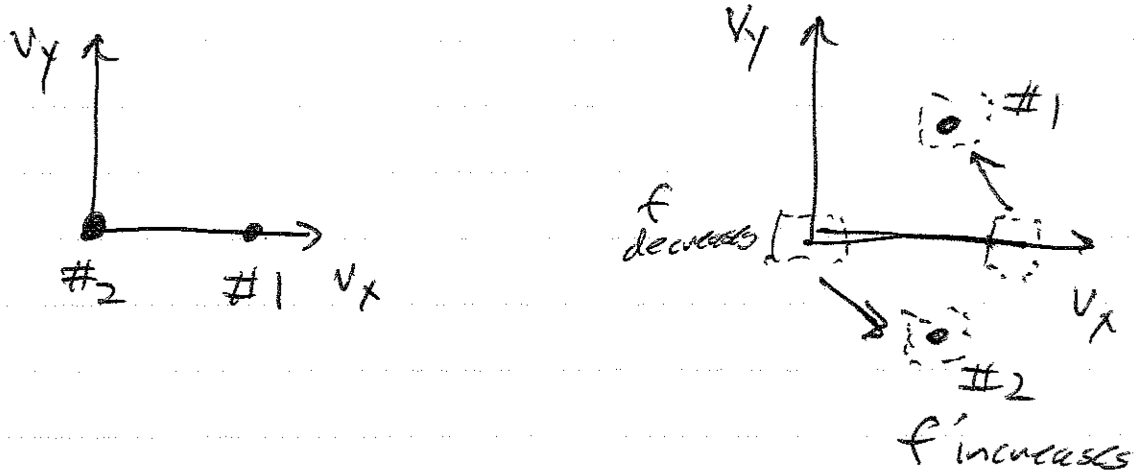
~~[illegible]~~

How do collisions act on distribution functions?

Consider a simple large-angle scattering event.



velocity space



Actually collisions are mostly small angle so they typically move ~~[illegible]~~ in small steps in velocity space.

Include collisions with a collision operator as follows:

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{\underset{\sim}{F}_{ext}}{m} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f = C(f)$$

$C(f)$ is the collision operator.

$\Longrightarrow$ typically a nonlinear integral operator.

Properties of $C(f)$

① Coulomb collisions conserve

a) particle number

$$\int d\underset{\sim}{v}\, C(f) = 0$$

b) momentum (when summed over species)

c) energy (when summed over species)

② If f is in thermal equilibrium

$$f = \frac{n_0}{\left(\frac{2\pi T}{m}\right)^{3/2}} e^{-\frac{\frac{1}{2}mv^2 + q\varphi}{T}}$$

then $C(f) = 0 \Longrightarrow \frac{\partial f}{\partial t} = 0$

Collisions drive f to TE

Entropy increases until TE is reached (Boltzmann's H theorem)

3 Collisions act locally in physical space

$\Rightarrow$ particles within λ_D of each other

nonlocally in velocity space

$\Rightarrow$ particles with very different velocities can interact.

Fokker-Planck Equation for collisions:

Want to describe the evolution of ~~f(v)~~ $f(\underset{\sim}{v}, t)$ due to the action of small angle collisions (ignore space variation for now).

Define $P(\underset{\sim}{v}, \Delta\underset{\sim}{v})$ as the probability that a particle with velocity $\underset{\sim}{v}$ with undergo an increment $\Delta\underset{\sim}{v}$ in a time Δt. Thus at a time t we can write

$$(1) \qquad f(\underset{\sim}{v}, t) = \int d^3\Delta v \, P(\underset{\sim}{v} - \Delta\underset{\sim}{v}, \Delta\underset{\sim}{v}) \, f(\underset{\sim}{v} - \Delta\underset{\sim}{v}, t - \Delta t)$$

Since the ~~sum~~ sum of all probabilities must be 1

$$\int d^3\Delta v \, P(\underset{\sim}{v}, \Delta\underset{\sim}{v}) = 1$$

For small angle collisions $\Delta v \ll v_{te}$ and can expand RHS of (1) for small $\Delta\underline{v}$ (compared with $\underline{v}$)

$$f(\underline{v}-\Delta\underline{v}, t-\Delta t) \simeq -\frac{\partial f}{\partial t}\Delta t + f(\underline{v},t) - \Delta\underline{v}\cdot\frac{\partial}{\partial\underline{v}} f(\underline{v},t) + \frac{1}{2}\Delta\underline{v}\Delta\underline{v} : \frac{\partial^2}{\partial\underline{v}\partial\underline{v}} f$$

$$P(\underline{v}-\Delta\underline{v}, \Delta\underline{v}) = P(\underline{v}, \Delta\underline{v}) - \Delta\underline{v}\cdot\frac{\partial}{\partial\underline{v}} P + \frac{1}{2}\Delta\underline{v}\Delta\underline{v} : \frac{\partial^2}{\partial\underline{v}\partial\underline{v}} P$$

$$\cancel{f(v,t)} = \left[\cancel{f(\underline{v},t)} - \frac{\partial f}{\partial t}\Delta t\right] \underbrace{\int d^3\Delta v\, P(\underline{v},\Delta\underline{v})}_{=1}$$

$$- \int d^3\Delta v\, \Delta\underline{v}\cdot\left(\frac{\partial f}{\partial\underline{v}}P + \frac{\partial P}{\partial\underline{v}} f\right)$$

$$+ \frac{1}{2}\int d^3\Delta v\, \Delta\underline{v}\Delta\underline{v} : \left(\frac{\partial^2 f}{\partial\underline{v}\partial\underline{v}}P + 2\frac{\partial f}{\partial\underline{v}}\frac{\partial P}{\partial\underline{v}} + f\frac{\partial^2 P}{\partial\underline{v}\partial\underline{v}}\right)$$

$$\frac{\partial f}{\partial t}\Delta t = -\frac{\partial}{\partial\underline{v}}\cdot\int d^3\Delta v\, \Delta\underline{v}\, fP + \frac{1}{2}\frac{\partial^2}{\partial\underline{v}\partial\underline{v}} : \int d^3\Delta v\, \Delta\underline{v}\Delta\underline{v}\, fP$$

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\frac{\partial}{\partial\underline{v}}\cdot\frac{d\langle\Delta\underline{v}\rangle}{dt} f + \frac{1}{2}\frac{\partial^2}{\partial\underline{v}\partial\underline{v}} : \frac{d\langle\Delta\underline{v}\Delta\underline{v}\rangle}{dt} f$$

$$\frac{d}{dt}\langle \Delta \underset{\sim}{v} \rangle = \frac{1}{\Delta t}\int P \Delta \underset{\sim}{v}\, d^3 \Delta v$$

$$\frac{d}{dt}\langle \Delta \underset{\sim}{v} \Delta \underset{\sim}{v} \rangle = \frac{1}{\Delta t}\int P \Delta \underset{\sim}{v} \Delta \underset{\sim}{v}\, d^3 \Delta v$$

note that are functions of $\underset{\sim}{V}$

Note that there are two distinct terms associated with collisions. ~~The first~~
Must have

$$\frac{d}{dt}\langle \Delta \underset{\sim}{v} \rangle \sim -(\ )\ \underset{\sim}{V}$$

since have no other direction. Recall that earlier found (for e-i collisions)

$$\frac{d}{dt}\langle \Delta \underset{\sim}{v} \rangle = \frac{-4\pi n_i Z^2 e^4 \ln \Lambda}{m^2 V^3}\ \underset{\sim}{V}$$

The first term is the drag or friction term. Note that if we ~~drag~~ mult by $\underset{\sim}{V}$ and integrate the second term drops out

$$\frac{\partial}{\partial t}\overbrace{\int d\underset{\sim}{v}\, f \underset{\sim}{v}}^{n_0 \underset{\sim}{u}} = \int d\underset{\sim}{v}\, f \frac{d}{dt}\langle \Delta \underset{\sim}{v} \rangle$$

~~The second term is a diffusion~~
Causes the slowing down of the distribution

$$n_0 \frac{\partial}{\partial t}\underset{\sim}{u} \sim -\nu\, \underset{\sim}{u}\, n_0$$

The second term causes ~~a spreading~~ diffusion of the distribution in velocity space.

The F-P equation is the generic form of the collision operator in an ionized plasma $\Rightarrow$ no neutral collisions.

Landau form of collision operator

$$\frac{\partial}{\partial t} f^\alpha + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} f^\alpha + \frac{F}{m_\alpha} \cdot \frac{\partial}{\partial v} f^\alpha = \sum_\beta C(f^\alpha, f^\beta)$$

$C(f^\alpha, f^\beta)$ rate of change of f^α due to collisions with β.

$$C(f^\alpha, f^\beta) = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot \frac{2\pi q_\alpha^2 q_\beta^2 \ln \Lambda}{m_\alpha}$$

$$\int d\underset{\sim}{v}' \left(\frac{u^2 \underset{\approx}{I} - \underset{\sim}{u}\,\underset{\sim}{u}}{u^3} \right) \cdot \left[\frac{1}{m_\beta} f^\alpha(\underset{\sim}{v}) \frac{\partial}{\partial \underset{\sim}{v}'} f^\beta(\underset{\sim}{v}') - \frac{1}{m_\alpha} f^\beta(\underset{\sim}{v}') \frac{\partial}{\partial \underset{\sim}{v}} f^\alpha \right]$$

$$\underset{\sim}{u} = \underset{\sim}{v} - \underset{\sim}{v}'$$

What about the limit where

f^α, f^β are Maxwellian with the same temperature?

$$\frac{\partial}{\partial \underset{\sim}{v}} f^\alpha = -f^\alpha \frac{m_\alpha \underset{\sim}{v}}{T}$$

$$- \int d\underset{\sim}{v}' \frac{u^2 \underset{\approx}{I} - \underset{\sim}{u}\underset{\sim}{u}}{u^3} \cdot \left[\frac{1}{\cancel{m_\beta}} f^\alpha \frac{\cancel{m_\beta} \underset{\sim}{v}'}{T} f^\beta - \frac{1}{\cancel{m_\alpha}} f^\beta \frac{\cancel{m_\alpha} \underset{\sim}{v}}{T} f^\alpha \right]$$

$$= \frac{1}{T} \int d\underset{\sim}{v}' \underbrace{\left(u^2 \underset{\approx}{I} - \underset{\sim}{u}\underset{\sim}{u}\right) \cdot \underset{\sim}{u}}_{\underbrace{u^2 \underset{\sim}{u} - \underset{\sim}{u} u^2}_{0}} f^\alpha f^\beta$$

$\Rightarrow$ collisions have no effect when have TE.

Simple result for electrons colliding with ions.

$$\left(\frac{\partial f_e}{\partial t}\right)_{ei} = \frac{2\pi n_i Z^2 e^4 \ln \Lambda}{m^2} \frac{\partial}{\partial \underset{\sim}{v}} \cdot \frac{\underset{\approx}{I} v^2 - \underset{\sim}{v}\underset{\sim}{v}}{v^3} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f^e$$

Ignoring e-e collisions but retains e-i collisions

$\Rightarrow$ Lorentz gas

note that this operator only scatters the pitch angle of a particle

$\Rightarrow$ no energy scattering

$$\frac{\partial}{\partial \underset{\sim}{v}} \cdot \frac{\left(I v^2 - \underset{\sim}{v}\underset{\sim}{v}\right)}{v^3} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f(v^2) = 0$$

$\Rightarrow$ solution arbitrary function of v^2

Krook Model

$$C(f) = -\nu\left(f - \frac{n(x)}{\left(\frac{2\pi T_0}{m}\right)^{3/2}} e^{-\frac{1}{2}\frac{m v_0^2}{T_0}}\right)$$

This form conserves the density.

$\Rightarrow$ can also use a Krook model that conserves momentum and energy

Characteristics of the Vlasov Eqn

Characteristic curves are trajectories in phase space of individual particles

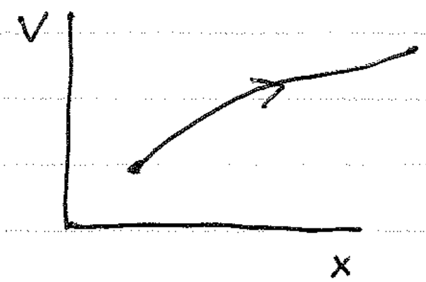
$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{F}{m} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f = 0$$

Can re-write the equation as

$$\frac{\partial f}{\partial t} + \frac{d\underset{\sim}{x}}{dt} \cdot \frac{\partial}{\partial \underset{\sim}{x}} f + \frac{d\underset{\sim}{v}}{dt} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f = 0$$

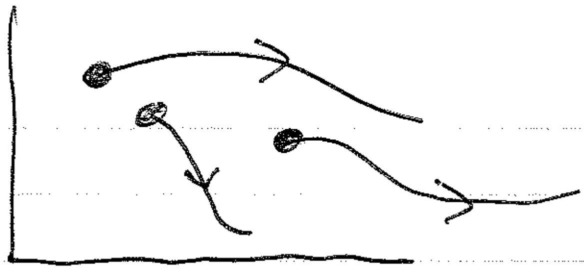
$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v}$$

$$\frac{d\underset{\sim}{v}}{dt} = \frac{\underset{\sim}{F}}{m}$$



$\Rightarrow \quad \frac{df}{dt} = 0$ ~~can~~ f is a constant ~~[illegible]~~ along the trajectory of a particle

$\Rightarrow$ patches of phase space density move around such that their values don't change



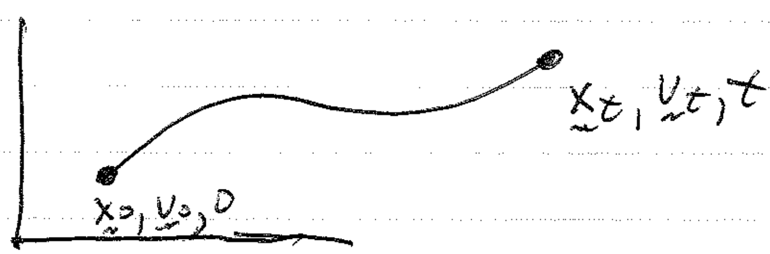
e.g., suppose that f has a maximum value f_{max}

$$f \leq f_{max} \text{ for all } t.$$

Define a set of trajectories for all particles.

$$x_t(\underset{\sim}{x}_0, \underset{\sim}{v}_0, t)$$
$$v_t(\underset{\sim}{x}_0, \underset{\sim}{v}_0, t)$$

$$\left.\begin{aligned} \frac{d\underset{\sim}{x}_t}{dt} &= \underset{\sim}{v}_t \\ \frac{d\underset{\sim}{v}_t}{dt} &= \frac{F(\underset{\sim}{x}_t, \underset{\sim}{v}_t, t)}{m} \end{aligned}\right\} \quad \begin{aligned} &\text{at } t = 0 \\ &\underset{\sim}{x}_t = \underset{\sim}{x}_0 \\ &\underset{\sim}{v}_t = \underset{\sim}{v}_0 \end{aligned}$$



$$f(x_t, \underset{\sim}{v}_t, t) = f(\underset{\sim}{x}_0, \underset{\sim}{v}_0, 0)$$

Define an inverse problem:

Know $\underset{\sim}{x}_t$, $\underset{\sim}{v}_t$ at a time t

find $\underset{\sim}{x}_0$, $\underset{\sim}{v}_0$ at time ~~...~~ t earlier

$$\underset{\sim}{x}_0 = \underset{\sim}{x}_0(\underset{\sim}{x}_t, \underset{\sim}{v}_t, t)$$
$$\underset{\sim}{v}_0 = \underset{\sim}{v}_0(\underset{\sim}{x}_t, \underset{\sim}{v}_t, t)$$

Know that

$$f(\underset{\sim}{x}_t, \underset{\sim}{v}_t, t) = f(\underset{\sim}{x}_0, \underset{\sim}{v}_0, 0)$$

$$= f(\underset{\sim}{x}_0(\underset{\sim}{x}_t, \underset{\sim}{v}_t, t), \underset{\sim}{v}_0(\underset{\sim}{x}_t, \underset{\sim}{v}_t, t), 0)$$

$x_t \to x$
$v_t \to v$

$$f(x, v, t) = f(\underset{\sim}{x}_0(\underset{\sim}{x}, \underset{\sim}{v}, t), v_0(\underset{\sim}{x}, \underset{\sim}{v}, t), 0)$$

This must be a solution of the VE.
At time t the value of f is given by phase space location at $t=0$.
Simple case: $F=0$, 1-D

~~x_0~~ $x_0 = x - vt$
$v_0 = v$

$x - vt$

$f(x, v, t) = f(x - vt, v, 0)$
value of f at t is from f that was earlier at

Solution of the VE in terms of constants of the motion

Suppose have a constant of motion of a particle

Example $\quad H = \frac{1}{2} m v^2 + q \varphi(\underset{\sim}{x})$

$$\underset{\sim}{E} = -\nabla\varphi \;, \quad \frac{\partial \varphi}{\partial t} = 0$$

classical motion, $v(t)$, $x(t)$

$$\frac{d}{dt} H = m\, \underset{\sim}{v}\cdot\dot{\underset{\sim}{v}} + q\, \underset{\sim}{v}\cdot\nabla\varphi$$

$$= 0 \quad \text{since } m\dot{v} = -q\nabla\varphi$$

$\Rightarrow$ energy conserved.

Show that $f(\underset{\sim}{x}, \underset{\sim}{v}, t) = f[H(\underset{\sim}{x}, \underset{\sim}{v})]$

for any f satisfies the VE

$$\frac{\partial f}{\partial t} = 0$$

$$\underset{\sim}{v}\cdot\nabla f = (v\cdot\nabla H)\frac{\partial f}{\partial H} = q v\cdot\nabla\varphi\,\frac{\partial f}{\partial H}$$

$$\frac{F}{m}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f = -\frac{q}{m}\nabla\varphi\cdot\frac{\partial}{\partial \underset{\sim}{v}} H\left(\frac{\partial f}{\partial H}\right)$$

$$= -\frac{q}{\cancel{m}}\nabla\varphi\cdot\cancel{m}\underset{\sim}{v}\;\frac{\partial f}{\partial H}$$

$$\Rightarrow \frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{\underset{\sim}{F}}{m} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f = 0$$

Other examples: canonical momentum

$$\underset{\sim}{E} = -\nabla \varphi - \frac{1}{c} \frac{\partial \underset{\sim}{A}}{\partial t}, \quad \underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

$\underset{\sim}{A}$, φ indep. of x

$P_x = m v_x + \frac{q}{c} A_x$ is a const.

$\Rightarrow$ can write f in terms of P_x