R-G Ch 11

Collision~~al~~ Rates in Fully Ionized Plasma

First consider how electrons scatter from ions.

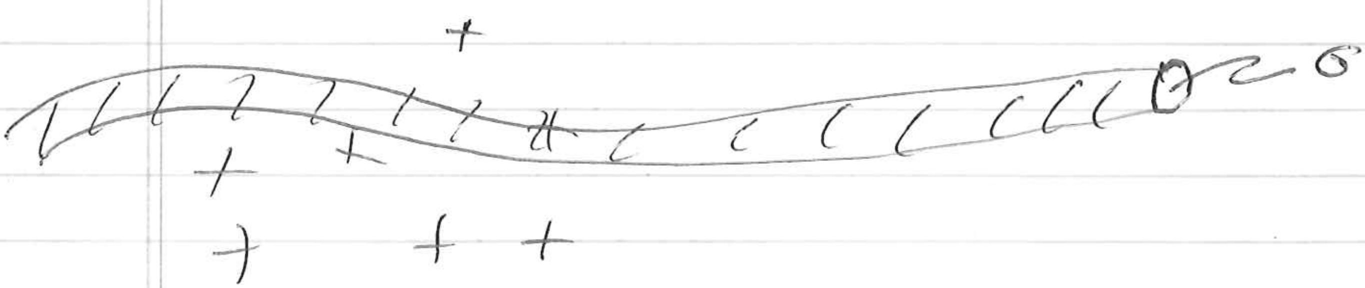
simple minded ~~actual~~ estimate:

How close does the electron have to approach the ion for its potential energy to be comparable to the thermal energy?

$$m_e v^2 \sim \frac{e^2}{r} \Rightarrow r \sim \frac{e^2}{m_e v^2}$$

$$\sigma \sim \pi r^2 \sim \pi \frac{e^4}{m_e^2 v^4}$$

$$\nu_{ei} \sim \overbrace{n(\underbrace{\sigma v}_{\text{rate which volume swept out}})}^{\text{rate at which encounter an ion.}}$$



$$\nu_{ei} \sim n\frac{\pi e^4}{m_e^2 v^4} v \sim \frac{n\pi e^4}{m_e^2 v^3}$$

$$\sim \frac{n\pi e^4}{m_e^{1/2} T_e^{3/2}}$$

$$\frac{4\pi n e^2}{T_e} \sim k_D^2 \sim \frac{1}{\lambda_D^2}$$

$$T_e \sim \frac{4\pi n e^2}{\cancel{\lambda_D^2}} \lambda_D^2$$

$$\frac{\nu_{ei}}{\omega_{pe}} \sim \frac{n\pi e^4 \qquad \cancel{m_e^{1/2}}}{\cancel{m_e^{1/2}} \left(\frac{4\pi n e^2}{\cancel{\lambda_D^2}}\right)^{3/2} \lambda_D^3 \sqrt{4\pi n e^2}}$$

$$\sim \frac{n\pi \cancel{e^4}}{(4\pi n \cancel{e^2})^2 \lambda_D^3} \sim \frac{1}{n\lambda_D^3}$$

For a weakly coupled plasma

$$\frac{\nu_{ei}}{\omega_{pe}} \sim \frac{1}{n\lambda_D^3} << 1$$

$\Rightarrow$ collisions are weak at least on the plasma frequency time scale.

Consider now a more careful discussion of collisional processes in which we will calculate the ~~time scales of~~ rates of several processes

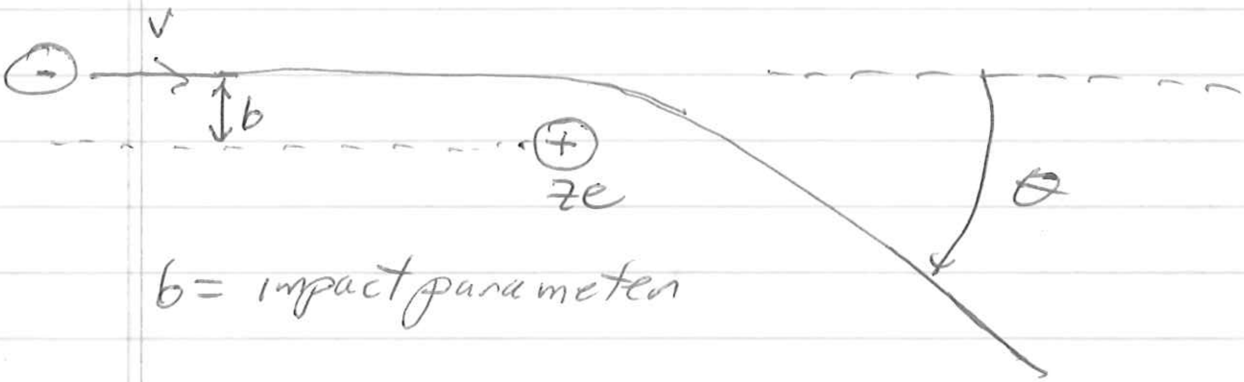
$\Rightarrow$ electron momentum scattering

$\Rightarrow$ ion momentum scattering

$\Rightarrow$ electron-ion energy exchange by ?

Electron scattering off of ions

Consider ions of charge Ze with Z possibly different from 1.



b = impact parameter

The orbit is hyperbolic and is given by

$$\tan\frac{\theta}{2} = \frac{(Ze)(e)}{\mu b v^2} \qquad \underline{\text{CM frame}}$$

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{m_i} \qquad \mu = \text{reduced mass}$$

For electron scattering off ions $\mu \cong m_e$

For 90° scattering

$$b_0 = \frac{Ze^2}{m_e v^2}$$

and

$$\sigma_0 = \pi b_0^2 = \frac{\pi Z^2 e^4}{m_e^2 v^4}$$

as before

The cross section for large angle scattering is given by σ_0. We now want to explore the contributions due to many interactions that deflect the electron by ~~a small~~ many small angles

$$\langle \Delta v_x \rangle = 0$$
$$\langle \Delta v_y \rangle = 0$$
$$\langle \Delta v_z \rangle \neq 0$$

$$\langle \Delta v_x^2 \rangle = \langle \Delta v_y^2 \rangle \neq 0$$

Let $\langle \Delta v_\perp^2 \rangle = \langle \Delta v_x^2 \rangle + \langle \Delta v_y^2 \rangle$

$$\Delta v_\perp^2 = v^2 \sin^2\theta \simeq v^2 \theta^2$$

$\Delta v_\perp^2$ will increase in time due to scattering off of many ions.

$\Rightarrow$ random walk in $v_\perp$

$\Rightarrow$ random walk in θ

In a single scattering with impact parameter b will scatter through

$$\tan\left(\frac{\theta_b}{2}\right) = \frac{b_0}{b} \quad \text{or} \quad \theta_b = \frac{2b_0}{b}$$

For N scatterings will have

$$\theta = \sum_i \theta_{b_i} \quad \text{where } \theta_{b_i} = \pm\theta_b$$

$\Rightarrow$ θ can increase or decrease for a given scattering

$\langle \theta \rangle = \langle \sum_i \theta_{b_i} \rangle = 0$

$\langle \theta^2 \rangle = \langle \sum_i \theta_{b_i} \sum_j \theta_{b_j} \rangle = \sum_i \theta_{b_i}^2$

$\Rightarrow$ since any successive collision will not be correlated with the previous collision

$\langle \theta^2 \rangle = \theta_b^2 \sum_i = N \theta_b^2$

Need to calculate ~~N~~ N. Over a time Δt have

$\Delta N = d\sigma_b n_i v \Delta t$ where $d\sigma_b = 2\pi b \, db$

$d\sigma$ is the cross section for a ring of radius b and thickness db

Thus, over a time Δt

$$d\langle \theta^2 \rangle = d\sigma_b n_i v \Delta t \, \theta_b^2$$

or $$\frac{d\langle \theta^2 \rangle}{dt} = d\sigma_b n_i v \, \theta_b^2$$

This is for the ring so need to sum over b from b_0 to some b_{max}

$$\frac{d\langle \theta^2 \rangle}{dt} = n_i v \int_{b_0}^{b_{max}} 2\pi b \, db \, \frac{4 b_0^2}{b^2}$$

with $\frac{d\langle v_\perp^2 \rangle}{dt} = v^2 \frac{d\langle \theta^2 \rangle}{dt}$

Note that integral diverges logarithmically as $b \to \infty$ $\Rightarrow$ small angle scattering dominates ($\lambda_D \gg b_0$)

$$\frac{d}{dt}\langle \Delta v_\perp^2 \rangle \simeq 8\pi n_i v^3 b_0^2 \ln\left(\frac{b_{max}}{b_0}\right)$$

The Debye length provides a natural upper limit on b since electrons don't feel the ion charge beyond λ_D

$$\Lambda \equiv \frac{\lambda_D}{b_0}$$

Want to find how electron slows down $\Rightarrow$ but energy is conserved.
$\Rightarrow$ consider:

$$v_\perp^2 + v_z^2 = \text{const}$$

$$\Delta v_\perp^2 + 2 v_z \Delta v_z = 0$$

$$\langle \Delta v_z \rangle = -\frac{\langle \Delta v_\perp^2 \rangle}{2v}$$

$$\frac{d}{dt}\langle \Delta v_z \rangle = -\frac{1}{v} \cancel{8}4\pi n_i v^3 b_0^2 \ln \Lambda \equiv -\nu_{ei} v$$

$$\nu_{ei} = \cancel{8}4\pi n_i \cancel{v} \frac{Z^2 e^4}{m_e^2 v \cancel{{}^4} 3} \ln \Lambda$$

$$\boxed{\nu_{ei} = \frac{4\pi n_i Z^2 e^4}{m_e^2 v^3} \ln \Lambda}$$

$\langle \frac{1}{2} m v^2 \rangle \sim \frac{3}{2} T$

What about λ_D / b_0?

$$b_0 = \frac{Z e^2}{m_e V^2} \longrightarrow b_0 = \frac{Z e^2}{3 T_e} \frac{4\pi n}{4\pi n}$$

$$= \frac{Z}{12\pi n \lambda_D^2}$$

$$\Lambda = \frac{\lambda_D}{b_0}$$

$$\Lambda = \frac{12\pi \lambda_D^3 n}{\cancel{Z}} \gg 1 \text{ for } n\lambda_D^3 \text{ large}$$

Typically $\ln\Lambda \sim 20$

Note that $\nu_{ei} \sim \frac{1}{v^3}$ $\Rightarrow$ higher energy particles suffer fewer collisions.

Electron-electron collisions are similar to electron-ion $\Rightarrow$ used reduced mass.

~~Ion-Ion~~ $\nu_{ee} \sim \frac{\nu_{ei}}{Z^2 n_i} n_e$

$\Rightarrow$ electrons exchange energy

Ion-Ion collisions

What dominates momentum scattering of ions? $\Rightarrow$ other ions.

$\Rightarrow$ use reduced mass $\mu = \frac{m_i}{2}$

$$\nu_{ii} \simeq \frac{16\pi n_i Z^4 e^4 \ln\Lambda}{m_i^2 V_i^3}$$

taking $m_i V_i^2 \sim m_e V_e^2$

$$\nu_{ii} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \nu_{ei}$$

What about energy exchange times?

During electron-electron collisions the change in energy can be of order unity so electrons equalize their energy ~~...~~ at a rate given by ν_{ee}, same for ions $\sim \nu_{ii}$.

How long does it take for ions and electrons to exchange energy?

$\Rightarrow$ consider electron colliding with ions.

$\Delta V_e \sim 2 V_e$ $\qquad\qquad$ $2 V_e m_e \sim m_i \Delta V_i$

$$\cancel{\Delta E_i \sim \frac{1}{2} m_i \Delta V_i^2 \sim \frac{1}{2} m_i \frac{4 V_e^2 m_e^2}{m_i^2}}$$

$$\cancel{\sim \frac{m_e}{m_i} m_e V_e^2}$$

$$\frac{\Delta E_i}{E_e} \sim \frac{m_e}{m_i}$$

~~Need $\frac{m_i}{m_e}$ collisions for $E_i \Rightarrow E_e$~~

$\cancel{\nu_{Eei} \sim \frac{m_e}{m_i} \nu_{ei}}$ ~~Energy exchange~~ slowest process.

$$\Delta V_i \sim \frac{2 V_e m_e}{m_i}$$

$$\langle V_i^2 \rangle \approx \frac{(\Delta V_i)^2}{\tau} t$$

$$\sim \frac{V_e^2 m_e^2}{m_i^2} \nu_{ei} t$$

$$m_i \langle V_i^2 \rangle \sim \frac{V_e^2 m_e^2}{m_i} \nu_{ei} t \sim m_e V_e^2$$

$$\frac{m_e}{m_i} \nu_{ei} t \sim 1$$

$$\boxed{\nu_{Eei} \sim \nu_{ei} \frac{m_e}{m_i}}$$

⇒ energy exchange is slowest process